Design and Analysis of Algorithms III

Minimum Spanning Trees

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Minimum Spanning Trees

Cycles

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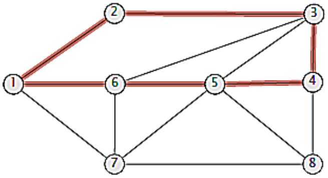

path $P = \{(1,2),(2,3),(3,4),(4,5),(5,6)\}$
cycle $C = \{(1,2),(2,3),(3,4),(4,5),(5,6),(6,1)\}$

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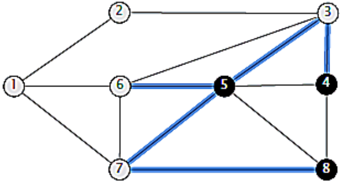

cut $S = \{4, 5, 8\}$

cutset $D = \{(3,4), (3,5), (5,6), (5,7), (8,7)\}$

Cycle-Cut Intersection

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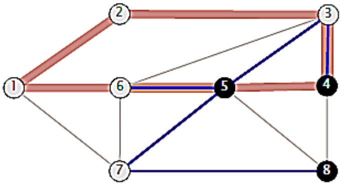
Proposition

A cycle and a cutset intersect in an even number of edges.



$$\begin{aligned}
\text{cycle } C &= \{(1,2),(2,3),(3,4),(4,5),(5,6),(6,1)\} \\
\text{cutset } D &= \{(3,4),(3,5),(5,6),(5,7),(8,7)\} \\
\text{intersection } C \cap D &= \{(3,4),(5,6)\}
\end{aligned}$$

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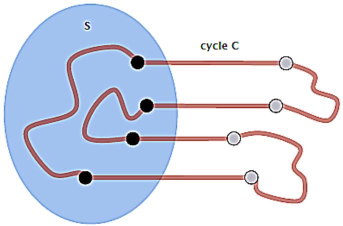

Spanning Tree Definition

Let $H = (V, T)$ be a subgraph of an undirected graph $G = (V, E)$. H is a spanning tree of G if H is both acyclic and connected.

Spanning Tree Properties

Proposition

Let $H = (V, T)$ be a subgraph of an undirected graph $G = (V, E)$. Then, the following are equivalent:

- H *is a spanning tree of* G.
- H *is acyclic and connected.*
- H *is connected and has* $|V| - 1$ *edges.*
- H *is acyclic and has* $|V| - 1$ *edges.*
- H *is minimally connected: removal of any edge disconnects it.*
- H *is maximally acyclic: addition of any edge creates a cycle.*

Minimum Spanning Tree (MST)

Given a connected, undirected graph $G = (V, E)$ with edge costs c_e, a minimum spanning tree (V, T) is a spanning tree of G such that the sum of the edge costs in T is minimized.

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Cayley's theorem. The complete graph on n nodes has n^{n-2} spanning trees.

Applications

MST is fundamental problem with diverse applications.

- Dithering.
- Cluster analysis.
- Max bottleneck paths.
- Real-time face verification.
- LDPC codes for error correction.
- Image registration with Renyi entropy.
- Find road networks in satellite and aerial imagery.
- Model locality of particle interactions in turbulent fluid flows.
- Reducing data storage in sequencing amino acids in a protein.
- Autoconfig protocol for Ethernet bridging to avoid cycles in a network.
- Approximation algorithms for NP-hard problems.
- Network design (communication, electrical, hydraulic, computer, road).

Fundamental Cycle

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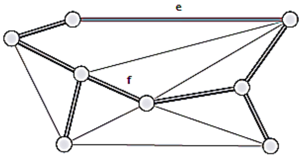
Fundamental cycle. Let $H = (V, T)$ be a spanning tree of $G = (V, E)$.

- For any non tree-edge $e \in E : T \cup \{e\}$ contains a unique cycle, say C.
- For any edge $f \in C : T \cup \{e\} - \{f\}$ is a spanning tree.



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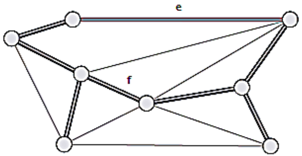

graph $G = (V, E)$
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Observation. If $c_e < c_f$, then (V, T) is not an MST.

Fundamental Cutset

Fundamental cutset. Let $H = (V, T)$ be a spanning tree of $G = (V, E)$.

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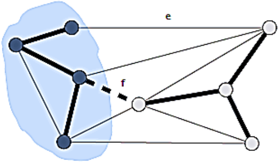

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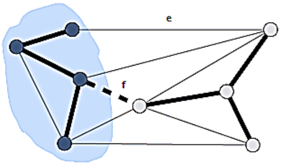

$$\begin{aligned} \textbf{graph } G &= (V, E) \\ \textbf{spanning tree } H &= (V, T) \end{aligned}$$

Observation. If $c_e < c_f$, then (V, T) is not an MST.

The Greedy Algorithm

Red rule.

- Let C be a cycle with no red edges.
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Greedy algorithm.

- Apply the red and blue rules (nondeterministically!) until all edges are colored. The blue edges form an MST.
- Note: can stop once $|V| - 1$ edges colored blue.

Proof of Correctness

Color invariant. There exists an $MST(V, T^*)$ containing every blue edge and no red edge.

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Base case. No edges colored $\implies$ every MST satisfies invariant.

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Proof. We need to show that either the red or blue rule (or both) applies.

- Suppose edge e is left uncolored.
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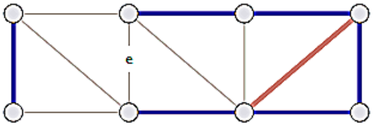

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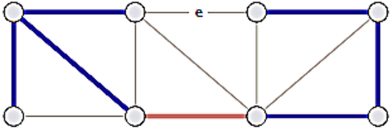

Prim, Kruskal, Borůvka

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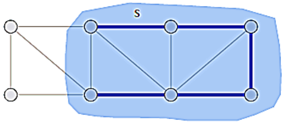
Proof. Special case of greedy algorithm (blue rule repeatedly applied to S).



Prim's Algorithm: Implementation


```
PRIM(G, w)
input  : A connected undirected graph G = (V, E), with edge weights w_e
output: A minimum spanning tree defined by the array prev

for all u ∈ V do
    cost(u) = ∞;
    prev(u) = nil;
end
pick any initial node u_0;
cost(u_0) = 0;
H =makequeue(V)\\ using cost-values as keys;
while H is not empty do
    v=deletemin(H);
    for each (v, z) ∈ E do
        if cost(z) > w(v, z) then
            cost(v) = w(v, z); prev(z) = v;
            decreasekey (H,z);
        end
    end
end
```

Prim's Algorithm: Analysis

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Prim's algorithm can be implemented to run in $O(|E| \log |V|)$ *time.*

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By priority queue implementation.

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  $\Rightarrow$ color e blue by applying blue rule to cutset defined by either tree.

Kruskal's Algorithm

KRUSKAL(V, E, c)

SORT m edges by cost and renumber so that
$c(e_1) \leq c(e_2) \leq \ldots \leq c(e_m)$;
$T \leftarrow \varnothing$;
for *each* $v \in V$ **do** MAKESET(v);
for $i = 1$ TO m **do**
  $(u, v) \leftarrow e_i$;
  if FINDSET(u) $\neq$ FINDSET(v) **then**
    $T \leftarrow T \cup \{e_i\}$;
    UNION(u,v);
  end
end
RETURN T;

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Kruskal's Algorithm: Analysis

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Theorem

Kruskal's algorithm can be implemented to run in $O(|E| \log |E|)$ time.

- Sort edges by cost.
- Use disjoint set data structure to dynamically maintain connected components.

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  $\Rightarrow$ apply red rule to cycle C formed by adding e to another path in T between its two endpoints
- Case 2. [deleting edge e disconnects T]
  $\Rightarrow$ apply blue rule to cutset D induced by either component

Fact. [Thorup 2000] Can be implemented to run in $O(|E| \log |V| (\log \log |V|)^3)$ time.

Review: the Greedy MST Algorithm

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Red rule.

- Let C be a cycle with no red edges.
- Select an uncolored edge of C of max cost and color it red.

Blue rule.

- Let D be a cutset with no blue edges.
- Select an uncolored edge in D of min cost and color it blue.

Greedy algorithm.

- Apply the red and blue rules (nondeterministically!) until all edges are colored. The blue edges form an MST.
- Note: can stop once $|V| - 1$ edges colored blue.

Review: the Greedy MST Algorithm

Theorem

The greedy algorithm is correct.

Review: the Greedy MST Algorithm

Theorem

The greedy algorithm is correct.

Special cases. Prim, Kruskal, reverse-delete, ...

Borůvka's Algorithm

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Repeat until only one tree.

- Apply blue rule to cutset corresponding to each blue tree.
- Color all selected edges blue.

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Borůvka's algorithm computes the MST. ⟵ *assume edge costs are distinct*

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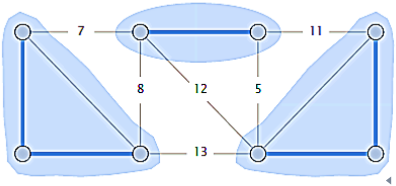
Theorem

Borůvka's algorithm computes the MST. ← *assume edge costs are distinct*

Proof. Special case of greedy algorithm (repeatedly apply blue rule).



Borůvka's Algorithm

Theorem

Borůvka's algorithm can be implemented to run in $O(|E| \log |V|)$ *time.*

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To implement a phase in $O(|E|)$ time:

- compute connected components of blue edges
- for each edge $(u, v) \in E$, check if u and v are in different components; if so, update each component's best edge in cutset

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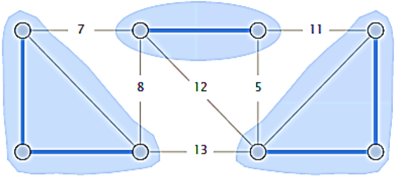
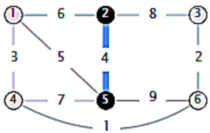
$\leq \log_2 |V|$ phases since each phase (at least) halves total $\#$ components.



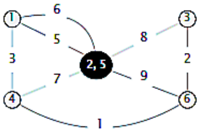
Contraction version.

- After each phase, contract each blue tree to a single supernode.
- Delete self-loops and parallel edges (keeping only cheapest one).
- Borůvka phase becomes: take cheapest edge incident to each node.

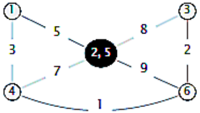
graph G



contract edge 2–5



delete self–loops and parallel edges



A Question

Q. How to contract a set of edges?

Contract a Set of Edges

Problem. Given a graph $G = (V, E)$ and a set of edges F, contract all edges in F, removing any self-loops or parallel edges.

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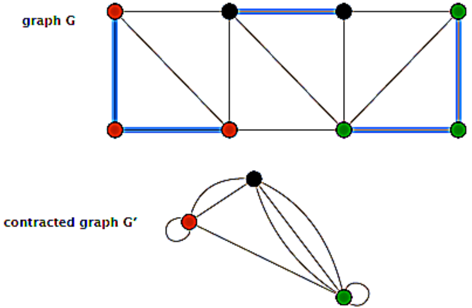
Goal. $O(|V| + |E|)$ time.



Contract a Set of Edges

1. mark the edges to be contracted;
2. determine the connected components formed by the marked edges;
3. replace each connected component by a single vertex;
4. finally, eliminate the self-loops and multiple edges created by these contractions.

Contract a Set of Edges

1. mark the edges to be contracted;
    - To find the minimum weight edge incident on each node, takes $O(|E| + |V|)$ time;
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   - To eliminate edges, takes $O(|E|)$ time.

Borůvka's on Planar Graphs

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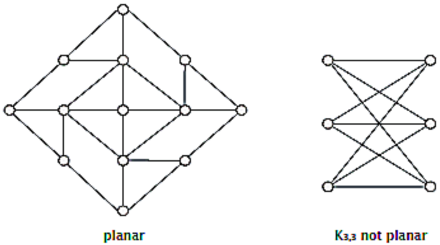
Theorem

Borůvka's algorithm (contraction version) can be implemented to run in $O(|V|)$ time on planar graphs.



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Each Borůvka phase takes $O(|V|)$ time:

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Each Borůvka phase takes $O(|V|)$ time:

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- Fact 2: planar graphs remains planar after edge contractions/deletions.

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Thus, overall running time $\leq c \cdot |V| + c \cdot |V|/2 + c \cdot |V|/4 + c \cdot |V|/8 + \cdots = O(|V|)$.

A Hybrid Algorithm

Borůvka-Prim algorithm.

- Run Borůvka (contraction version) for $\log_2 \log_2 |V|$ phases.
- Run Prim on resulting, contracted graph.

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Borůvka-Prim computes an MST.

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Theorem

Borůvka-Prim computes an MST.

Proof. Special case of the greedy algorithm.

A Hybrid Algorithm

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Theorem

Borůvka-Prim can be implemented to run in $O(|E| \log \log |V|)$ *time.*

A Hybrid Algorithm

SHANGHAI JIAO TONG UNIVERSITY

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Proof.

A Hybrid Algorithm

Theorem

Borůvka-Prim can be implemented to run in $O(|E| \log\log|V|)$ *time.*

Proof.

- The $\log_2 \log_2 |V|$ phases of Borůvka's algorithm take $O(|E| \log\log|V|)$ time; resulting graph has $\leq |V|/\log_2|V|$ nodes and $\leq |E|$ edges.
- Prim's algorithm (using Fibonacci heaps) takes $O(|E| + |V|)$ time on a graph with $|V|/\log_2|V|$ nodes and $|E|$ edges.

Linear-Time Algorithm?

year	worst case	discoverec by
1975	$O(\lvert E\rvert \log\log \lvert V\rvert)$	Yao
1976	$O(\lvert E\rvert \log\log \lvert V\rvert)$	Cheriton-Tarjan
1984	$O(\lvert E\rvert \log^* \lvert V\rvert), O(\lvert E\rvert + \lvert V\rvert \log \lvert V\rvert)$	Fredman-Tarjan
1986	$(\lvert E\rvert \log(\log^* \lvert V\rvert))$	Gabow-Galil-Spencer-Tarjan
1997	$O(\lvert E\rvert \alpha(\lvert V\rvert) \log \alpha(\lvert V\rvert))$	Chazelle
2000	$O(\lvert E\rvert \alpha(\lvert V\rvert))$	Chazelle
2002	asymptotically optimal	Pettie-Ramachandran
20xx	$O(\lvert E\rvert)$	???

deterministic compare-based MST algorithms

iterated logarithm function

$$\lg^* n = \begin{cases} 0 & \text{if } n \le 1 \\ 1 + \lg^*(\lg n) & \text{if } n > 1 \end{cases}$$

n	$\lg^* n$
$(-\infty, 1]$	0
$(1, 2]$	1
$(2, 4]$	2
$(4, 16]$	3
$(16, 2^{16}]$	4
$(2^{16}, 2^{65536}]$	5

Minimum Bottleneck Spanning Tree

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Problem. Given a connected graph G with positive edge costs, find a spanning tree that minimizes the most expensive edge.

Goal. $O(|E| \log |E|)$ time or better.