Design and Analysis of Algorithms XV

Complexity Classes

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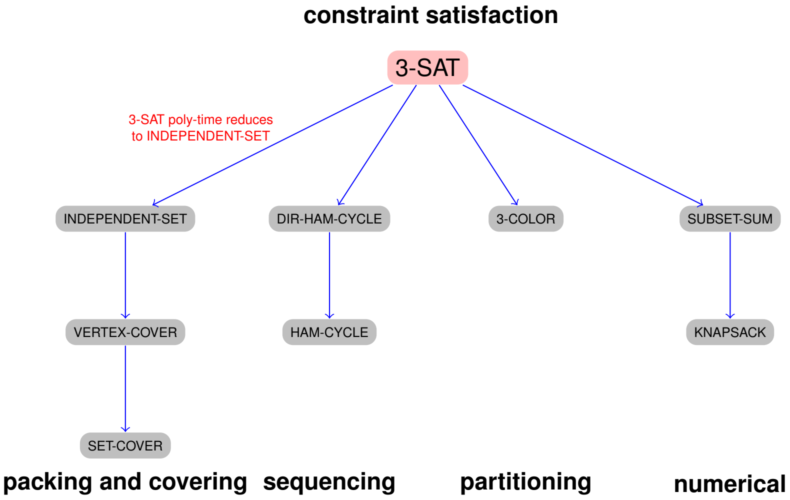
constraint satisfaction
3-SAT
3-SAT poly-time reduces
to INDEPENDENT-SET
INDEPENDENT-SET
DIR-HAM-CYCLE
3-COLOR
SUBSET-SUM
VERTEX-COVER
HAM-CYCLE
KNAPSACK
SET-COVER
packing and covering
sequencing
partitioning
numerical

P VS. NP

Decision problem.

- Problem X is a set of strings.
- Instance s is one string.
- Algorithm A solves problem X: $A(s) = \begin{cases} yes & \text{if } s \in X \\ no & \text{if } s \notin X \end{cases}$

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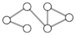
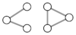
problem PRIMES**:**	$\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \ldots\}$
instance s**:**	592335744548702854681
algorithm:	Agrawal-Kayal-Saxena (2002)

Some problems in P

P. Decision problems for which there exists a poly-time algorithm.

problem	description	poly-time algorithm	yes	no
MULTIPLE	Is x a multiple of y?	grade-school division	51, 17	51, 16
REL-PRIME	Are x and y relatively prime?	Euclid's algorithm	34, 39	34, 51
PRIMES	Is x prime?	Agrawal-Kayal-Saxena	53	51
EDIT-DISTANCE	Is the edit distance between x and y less than 5?	Needleman–Wunsch	niether neither	acgggt ttttta
L-SOLVE	Is there a vector x that satisfies $Ax = b$?	Gauss–Edmonds elimination	$\begin{bmatrix} 0 & 1 & 1 \\ 2 & 4 & -2 \\ 0 & 3 & 15 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 36 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
U-CONN	Is an undirected graph G connected?	depth-first search		

NP

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problem COMPOSITES:	$\{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, \ldots\}$
instance s:	437669
certificate t:	$541 \longleftarrow 437,669 = 541 \times 809$
certifier $\mathrm{C(s,t)}$:	grade school division

Certifiers and certificates: satisfiability

SAT. Given a CNF formula Φ, does it have a satisfying truth assignment?

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instance s $\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$

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Conclusions. SAT$\in$ **NP**, 3-SAT $\in$ **NP**

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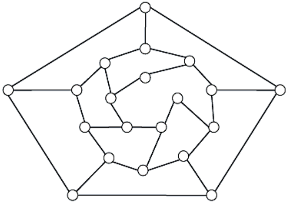

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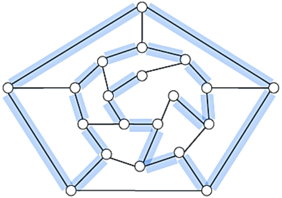

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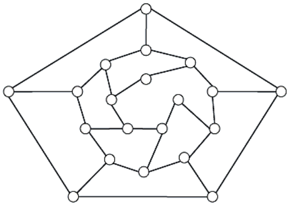

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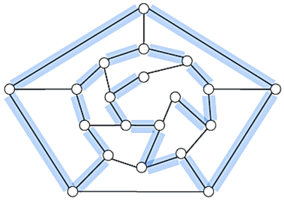

certificate t

Conclusion. Hamilton path $\in$ **NP.**

Some problems in NP

NP. Decision problems for which there exists a poly-time certifier.

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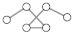
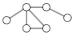
L-solve	Is there a vector x that satisfies $Ax = b$?	Gauss–Edmonds elimination	$\begin{bmatrix} 0 & 1 & 1 \\ 2 & 4 & -2 \\ 0 & 3 & 15 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 36 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
Composites	Is x composite ?	Agrawal-Kayal-Saxena	51	53
Factor	Does x have a nontrivial factor less than y ?	???	(56159, 50)	(55687, 50)
SAT	Given a CNF formula, does it have a satisfying truth assignment?	???	$\neg x_1 \vee x_2 \vee \neg x_3$ $x_1 \vee \neg x_2 \vee x_3$ $\neg x_1 \vee \neg x_2 \vee x_3$	$\neg x_2$ $x_1 \vee x_2$ $\neg x_1 \vee x_2$
Hamilton path	Is there a simple path between u and v that visits every node?	???		

Quiz

Which of the following graph problems are known to be in NP?

- A. Is the length of the longest simple path $\leq k$?
- B. Is the length of the longest simple path $\geq k$?
- C. Is the length of the longest simple path $= k$?
- D. Find the length of the longest simple path.
- E. All of the above.

Quiz

In complexity theory, the abbreviation NP stands for . . .

A. Nope.
B. No problem.
C. Not polynomial time.
D. Not polynomial space.
E. Nondeterministic polynomial time.

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"In an ideal world it would be renamed P vs VP. "

–Clyde Kruskal

P, NP, and EXP

P. Decision problems for which there exists a poly-time algorithm.

NP. Decision problems for which there exists a poly-time certifier.

EXP. Decision problems for which there exists an exponential-time algorithm.

P, NP, and EXP

Proposition. **P** $\subseteq$ **NP**.

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Proof.

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Proof. Consider any problem $X \in$ **P**.

- By definition, there exists a poly-time algorithm $A(s)$ that solves X.
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Fact. **P** $\neq$ **EXP** $\Rightarrow$ either **P** $\neq$ **NP**, or **NP** $\neq$ **EXP**, or both.

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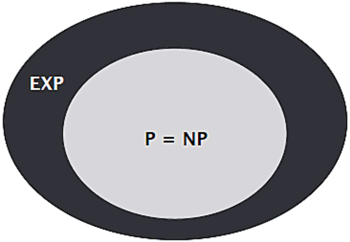
Conjecture. No poly-time algorithm for 3-SAT.

"intractable"

The main question: P vs. NP

Does **P** = **NP**? [Cook 1971, Edmonds, Levin, Yablonski, Gödel]
Is the decision problem as easy as the certification problem?



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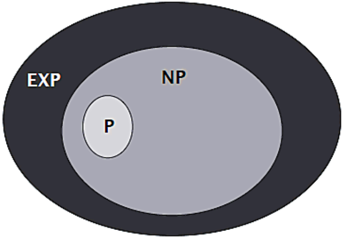

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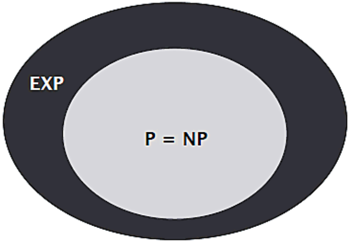

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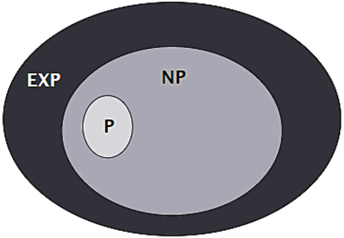

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If yes. . . Efficient algorithms for 3-SAT, TSP, VERTEX-COVER, FACTOR. . .

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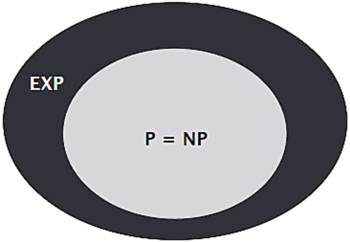

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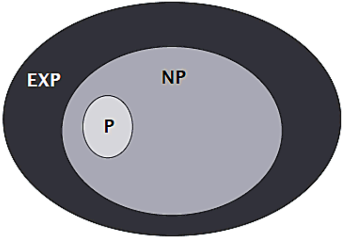

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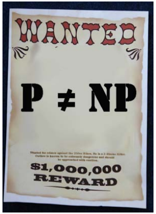
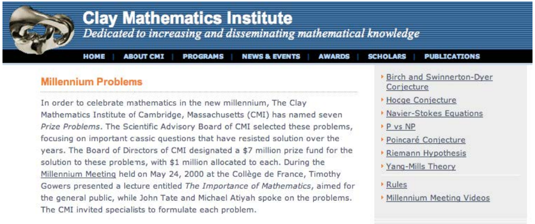
Consensus opinion. Probably no.

Millennium prize

Millennium prize. \$1 million for resolution of $\mathbf{P} \neq \mathbf{NP}$ problem.





NP-complete

Polynomial transformations

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Problem X polynomial (Cook) reduces to problem Y if arbitrary instances of problem X can be solved using:

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Open question. Are these two concepts the same with respect to **NP**?

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- This implies **NP** $\subseteq$ **P**.
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Fundamental question. Are there any "natural" **NP**-complete problems?

The "first" NP-complete problem

Theorem (Cook 1971, Levin 1973)

SAT $\in$ ***NP***-*complete.*

The Complexity of Theorem-Proving Procedures

Stephen A. Cook

University of Toronto

Summary

It is shown that any recognition problem solved by a polynomial time-bounded nondeterministic Turing machine can be "reduced" to the problem of determining whether a given propositional formula is a tautology. Here "reduced" means, roughly speaking, that the first problem can be solved deterministically in polynomial time provided an oracle is available for solving the second. From this notion of reducible, polynomial degrees of difficulty are defined, and it is shown that the problem of determining tautologyhood has the same polynomial degree as the problem of determining whether the first of two given graphs is isomorphic to a subgraph of the second. Other examples are discussed. A method of measuring the complexity of proof procedures for the predicate calculus is introduced and discussed.

Throughout this paper, a set of strings means a set of strings on some fixed, large, finite alphabet Σ. This alphabet is large enough to include symbols for all sets described here. All Turing machines are deterministic recognition devices, unless the contrary is explicitly stated.

1. Tautologies and Polynomial Re-Reducibility.

Let us fix a formalism for the propositional calculus in which formulas are written as strings on Σ. Since we will require infinitely many proposition symbols (atoms), each such symbol will consist of a member of Σ followed by a number in binary notation to distinguish that symbol. Thus a formula of length n can only have about n/logn distinct function and predicate symbols. The logical connectives are & (and), ∨ (or), and ¬(not).

The set of tautologies (denoted by {tautologies}) is a certain recursive set of strings on this alphabet, and we are interested in the problem of finding a good lower bound on its possible recognition times. We provide no such lower bound here, but theorem 1 will give evidence that {tautologies} is a difficult set to recognize, since many apparently difficult problems can be reduced to determining tautologyhood. By reduced we mean, roughly speaking, that if tautologyhood could be decided instantly (by an "oracle") then these problems could be decided in polynomial time. In order to make this notion precise, we introduce query machines, which are like Turing machines with oracles in [1].

A query machine is a multitape Turing machine with a distinguished tape called the query tape, and three distinguished states called the query state, yes state, and no state, respectively. If M is a query machine and T is a set of strings, then a T-computation of M is a computation of M in which initially M is in the initial state and has an input string w on its input tape, and each time M assumes the query state there is a string u on the query tape, and the next state M assumes is the yes state if u∈T and the no state if u∉T. We think of an "oracle", which knows T, placing M in the yes state or no state.

Definition

A set S of strings is P-reducible (P for polynomial) to a set T of strings iff there is some query machine M and a polynomial Q(n) such that for each input string w, the T-computation of M with input w halts within Q(|w|) steps (|w| is the length of w), and ends in an accepting state iff w∈S.

It is not hard to see that P-reducibility is a transitive relation. Thus the relation E on

ПРОБЛЕМЫ ПЕРЕДАЧИ ИНФОРМАЦИИ

Том IX *1973* *Вып. 3*

КРАТКИЕ СООБЩЕНИЯ

УДК 519.14

УНИВЕРСАЛЬНЫЕ ЗАДАЧИ ПЕРЕБОРА

Л. А. Левин

В статье рассматривается несколько известных массовых задач «переборного типа» и доказывается, что эти задачи можно решать лишь за такое время, за которое можно решать вообще любые задачи указанного типа.

После уточнения понятия алгоритма была доказана алгоритмическая неразрешимость ряда классических массовых проблем (например, проблем тождества элементов групп, гомеоморфности многообразий, разрешимости диофантовых уравнений и других). Тем самым был снят вопрос о нахождении практического способа их решения. Однако существование алгоритмов для решения других задач не снимает для них аналогичного вопроса из-за фантастически большого объема работы, предписываемого этими алгоритмами. Такова ситуация с так называемыми переборными задачами: минимизации булевых функций, поиска доказательств ограниченной длины, выяснения изоморфности графов и другими. Все эти задачи решаются тривиальными алгоритмами, состоящими в переборе всех возможностей. Однако эти алгоритмы требуют экспоненциального времени работы и у математиков сложилось убеждение, что более простые алгоритмы для них невозможны. Был получен ряд серьезных аргументов в пользу его справедливости (см.[1, 2]), однако доказать это утверждение не удалось никому. (Например, до сих пор не доказано, что для нахождения математических доказательств нужно больше времени, чем для их проверки.)

Однако если предположить, что вообще существует какая-нибудь (хотя бы искусственно построенная) массовая задача переборного типа, неразрешимая простыми (в смысле объема вычислений) алгоритмами, то можно показать, что этим же свойством обладают и многие «классические» переборные задачи (в том числе задача минимизации, задача поиска доказательств и др.). В этом и состоят основные результаты статьи.

Функции $f(n)$ и $g(n)$ будем называть сравнимыми, если при некотором k

$$f(n) \leqslant (g(n)+2)^k \quad \text{и} \quad g(n) \leqslant (f(n)+2)^k.$$

Аналогично будем понимать термин «меньше или сравнимо».

Определение. Задачей переборного типа (или просто переборной задачей) будем называть задачу вида «по данному x найти какое-нибудь y длины, сравнимой с длиной x, такое, что выполняется $A(x, y)$», где $A(x, y)$ – какое-нибудь свойство, проверяемое алгоритмом, время работы которого сравнимо с длиной x. (Под алгоритмом здесь можно понимать, например, алгоритмы Колмогорова — Успенского или машины Тьюринга, или нормальные алгоритмы; x, y – двоичные слова). Квазипереборной задачей будем называть задачу выяснения, существует ли такое y.

Мы рассмотрим шесть задач этих типов. Рассматриваемые в них объекты кодируются естественным образом в виде двоичных слов. При этом выбор естественной кодировки не существен, так как все они дают сравнимые длины кодов.

Задача 1. Заданы списком конечное множество и покрытие его 500-элементными подмножествами. Найти подпокрытие заданной мощности (соответственно выяснить существует ли оно).

Задача 2. Таблично задана частичная булева функция. Найти заданного размера дизъюнктивную нормальную форму, реализующую эту функцию в области определения (соответственно выяснить существует ли она).

Задача 3. Выяснить, выводима или опровержима данная формула исчисления высказываний. (Или, что то же самое, равна ли константе данная булева формула.)

Задача 4. Даны два графа. Найти гомоморфизм одного на другой (выяснить его существование).

Задача 5. Даны два графа. Найти изоморфизм одного в другой (на его часть).

Задача 6. Рассматриваются матрицы из целых чисел от 1 до 100 и некоторое условие о том, какие числа в них могут соседствовать по вертикали и какие по горизонтали. Заданы числа на границе и требуется продолжить их на всю матрицу с соблюдением условия.

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Recipe. To prove that $Y \in$ **NP**-complete:

- Step 1. Show that $Y \in$ **NP**.
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Proposition

*If $X \in$ **NP**-complete, $Y \in$ **NP**, and $X \leq_P Y$, then $Y \in$ **NP**-complete.*

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Proof. Consider any problem $W \in$ **NP**. Then, both $W \leq_P X$ and $X \leq_P Y$.

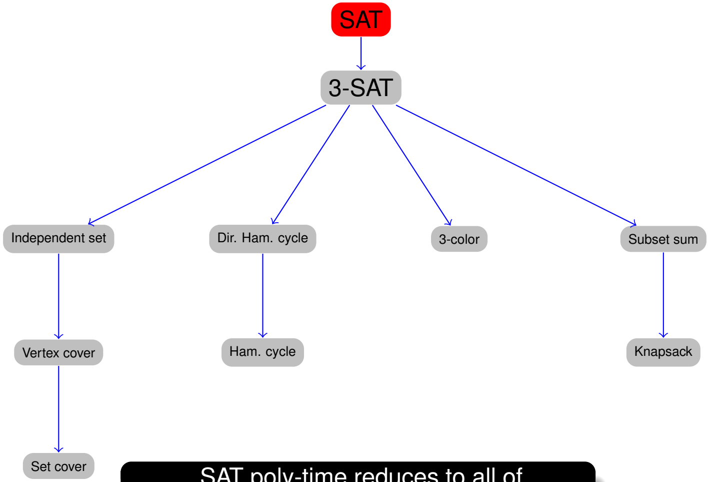
- By transitivity, $W \leq_P Y$.
- Hence $Y \in$ **NP**-complete.

Quiz

Suppose that $X \in$ **NP**-Complete, $Y \in$ **NP**, and $X \leq_P Y$. Which can you infer?

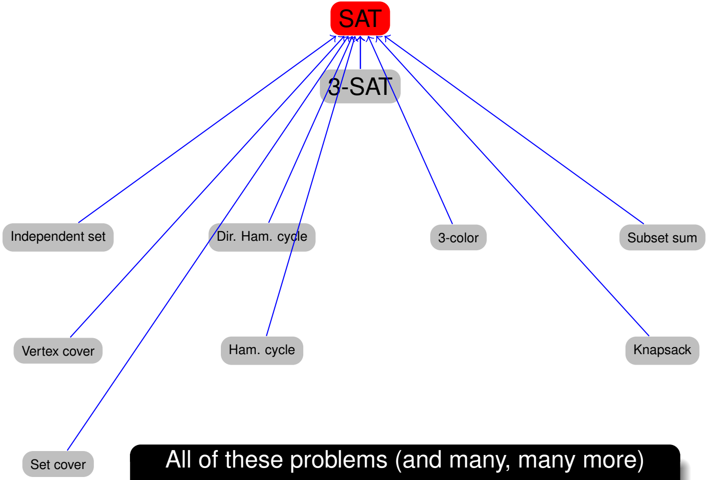
A. Y is **NP**-complete.
B. If $Y \notin$ **P**, then **P**$\neq$ **NP**.
C. If **P**$\neq$**NP**, then neither X nor Y is in **P**.
D. All of the above.

Implications of Karp



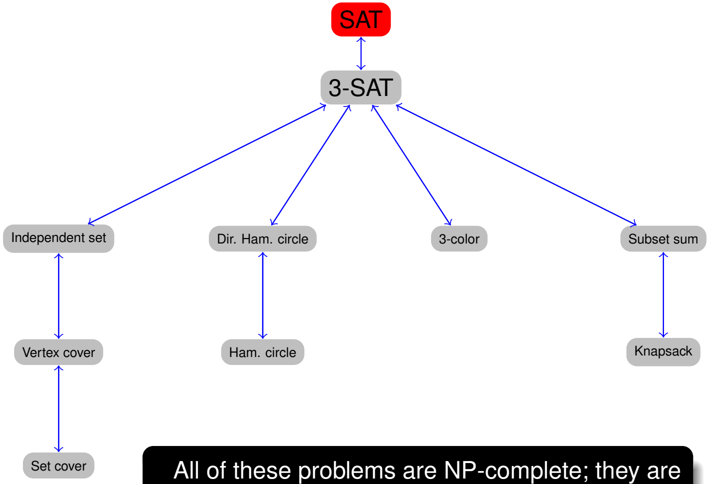
SAT poly-time reduces to all of these problems (and many, many more)

Implications of Cook–Levin



All of these problems (and many, many more) poly-time reduce to SAT .

Implications of Karp + Cook–Levin



All of these problems are NP-complete; they are manifestations of the same really hard problem.

Some NP-complete problems

Basic genres of **NP**-complete problems and paradigmatic examples.

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Theorem (Ladner 1975)

*Unless **P = NP**, there exist problems in **NP** that are in neither **P** nor **NP**-complete.*

More hard computational problems

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Garey and Johnson. *Computers and Intractability*.

- Appendix includes over 300 **NP**-complete problems.
- Most cited reference in computer science literature.

Most Cited Computer Science Citations

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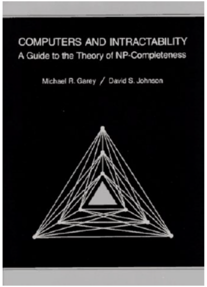

More hard computational problems

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Aerospace engineering. Optimal mesh partitioning for finite elements.
Biology. Phylogeny reconstruction.
Chemical engineering. Heat exchanger network synthesis.
Chemistry. Protein folding.
Civil engineering. Equilibrium of urban traffic flow.
Economics. Computation of arbitrage in financial markets with friction.
Electrical engineering. VLSI layout.
Environmental engineering. Optimal placement of contaminant sensors.
Financial engineering. Minimum risk portfolio of given return.
Game theory. Nash equilibrium that maximizes social welfare.
Mathematics. Given integer $a_1, \ldots, a_n$, compute $\int_0^{2\pi} \cos(a_1\theta) \times \cos(a_2\theta) \times \cdots \times \cos(a_n\theta)\, d\theta$
Mechanical engineering. Structure of turbulence in sheared flows.
Medicine. Reconstructing 3d shape from biplane angiocardiogram.
Operations research. Traveling salesperson problem.
Physics. Partition function of 3d Ising model.
Politics. Shapley–Shubik voting power.
Recreation. Versions of Sudoku, Checkers, Minesweeper, Tetris, Rubik's Cube.
Statistics. Optimal experimental design.

co-NP

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- How could we prove that a formula is not satisfiable?

SAT. Given a CNF formula Φ, is there a satisfying truth assignment?

Un-SAT. Given a CNF formula Φ, is there no satisfying truth assignment?

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Example 2. Hamilton cycle vs. No Hamilton cycle.

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HAMILTON CYCLE. Given a graph $G = (V, E)$, is there a simple cycle Γ that contains every node in V?

NO HAMILTON CYCLE. Given a graph $G = (V, E)$, is there no simple cycle Γ that contains every node in V?

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Q. How to classify Un-SAT and No Hamilton cycle?

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Asymmetry of **NP**. We need short certificates only for yes instances.

Q. How to classify Un-SAT and No Hamilton cycle?

- SAT $\in$ **NP**-complete and SAT$\equiv_P$ Un-SAT.
- Hamilton circle $\in$ **NP**-complete and Hamilton circle $\equiv_P$ No Hamilton circle.
- But neither Un-SAT nor No Hamilton circle are known to be in **NP**.

NP and co-NP

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co-NP. Complements of decision problems in **NP**.
Example. Un-SAT, No Hamilton cycle, and Primes.

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Proof idea.

- **P** is closed under complementation.
- If **P = NP**, then **NP** is closed under complementation.
- In other words, **NP = co-NP**.
- This is the contrapositive of the theorem.

Good characterizations

Good characterization.[Edmonds 1965] **NP** $\cap$ **co-NP**.

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- If problem X is in both **NP** and **co-NP**, then:
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Example. Given a bipartite graph, is there a perfect matching?

- If yes, can exhibit a perfect matching.
- If no, can exhibit a set of nodes S such that $|N(S)| < |S|$.

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Observation. $\mathbf{P} \subseteq \mathbf{NP} \cap \mathbf{co\text{-}NP}$.

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- Mixed opinions.
- Many examples where problem found to have a nontrivial good characterization, but only years later discovered to be in **P**.

Factoring is in NP $\cap$ co-NP

Linear programming. Given $A \in \mathcal{R}^{m \times n}, b \in \mathcal{R}^{m}, c \in \mathcal{R}^{n}$, and $\alpha \in R$, does there exist $x \in \mathcal{R}^{n}$ such that $Ax \leq b, x \geq 0$ and $c^T x \geq \alpha$?

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Theorem (Gale–Kuhn–Tucker 1948)

LINEAR PROGRAMMING ∈ ***NP*** ∩ ***Co-NP*** .

Proof sketch. If (P) and (D) are nonempty, then $\max = \min$.

$$\text{(P)} \max c^T x \quad \text{s.t. } Ax \leq b, \quad x \geq 0$$

$$\text{(D)} \min y^T b \quad \text{s.t. } A^T y \geq c, \quad y \geq 0$$

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Theorem (Khachiyan 1979)

LINEAR PROGRAMMING $\in$ ***P***.

Factoring is in NP $\cap$ co-NP

Theorem (Pratt 1975)

PRIMES $\in$ ***NP*** $\cap$ ***co-NP***.

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Theorem (Pratt 1975)

PRIMES $\in$ ***NP*** $\cap$ ***co-NP***.

Proof sketch. An odd integer s is prime iff there exists an integer $1 < t < s$ s.t.

$$t^{s-1} \equiv 1 \pmod{s}$$
$$t^{(s-1)/p} \not\equiv 1 \pmod{s}$$

for all prime divisors p of $s - 1$.

Primality testing is in P

Theorem (Agrawal–Kayal–Saxena 2004)

PRIMES $\in$ ***P***.

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FACTORIZE. Given an integer x, find its prime factorization.
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FACTOR $\in$ ***NP*** $\cap$ ***co-NP***.

Proof.

Factoring is in NP ∩ co-NP

FACTORIZE. Given an integer x, find its prime factorization.
FACTOR. Given two integers x and y, does x have a nontrivial factor $< y$?

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Theorem

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Proof.

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- Disqualifier: the prime factorization of x (where each prime factor is greater than y).

Is factoring in P?

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Fundamental question. Is FACTOR $\in$ **P**?

Is factoring in P?

Fundamental question. Is FACTOR ∈ **P**?

Challenge. Factor this number.

74037563479561712828046796097429573142593188889231289
08493623263897276503402826627689199641962511784399589
43305021275853701189680982867331732731089309005525051
16877063299072396380786710086096962537934650563796359

RSA-704

($30,000 prize if you can factor)

Exploiting intractability

Modern cryptography.

- Example. Send your credit card to Amazon.
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RSA. Based on dichotomy between complexity of two problems.

- To use: generate two random n-bit primes and multiply.
- To break: suffices to factor a $2n$-bit integer.

Factoring on a quantum computer

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Theorem (Shor 1994)

Can factor an n-bit integer in $O(n^3)$ steps on a "quantum computer".

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Fundamental question. Does **P = BQP**?

NP-hard

A note on terminology: consensus

NP-complete. A problem in **NP** such that every problem in **NP** poly-time reduces to it.

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NP-hard. [[Bell Labs, Steve Cook, Ron Rivest, Sartaj Sahni] A problem such that every problem in **NP** poly-time reduces to it.