Design and Analysis of Algorithms (II)

Strongly Connected Components

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Depth-First Search in Graphs

Exploring Graphs

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```
EXPLORE(G, v)
input  : G = (V, E) is a graph; v ∈ V
output: visited(u) to true for all nodes u reachable from v

visited(v) = true;
PREVISIT(v);
for each edge (v, u) ∈ E do
    if not visited(u) then EXPLORE(G, u);
end
POSTVISIT(v);
```

Types of Edges in Undirected Graphs

Those edges in G that are traversed by `EXPLORE` are tree edges.

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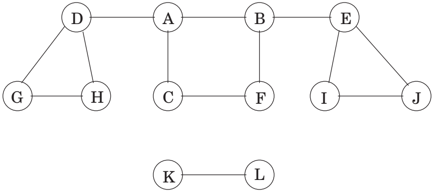
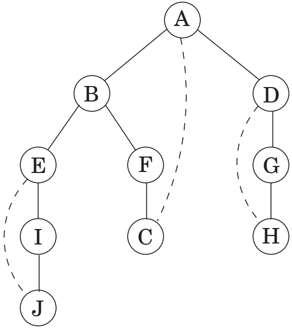
The rest are back edges.





Depth-First Search


```
DFS(G)

for all v ∈ V do
    visited(v) = false;
end
for all v ∈ V do
    if not visited(v) then Explore(G, v);
end
```

Previsit and Postvisit Orderings

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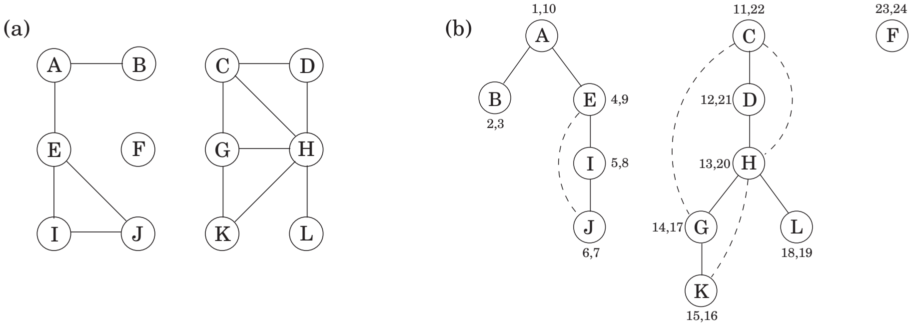
Lemma

For any nodes u and v, the two intervals $[pre(u), post(u)]$ and $[pre(v), post(v)]$ are either disjoint or one is contained within the other.

Previsit and Postvisit Orderings



Types of Edges in Directed Graphs

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- Forward edges lead from a node to a nonchild descendant in the DFS tree.
- Back edges lead to an ancestor in the DFS tree.
- Cross edges lead to neither descendant nor ancestor.

Directed Graphs

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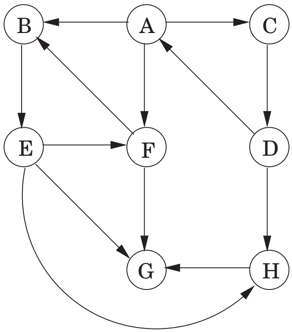

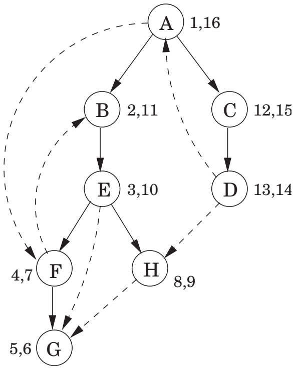

Types of Edges

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pre/post ordering for (u, v)				Edge type
$[_u$	$[_v$	$]_v$	$]_u$	Tree/forward
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Q: Is that all?

Directed Acyclic Graphs (DAG)

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Definition

A cycle in a directed graph is a circular path

$$v_0 \to v_1 \to v_2 \to \dots v_k \to v_0$$

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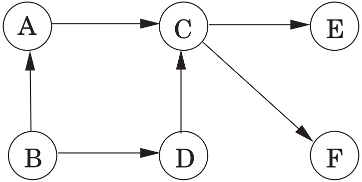
Lemma

A directed graph has a cycle if and only if its depth-first search reveals a back edge.

Directed Acyclic Graphs (DAG)



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Linearization/Topologically Sort: Order the vertices such that every edge goes from a earlier vertex to a later one.

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The only edges (u, v) in a graph for which $post(u) < post(v)$ are back edges, and we have seen that a DAG cannot have back edges.

Directed Acyclic Graphs (DAG)

Lemma

In a DAG, every edge leads to a vertex with a lower post number.

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There is a linear-time algorithm for ordering the nodes of a DAG.

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Symmetrically, the one with the highest post is a source, a node with no incoming edges.

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The guaranteed existence of a source suggests an alternative approach to linearization:

1. Find a source, output it, and delete it from the graph.
2. Repeat until the graph is empty.

Strongly Connected Components

Defining Connectivity for Directed Graphs

Definition

Two nodes u and v of a directed graph are connected if there is a path from u to v and a path from v to u.

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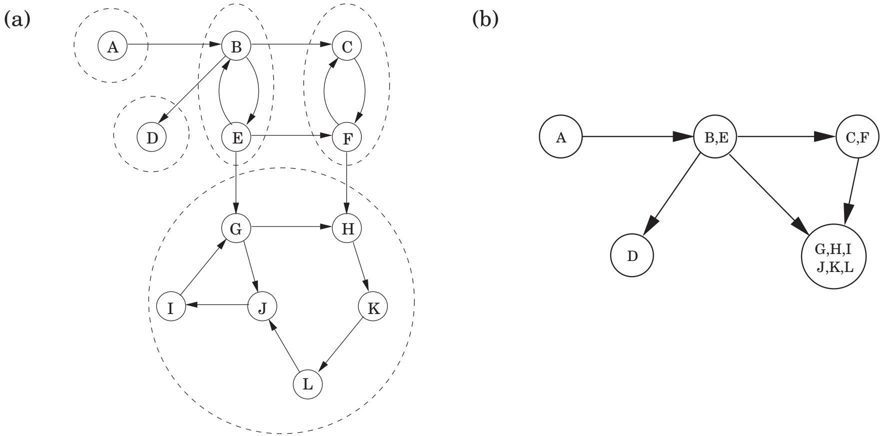
Lemma

Every directed graph is a DAG of its SCC.

Strongly Connected Components



An Efficient Algorithm

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If the `EXPLORE` *subroutine at node* u*, then it will terminate precisely when all nodes reachable from* u *have been visited.*

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1. How do we find a node that we know for sure lies in a sink SCC?

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If the `EXPLORE` *subroutine at node* u*, then it will terminate precisely when all nodes reachable from* u *have been visited.*

If we call explore on a node that lies somewhere in a sink SCC, then we will retrieve exactly that component.

We have two problems:

1. How do we find a node that we know for sure lies in a sink SCC?
2. How do we continue once this first component has been discovered?

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Lemma

The node that receives the highest post number in a depth-first search must lie in a source SCC.

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Lemma

If C and C' are SCC, and there is an edge from a node in C to a node in C', then the highest post number in C is bigger than the highest post number in C'.

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If C and C' are SCC, and there is an edge from a node in C to a node in C', then the highest post number in C is bigger than the highest post number in C'.

Hence the SCCs can be linearized by arranging them in decreasing order of their highest post numbers.

Solving Problem A

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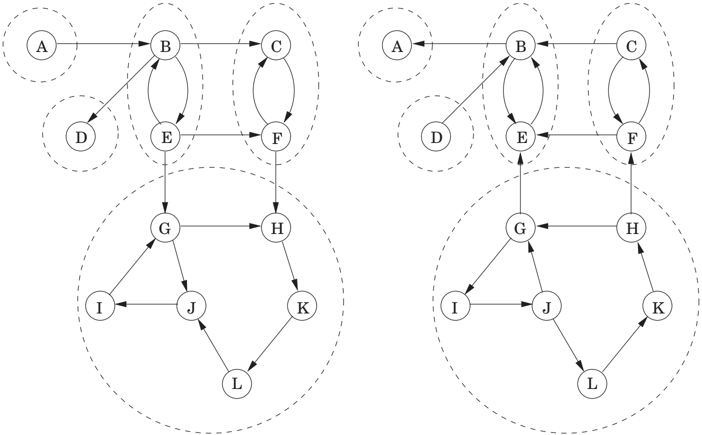
If we do a depth-first search of G^R, the node with the highest post number will come from a source SCC in G^R.

It is a sink SCC in G.

Strongly Connected Components



Solving Problem B

Once we have found the first SCC and deleted it from the graph, the node with the highest post number among those remaining will belong to a sink SCC of whatever remains of G.

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Once we have found the first SCC and deleted it from the graph, the node with the highest post number among those remaining will belong to a sink SCC of whatever remains of G.

Therefore we can keep using the post numbering from our initial depth-first search on G^R to successively output the second strongly connected component, the third SCC, and so on.

The Linear-Time Algorithm

1. Run depth-first search on G^R.

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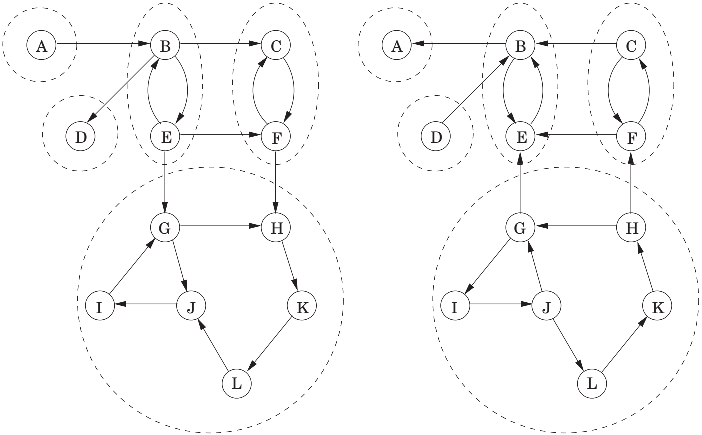
2. Run the EXPLORE algorithm on G, and during the depth-first search, process the vertices in decreasing order of their post numbers from step 1.

Strongly Connected Components



Think About

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How the SCC algorithm works when the graph is very, very huge?

Think About

How about edges instead of paths?

Exercises

Exercises 1

Suppose a CS curriculum consists of n courses, all of them mandatory. The prerequisite graph G has a node for each course, and an edge from course v to course w if and only if v is a prerequisite for w. Find an algorithm that works directly with this graph representation, and computes the minimum number of semesters necessary to complete the curriculum (assume that a student can take any number of courses in one semester). The running time of your algorithm should be linear.

Exercises 2

Give an efficient algorithm which takes as input a directed graph $G = (V, E)$, and determines whether or not there is a vertex $s \in V$ from which all other vertices are reachable.

Referred Materials

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Content of this lecture comes from section 3.2, 3.3 and 3.4 in [DPV07].