Design and Analysis of Algorithms (XIV)

Various Problems

Guoqiang Li
School of Software

SHANGHAI JIAO TONG UNIVERSITY

Poly-Time Reductions

Algorithm design patterns and antipatterns

Algorithm design patterns.

Algorithm design patterns and antipatterns

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Algorithm design patterns.

- Divide and conquer.
- Dynamic programming.
- Greedy.
- Duality.
- Reductions.
- Local search.
- Approximation.
- Randomization.

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Classify problems according to computational requirements

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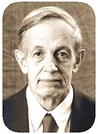
A working definition. Those with poly-time algorithms.


von Neumann (1953)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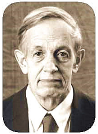
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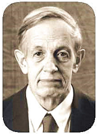
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Practice. Poly-time algorithms scale to huge problems.

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A working definition. Those with poly-time algorithms.

yes	probably no
shortest path	longest path
min cut	max cut
2-satisfiability	3-satisfiability
planar 4-colorability	planar 3-colorability
bipartite vertex cover	vertex cover
2D-matching	3D-matching
primality testing	factoring
linear programming	integer linear programming

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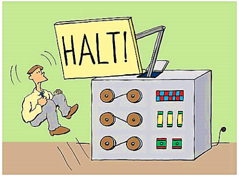
- Given a board position in an n-by-n generalization of checkers, can black guarantee a win?



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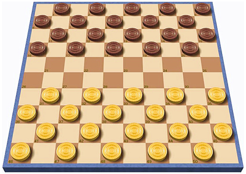

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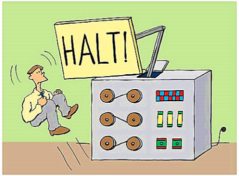

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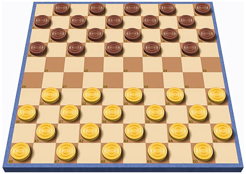

Frustrating news. Huge number of fundamental problems have defied classification for decades.

Poly-time reductions

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- Polynomial number of standard computational steps, plus
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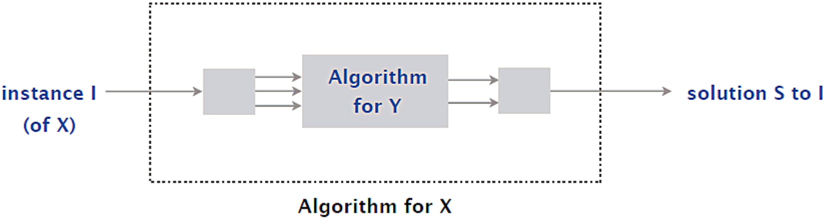

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Novice mistake. Confusing $X \leq_{\mathrm{P}} Y$ with $Y \leq_{\mathrm{P}} X$.

Quiz

Suppose that $X \leq_{\mathrm{P}} Y$. Which of the following can we infer?

A. If X can be solved in polynomial time, then so can Y.

B. X can be solved in poly time iff Y can be solved in poly time.

C. If X cannot be solved in polynomial time, then neither can Y.

D. If Y cannot be solved in polynomial time, then neither can X.

Quiz

Which of the following poly-time reductions are known?

A. FIND-MAX-FLOW $\leq_P$ FIND-MIN-CUT.

B. FIND-MIN-CUT $\leq_P$ FIND-MAX-FLOW.

C. Both A and B.

D. Neither A nor B.

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Design algorithms. If $X \le_{\mathrm{P}} Y$ and Y can be solved in polynomial time, then X can be solved in polynomial time.

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Establish intractability. If $X \leq_{\mathrm{P}} Y$ and X cannot be solved in polynomial time, then Y cannot be solved in polynomial time.

Establish equivalence. If both $X \leq_{\mathrm{P}} Y$ and $Y \leq_{\mathrm{P}} X$, we use notation $X \equiv_{P} Y$. In this case, X can be solved in polynomial time iff Y can be.

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Bottom line. Reductions classify problems according to relative difficulty.

Packing and Covering Problems

Independent set

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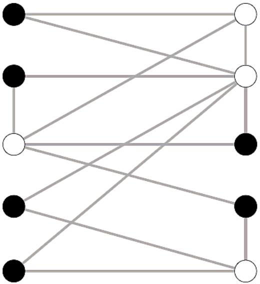
Example. Is there an independent set of size ≥ 6?
Example. Is there an independent set of size ≥ 7?



Vertex cover

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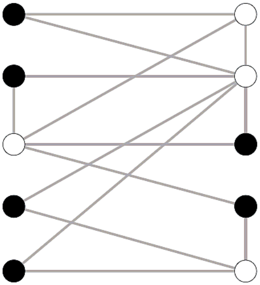
Example. Is there a vertex cover of size ≤ 4?
Example. Is there a vertex cover of size ≤ 3?



Quiz

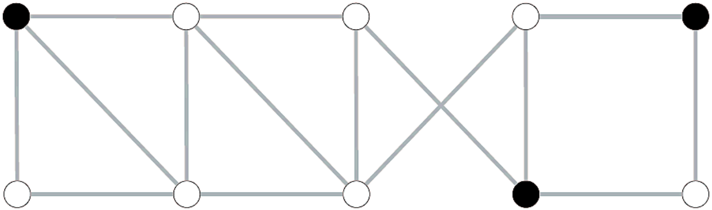
Consider the following graph G. Which are true?

A. The white vertices are a vertex cover of size 7.

B. The black vertices are an independent set of size 3.

C. Both A and B.

D. Neither A nor B.



Vertex cover and independent set reduce to one another

Theorem

Independent Set $\equiv_P$ *Vertex Cover*.

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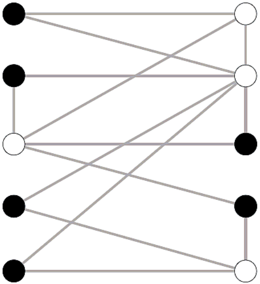
Proof. We show S is an independent set of size k iff $V - S$ is a vertex cover of size $n - k$.



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- Thus, $V - S$ covers (u, v).

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- Thus, S is an independent set.

Quiz

CLIQUE. Given a graph $G = (V, E)$ and an integer k, is there a subset of k (or more) vertices such that each of two are adjacent?

Set cover

SET COVER. Given a set U of elements, a collection S of subsets of U, and an integer k, are there $\leq k$ of these subsets whose union is equal to U?

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Sample application.

- m available pieces of software.
- Set U of n capabilities that we would like our system to have.
- The i^{th} piece of software provides the set $S_i \subseteq U$ of capabilities.
- Goal: achieve all n capabilities using fewest pieces of software.

$U = \{1,2,3,4,5,6,7\}$

$S_a = \{3,7\}$	$S_b = \{2,4\}$
$S_c = \{3,4,5,6\}$	$S_d = \{5\}$
$S_e = \{1\}$	$S_f = \{1,2,6,7\}$

$k = 2$

Quiz

Given the universe $U = \{1, 2, 3, 4, 5, 6, 7\}$ and the following sets, which is the minimum size of a set cover?

- A. 1
- B. 2
- C. 3
- D. None of the above.

$$
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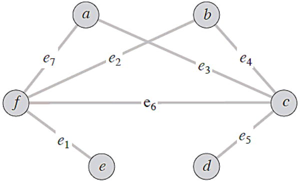
Construction.

- Universe $U = E$.
- Include one subset for each node $v \in V : S_v = \{e \in E :$ e incident to $v\}$.



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Lemma

$G = (V, E)$ *contains a vertex cover of size* k *iff* (U, S, k) *contains a set cover of size* k.

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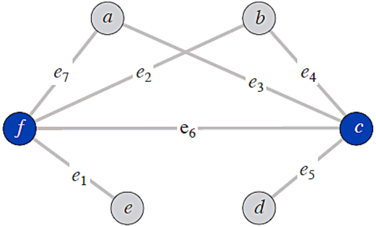
Lemma

$G = (V, E)$ contains a vertex cover of size k iff (U, S, k) contains a set cover of size k.

Proof. $\Rightarrow$

Let $X \subseteq V$ be a vertex cover of size k in G, then $Y = \{S_v : v \in X\}$ is a set cover of size k.



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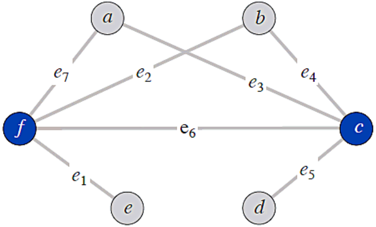

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Constraint Satisfaction Problems

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yes instance: x_1**= true,** x_2**= true,** x_3**= false** x_4**= false**

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Key application. Electronic design automation (EDA).

Satisfiability is hard

Scientific hypothesis. There does not exists a poly-time algorithm for 3-SAT.

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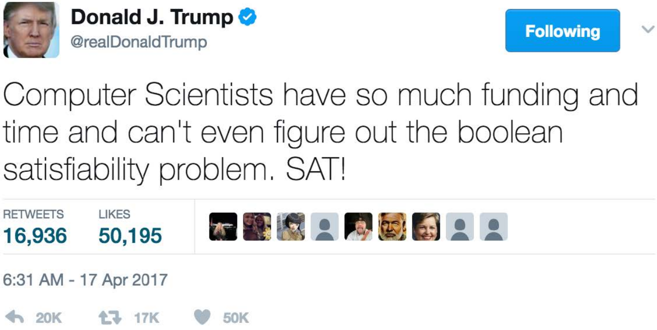

https://www.facebook.com/pg/npcompleteteens

3-satisfiability reduces to independent set

Theorem

3-SAT $\leq_P$ Independent Set.

3-satisfiability reduces to independent set

Theorem

3-SAT $\leq_P$ INDEPENDENT SET.

Proof.

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3-SAT $\leq_P$ Independent Set.

Proof. Given an instance Φ of 3-SAT, we construct an instance (G, k) of Independent Set that has an independent set of size $k = |\Phi|$ iff Φ is satisfiable.

3-satisfiability reduces to independent set

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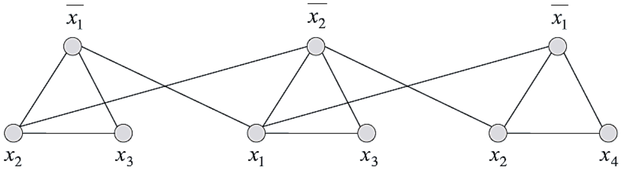
Proof. Given an instance Φ of 3-SAT, we construct an instance (G, k) of INDEPENDENT SET that has an independent set of size $k = |\Phi|$ iff Φ is satisfiable.

Construction.

- G contains 3 nodes for each clause, one for each literal.
- Connect 3 literals in a clause in a triangle.
- Connect literal to each of its negations.



$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

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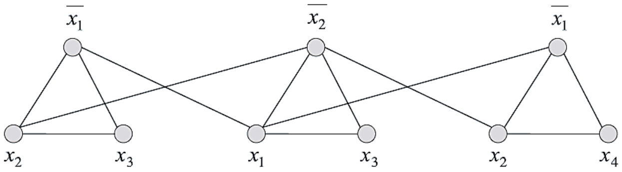
Theorem

3-SAT $\leq_P$ INDEPENDENT SET.

Proof. $\Rightarrow$

Consider any satisfying assignment for Φ.

- Select one true literal from each clause/triangle.
- This is an independent set of size $k = |\Phi|$.



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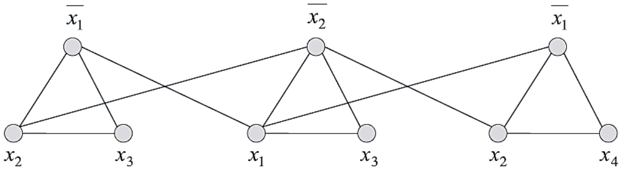
Theorem

3-SAT $\leq_P$ INDEPENDENT SET.

Proof. $\Leftarrow$

Let S be independent set of size k.

- S must contain exactly one node in each triangle.
- Set these literals to true and remaining literals consistently.
- All clauses in Φ are satisfied.



$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

Review

Basic reduction strategies.

- Simple equivalence: Independent Set $\equiv_P$ Vertex Cover
- Special case to general case: Vertex Cover $\leq_P$ Set Cover.
- Encoding with gadgets: 3-SAT $\leq_P$ Independent Set.

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Proof idea. Compose the two algorithms.

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Proof idea. Compose the two algorithms.

Example. 3-SAT $\leq_P$ Independent Set $\leq_P$ Vertex Cover $\leq_P$ Set Cover.

Decision, search and optimization problems

Decision problem. Does there exist a vertex cover of size $\leq k$?

Decision, search and optimization problems

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Search problem. Find a vertex cover of size $\leq k$.

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Decision, search and optimization problems

Decision problem. Does there exist a vertex cover of size $\leq k$?

Search problem. Find a vertex cover of size $\leq k$.

Optimization problem. Find a vertex cover of minimum size.

Goal. Show that all three problems poly-time reduce to one another.

Search problems VS. Decision problems

VERTEX COVER. Does there exist a vertex cover of size $\leq k$?

FIND VERTEX COVER. Find a vertex cover of size $\leq k$.

Search problems VS. Decision problems

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Search problems VS. Decision problems

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Search problems VS. Decision problems

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Search problems VS. Decision problems

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Proof.

$\leq_P$. Decision problem is a special case of search problem.

$\geq_P$. To find a vertex cover of size $\leq k$:

- Determine if there exists a vertex cover of size $\leq k$.
- Find a vertex v such that $G - \{v\}$ has a vertex cover of size $\leq k - 1$. (any vertex in any vertex cover of size $\leq k$ will have this property)
- Include v in the vertex cover.
- Recursively find a vertex cover of size $\leq k - 1$ in $G - \{v\}$.

Optimization problems VS. Search problems VS. Decision problems

Find vertex cover. Find a vertex cover of size $\leq k$.

Find min vertex cover. Find a vertex cover of minimum size.

Optimization problems VS. Search problems VS. Decision problems

FIND VERTEX COVER. Find a vertex cover of size $\leq k$.

FIND MIN VERTEX COVER. Find a vertex cover of minimum size.

Theorem. Find vertex cover $\equiv_P$ Find min vertex cover.

Optimization problems VS. Search problems VS. Decision problems

FIND VERTEX COVER. Find a vertex cover of size $\leq k$.

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Proof.

Optimization problems VS. Search problems VS. Decision problems

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Optimization problems VS. Search problems VS. Decision problems

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$\geq_P$. To find vertex cover of minimum size:

Optimization problems VS. Search problems VS. Decision problems

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FIND MIN VERTEX COVER. Find a vertex cover of minimum size.

Theorem. Find vertex cover $\equiv_P$ Find min vertex cover.

Proof.

$\leq_P$. Search problem is a special case of optimization problem.

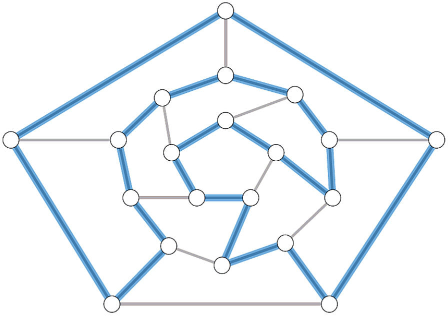
$\geq_P$. To find vertex cover of minimum size:

- Binary search (or linear search) for size k^* of min vertex cover.
- Solve search problem for given k^*.

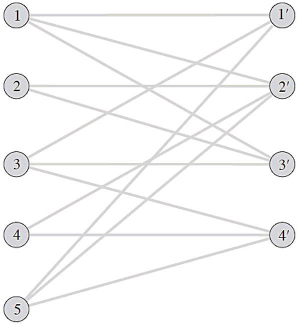
Sequencing Problems

Hamilton cycle

HAMILTON CYCLE. Given an undirected graph $G = (V, E)$, does there exist a cycle Γ that visits every node exactly once?



yes



no

Directed Hamilton cycle reduces to Hamilton cycle

DIRECTED HAMILTON CYCLE. Given a directed graph $G = (V, E)$, does there exist a directed cycle Γ that visits every node exactly once?

Theorem

DIRECTED HAMILTON CYCLE $\leq_P$ HAMILTON CYCLE.

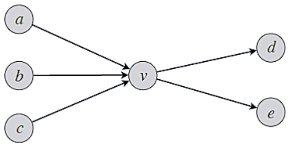
Directed Hamilton cycle reduces to Hamilton cycle

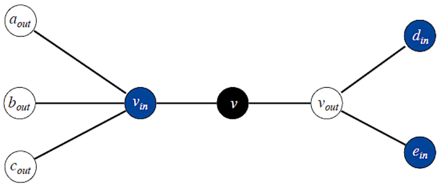
DIRECTED HAMILTON CYCLE. Given a directed graph $G = (V, E)$, does there exist a directed cycle Γ that visits every node exactly once?

Theorem

DIRECTED HAMILTON CYCLE $\leq_P$ HAMILTON CYCLE.

Proof. Given a directed graph $G = (V, E)$, construct a graph G' with $3n$ nodes.





Directed Hamilton cycle reduces to Hamilton cycle

Lemma

G has a directed Hamilton cycle iff G' has a Hamilton cycle.

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$\Rightarrow$

- Suppose G has a directed Hamilton cycle Γ.
- Then G' has an undirected Hamilton cycle (same order).

Directed Hamilton cycle reduces to Hamilton cycle

Lemma

G has a directed Hamilton cycle iff G' has a Hamilton cycle.

Proof.

$\Rightarrow$

- Suppose G has a directed Hamilton cycle Γ.
- Then G' has an undirected Hamilton cycle (same order).

$\Leftarrow$

- Suppose G' has an undirected Hamilton cycle Γ'.
- Γ' must visit nodes in G' using one of following two orders:
  $\ldots, black, white, blue, black, white, blue, black, white, blue, \ldots$
  $\ldots, black, blue, white, black, blue, white, black, blue, white, \ldots$
- Black nodes in Γ' comprise either a directed Hamilton cycle Γ in G, or reverse of one.

3-satisfiability reduces to directed Hamilton cycle

Theorem

3-SAT $\leq_P$ Directed Hamilton cycle.

3-satisfiability reduces to directed Hamilton cycle

Theorem

3-SAT $\leq_P$ Directed Hamilton cycle.

Proof.

3-satisfiability reduces to directed Hamilton cycle

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Theorem

3-SAT $\leq_P$ DIRECTED HAMILTON CYCLE.

Proof.

Given an instance Φ of 3-SAT, we construct an instance G of Directed Hamilton cycle that has a Hamilton cycle iff Φ is satisfiable.

3-satisfiability reduces to directed Hamilton cycle

Theorem

3-SAT $\leq_P$ DIRECTED HAMILTON CYCLE.

Proof.

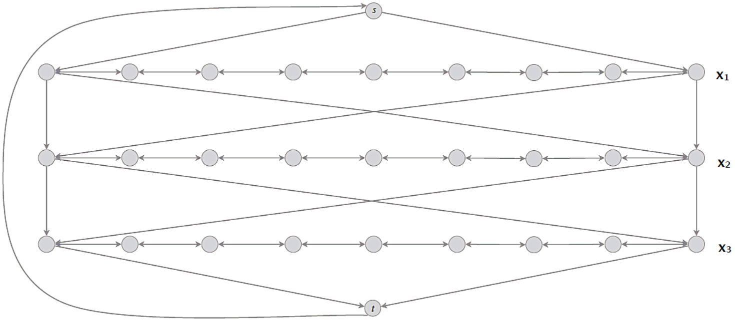
Given an instance Φ of 3-SAT, we construct an instance G of Directed Hamilton cycle that has a Hamilton cycle iff Φ is satisfiable.

Construction overview. Let n denote the number of variables in Φ. We will construct a graph G that has 2^n Hamilton cycles, with each cycle corresponding to one of the 2^n possible truth assignments.

3-satisfiability reduces to directed Hamilton cycle

Construction. Given 3-SAT instance Φ with n variables x_i and k clauses.

- Construct G to have 2^n Hamilton cycles.
- Intuition: traverse path i from left to right $\Leftrightarrow$ set variables x_i =true



Quiz

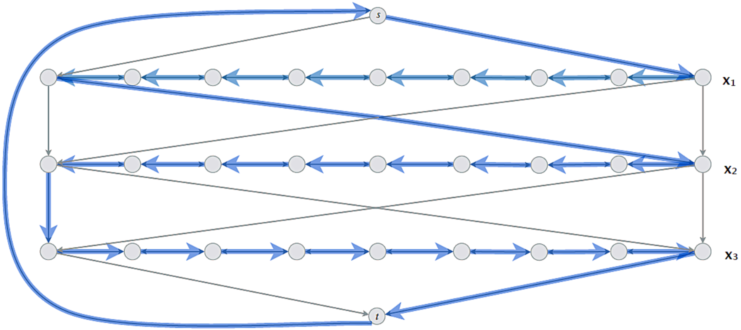
Which is truth assignment corresponding to Hamilton cycle below?

A. $x_1 = true, x_2 = true, x_3 = true$

B. $x_1 = true, x_2 = true, x_3 = false$

C. $x_1 = false, x_2 = false, x_3 = true$

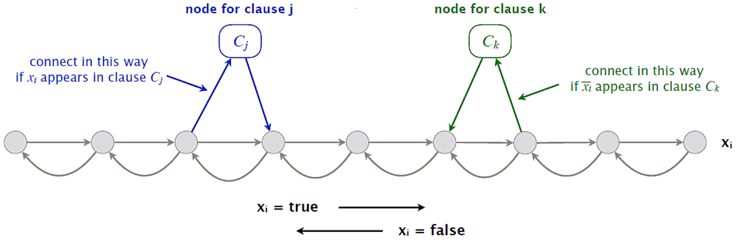
C. $x_1 = false, x_2 = false, x_3 = false$



3-satisfiability reduces to directed Hamilton cycle

Construction. Given 3-SAT instance Φ with n variables x_i and k clauses.

- For each clause: add a node and 2 edges per literal.



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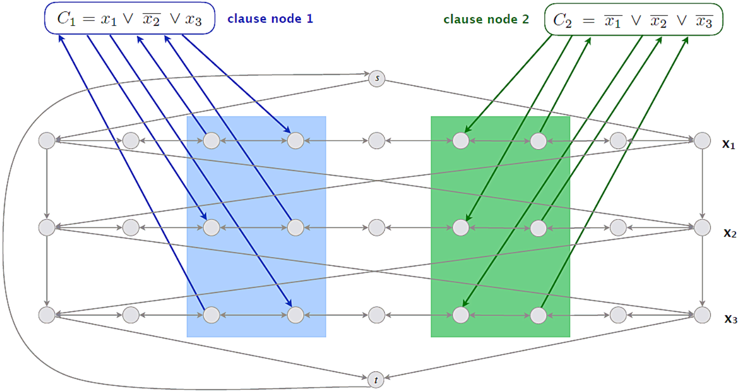

3-satisfiability reduces to directed Hamilton cycle

Lemma

Φ is satisfiable iff G has a Hamilton cycle.

3-satisfiability reduces to directed Hamilton cycle

Lemma

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Proof.

3-satisfiability reduces to directed Hamilton cycle

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Lemma

Φ *is satisfiable iff* G *has a Hamilton cycle.*

Proof. $\Rightarrow$

- Suppose 3-SAT instance Φ has satisfying assignment x^*.
- Then, define Hamilton cycle Γ in G as follows:
  - if $x_i^* = true$, traverse row i from left to right.
  - if $x_i^* = false$, traverse row i from right to left.
  - for each clause C_j, there will be at least one row i in which we are going in "correct" direction to splice clause node C_j into cycle (and we splice in C_j exactly once)

3-satisfiability reduces to directed Hamilton cycle

Lemma

Φ is satisfiable iff G has a Hamilton cycle.

Proof.

3-satisfiability reduces to directed Hamilton cycle

Lemma

Φ is satisfiable iff G has a Hamilton cycle.

Proof. $\Leftarrow$

- Suppose G has a Hamilton cycle Γ.
- If Γ enters clause node C_j, it must depart on mate edge.
  - nodes immediately before and after C_j are connected by an edge $e \in E$.
  - removing C_j from cycle, and replacing it with edge e yields Hamilton cycle on $G - \{C_j\}$.
- Continuing in this way, we are left with a Hamilton cycle Γ' in $G - \{C_1, C_2, \ldots, C_k\}$.
- Set $x_i^* = true$ if Γ' traverses row i left-to-right; otherwise, set $x_i^* = false$.
- traversed in "correct" direction, and each clause is satisfied.

Graph Coloring

Home reading!

Numerical Problems

Subset sum

Subset sum. Given n natural numbers $w_1, \ldots, w_n$ and an integer W, is there a subset that adds up to exactly W?

Subset sum

SUBSET SUM. Given n natural numbers $w_1, \ldots, w_n$ and an integer W, is there a subset that adds up to exactly W?

Example. $\{$ 215, 215, 275, 275, 355, 355, 420, 420, 580, 580, 655, 655 $\}$, W = 1505.

Subset sum

Subset sum. Given n natural numbers $w_1, \ldots, w_n$ and an integer W, is there a subset that adds up to exactly W?

Example. { 215, 215, 275, 275, 355, 355, 420, 420, 580, 580, 655, 655 }, W = 1505.

Yes. 215 + 355 + 355 + 580 = 1505.

Subset sum

SUBSET SUM. Given n natural numbers $w_1, \ldots, w_n$ and an integer W, is there a subset that adds up to exactly W?

Example. $\{$ 215, 215, 275, 275, 355, 355, 420, 420, 580, 580, 655, 655 $\}$, W = 1505.

Yes. 215 + 355 + 355 + 580 = 1505.

Remark. With arithmetic problems, input integers are encoded in binary. Poly-time reduction must be polynomial in binary encoding.

Subset sum

Theorem

3-SAT $\leq_P$ Subset sum.

Subset sum

Theorem

3-SAT $\leq_P$ SUBSET SUM.

Proof. Given an instance Φ of 3-SAT, we construct an instance of Subset sum that has solution iff Φ is satisfiable.

3-satisfiability reduces to subset sum

Construction. Given 3-SAT instance Φ with n variables and k clauses, form $2n + 2k$ decimal integers, each having $n + k$ digits:

- Include one digit for each variable x_i and one digit for each clause C_j.
- Include two numbers for each variable x_i.
- Include two numbers for each clause C_j.

3-satisfiability reduces to subset sum

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- Sum of each x_i digit is 1;
- Sum of each C_j digit is 4.

Key property. No carries possible $\Rightarrow$ each digit yields one equation.

$$
\begin{aligned}
C_1 &= \neg x_1 \vee x_2 \vee x_3 \\
C_2 &= x_1 \vee \neg x_2 \vee x_3 \\
C_3 &= \neg x_1 \vee \neg x_2 \vee \neg x_3
\end{aligned}
$$

dummies to get clause columns to sum to 4

	x_1	x_2	x_3	C_1	C_2	C_3	
x_1	1	0	0	0	1	0	100,010
$\neg x_1$	1	0	0	1	0	1	100,101
x_2	0	1	0	1	0	0	10,100
$\neg x_2$	0	1	0	0	1	1	10,011
x_3	0	0	1	1	1	0	1,110
$\neg x_3$	0	0	1	0	0	1	1,001
	0	0	0	1	0	0	100
	0	0	0	2	0	0	200
	0	0	0	0	1	0	10
	0	0	0	0	2	0	20
	0	0	0	0	0	1	1
	0	0	0	0	0	2	2
W	1	1	1	4	4	4	111,444

3-satisfiability reduces to subset sum

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Lemma

Φ is satisfiable iff there exists a subset that sums to W.

3-satisfiability reduces to subset sum

Lemma

Φ is satisfiable iff there exists a subset that sums to W.

Proof.

3-satisfiability reduces to subset sum

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Lemma

Φ is satisfiable iff there exists a subset that sums to W.

Proof. $\Rightarrow$ Suppose 3-SAT instance Φ has satisfying assignment x^*. If $x_i^* = true$, select integer in row x_i, otherwise, select integer in row $\neg x_i$.

- Each x_i digit sums to 1.
- Since Φ is satisfiable, each C_j digit sums to at least 1 from x_i and $\neg x_i$ rows.
- Select dummy integers to make C_j digits sum to 4.

$$
\begin{aligned}
C_1 &= \neg x_1 \vee x_2 \vee x_3 \\
C_2 &= x_1 \vee \neg x_2 \vee x_3 \\
C_3 &= \neg x_1 \vee \neg x_2 \vee \neg x_3
\end{aligned}
$$

dummies to get clause columns to sum to 4

	x_1	x_2	x_3	C_1	C_2	C_3	
x_1	1	0	0	0	1	0	100,010
$\neg x_1$	1	0	0	1	0	1	100,101
x_2	0	1	0	1	0	0	10,100
$\neg x_2$	0	1	0	0	1	1	10,011
x_3	0	0	1	1	1	0	1,110
$\neg x_3$	0	0	1	0	0	1	1,001
	0	0	0	1	0	0	100
	0	0	0	2	0	0	200
	0	0	0	0	1	0	10
	0	0	0	0	2	0	20
	0	0	0	0	0	1	1
	0	0	0	0	0	2	2
W	1	1	1	4	4	4	111,444

3-satisfiability reduces to subset sum

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Lemma

Φ is satisfiable iff there exists a subset that sums to W.

Proof.

3-satisfiability reduces to subset sum

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Lemma

Φ is satisfiable iff there exists a subset that sums to W.

Proof. $\Leftarrow$ Suppose there exists a subset S^* that sums to W. Digit x_i forces subset S^* to select either row x_i or row $\neg x_i$ (but not both). If row x_i selected, assign $x_i^* = true$; otherwise, assign $x_i^* = false$.

Digit C_j forces subset S^* to select at least one literal in clause.

$$
\begin{aligned}
C_1 &= \neg x_1 \vee x_2 \vee x_3 \\
C_2 &= x_1 \vee \neg x_2 \vee x_3 \\
C_3 &= \neg x_1 \vee \neg x_2 \vee \neg x_3
\end{aligned}
$$

dummies to get clause columns to sum to 4

	x_1	x_2	x_3	C_1	C_2	C_3	
x_1	1	0	0	0	1	0	100,010
$\neg x_1$	1	0	0	1	0	1	100,101
x_2	0	1	0	1	0	0	10,100
$\neg x_2$	0	1	0	0	1	1	10,011
x_3	0	0	1	1	1	0	1,110
$\neg x_3$	0	0	1	0	0	1	1,001
	0	0	0	1	0	0	100
	0	0	0	2	0	0	200
	0	0	0	0	1	0	10
	0	0	0	0	2	0	20
	0	0	0	0	0	1	1
	0	0	0	0	0	2	2
W	1	1	1	4	4	4	111,444

Subset sum reduces to knapsack

Subset sum. Given n natural numbers $w_1, \ldots, w_n$ and an integer W, is there a subset that adds up to exactly W?

Knapsack. Given a set of items X, weights $u_i \geq 0$, values $v_i \geq 0$, a weight limit U, and a target value V, is there a subset $S \subseteq X$ such that:

$$\sum_{i \in S} u_i \leq U, \quad \sum_{i \in S} v_i \geq V$$

Subset sum reduces to knapsack

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$$\sum_{i \in S} u_i \leq U, \quad \sum_{i \in S} v_i \geq V$$

Recall. $O(nU)$ dynamic programming algorithm for knapsack.

Subset sum reduces to knapsack

Subset sum. Given n natural numbers $w_1, \ldots, w_n$ and an integer W, is there a subset that adds up to exactly W?

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$$\sum_{i \in S} u_i \leq U, \quad \sum_{i \in S} v_i \geq V$$

Recall. $O(nU)$ dynamic programming algorithm for knapsack.

Challenge. Prove subset sum $\leq_P$ Knapsack.

SAT to 3SAT

From SAT problem to 3SAT problem

$$\left\{ \begin{array}{c} (a_1 \vee a_2 \vee \cdots \vee a_k) \\ \text{is satisfied} \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{there is a setting of the } y_i\text{'s for which} \\ (a_1 \vee a_2 \vee y_1)\ (\overline{y}_1 \vee a_3 \vee y_2)\ \cdots\ (\overline{y}_{k-3} \vee a_{k-1} \vee a_k) \\ \text{are all satisfied} \end{array} \right\}$$

From SAT problem to 3SAT problem

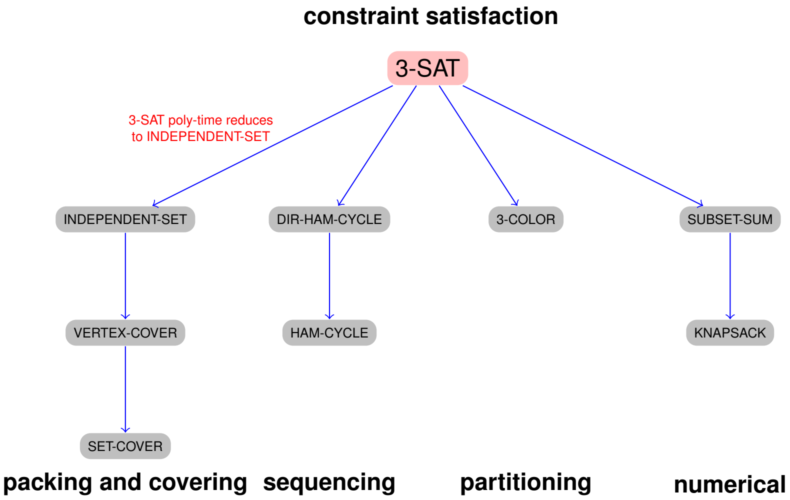
$$\left\{ \begin{array}{c} (a_1 \vee a_2 \vee \cdots \vee a_k) \\ \text{is satisfied} \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{there is a setting of the } y_i\text{'s for which} \\ (a_1 \vee a_2 \vee y_1)\ (\overline{y}_1 \vee a_3 \vee y_2)\ \cdots\ (\overline{y}_{k-3} \vee a_{k-1} \vee a_k) \\ \text{are all satisfied} \end{array} \right\}$$

Suppose that the clauses on the right are all satisfied. Then at least one of the literals $a_1, \ldots, a_k$ must be `true`.

Otherwise y_1 would have to be `true`, which would in turn force y_2 to be `true`, and so on.

Conversely, if $(a_1 \vee a_2 \vee \ldots \vee a_k)$ is satisfied, then some a_i must be `true`. Set $y_1, \ldots, y_{i-2}$ to `true` and the rest to `false`.

Poly-time reductions



Karp's 20 poly-time reductions from satisfiability

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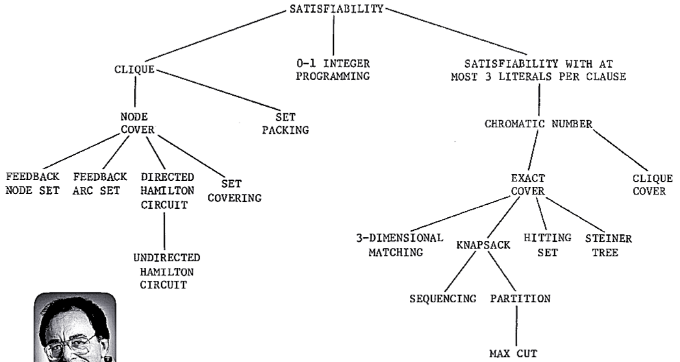

FIGURE 1 - Complete Problems

RICHARD M. KARP

Dick Karp (1972)
1985 Turing Award

Referred Materials

Referred Materials

- Content of this lecture comes from Chapter 8 in [KT05].