Design and Analysis of Algorithms (VIII)

Max-Flow Min-Cut Theorem

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School of Software

SHANGHAI JIAO TONG UNIVERSITY

Max-Flow and Min-Cut Problem

A Flow Network

A flow network is a tuple $G = (V, E, s, t, c)$.

- Diagraph (V, E) with source $s \in V$ and sink $t \in V$.
- Capacity $c(e) > 0$ for each $e \in E$.

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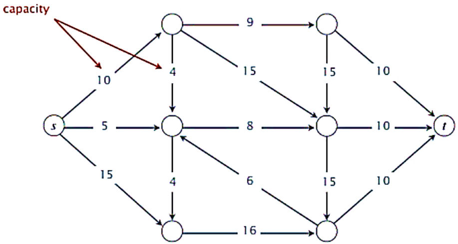
Intuition. Material flowing through a transportation network, which originates at source and is sent to sink.



Minimum-Cut Problem

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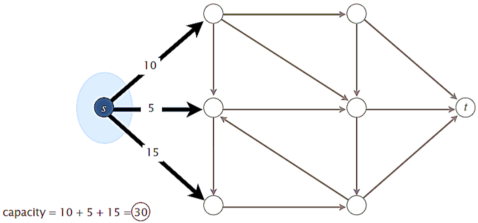

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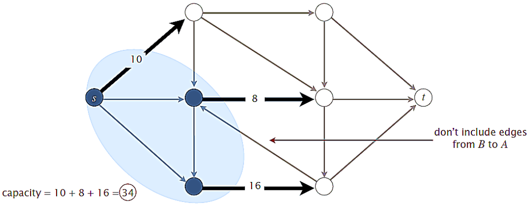
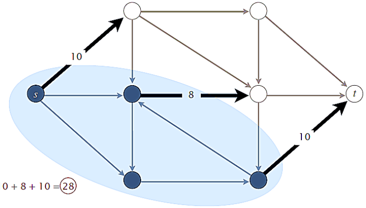


Minimum-Cut Problem

Min-cut problem. Find a cut of minimum capacity.



capacity = 10 + 8 + 10 = 28

Quiz 1

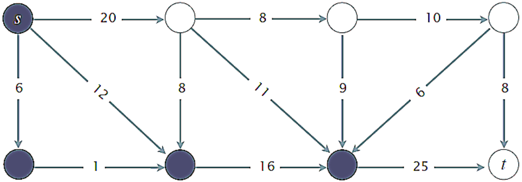
Which is the capacity of the given st-cut?

A. $11\ (20+25-8-11-9-6)$

B. $34\ (8+11+9+6)$

C. $45\ (20+25)$

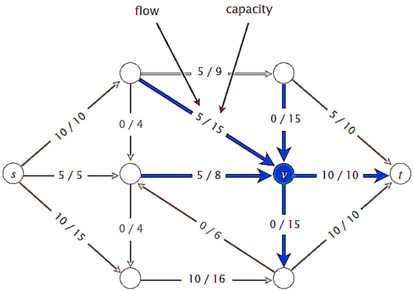
D. $79\ (20+25+8+11+9+6)$



Maximum-Flow Problem

An st-flow(flow) f is a function that satisfies:

- For each $e \in E$: $\quad 0 \leq f(e) \leq c(e)$
- For each $v \in V - \{s, t\}$: $\displaystyle\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$



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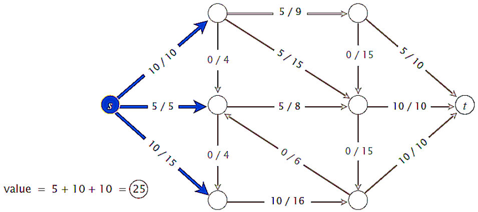
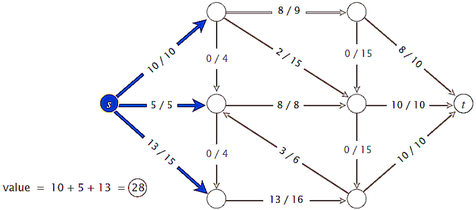
The value of a flow f is: $val(f) = \sum_{e \text{ out of } s} f(e) - \sum_{e \text{ in to } s} f(e)$



Maximum-Flow Problem

Max-flow problem. Find a flow of maximum value.

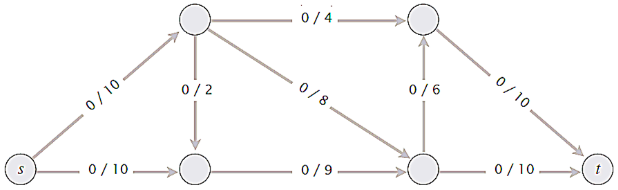


Ford-Fulkerson Algorithm

Toward a Max-Flow Algorithm

Greedy algorithm.

- Start with $f(e) = 0$ for each edge $e \in E$.
- Find an $s \rightsquigarrow t$ path P where each edge has $f(e) < c(e)$.
- Augment flow along path P.
- Repeat until you get stuck.



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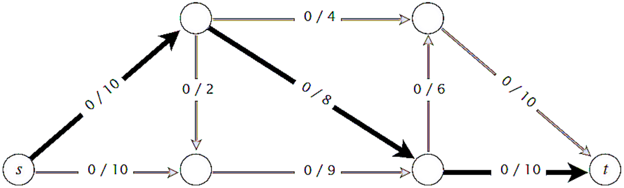

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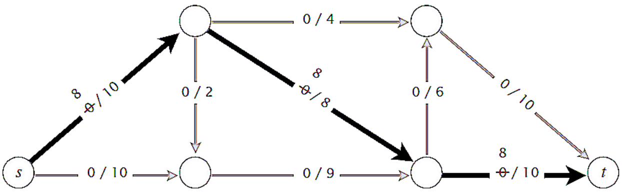

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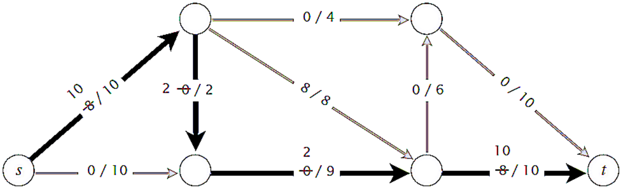

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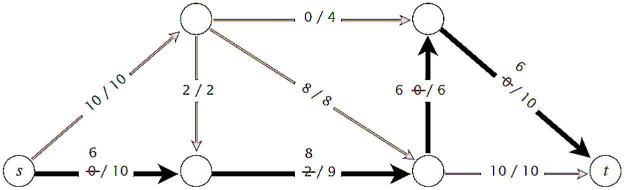

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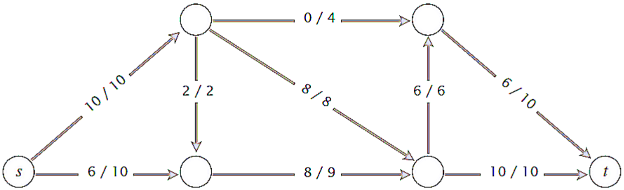

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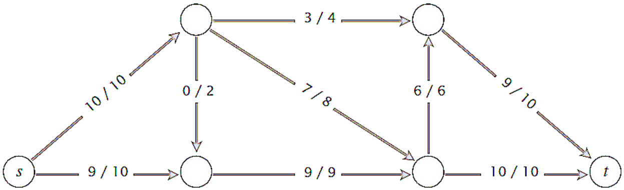

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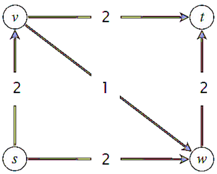
Q. Why does the greedy algorithm fail?

A. Once greedy algorithm increases flow on an edge, it never decreases it.

Ex. Consider flow network G.

- The unique max flow has $f^*(v, w) = 0$.
- Greedy algorithm could choose $s \to v \to w \to t$ as first augmenting path.



Why the Greedy Algorithm Fails

Bottom line. Need some mechanism to undo a bad decision.

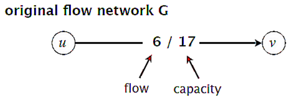
Residual Network

Original edge $e = (u, v) \in E$.

- Flow $f(e)$.
- Capacity $c(e)$

Reverse edge $e^{\text{reverse}} = (v, u)$

- Undo flow sent.



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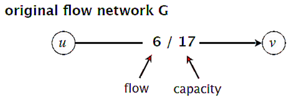

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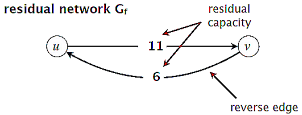
Residual capacity

$$c_f(e) = \begin{cases} c(e) - f(e) & \text{if } e \in E \\ f(e) & \text{if } e^{\text{reverse}} \in E \end{cases}$$



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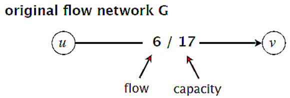

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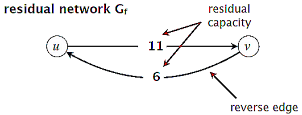

Residual network $G_f = (V, E_f, s, t, c_f)$

- $E_f = \{e : f(e) < c(e)\} \cup \{e^{\text{reverse}} : f(e) > 0\}$.
- Key property: f' is a flow in G_f iff $f + f'$ is a flow in G

Augmenting Path

An augmenting path is a simple $s \rightsquigarrow t$ path in the residual network G_f.

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The bottleneck capacity of an augmenting path P is the minimum residual capacity of any edge in P.

Augmenting Path

Key Property

Let f be a flow and let P be an augmenting path in G_f. After calling $f' \leftarrow \textsc{Augment}(f, P)$, the resulting f' is a flow and

$$val(f') = val(f) + bottleneck(G_f, P)$$

Augmenting Path

Key Property

Let f be a flow and let P be an augmenting path in G_f. After calling $f' \leftarrow \text{AUGMENT}(f, P)$, the resulting f' is a flow and

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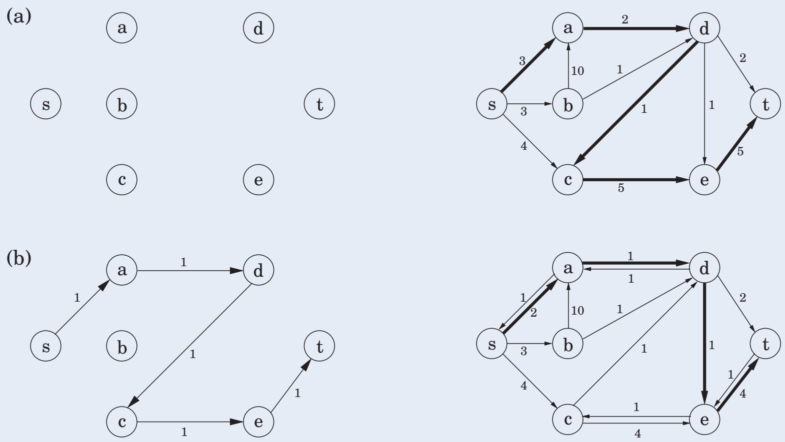
AUGMENT(f,P)

$\delta \leftarrow$ bottleneck capacity of augmenting path P;
for *each edge* $e \in P$ **do**
  if $(e \in E)$ **then** $f(e) \leftarrow f(e) + \delta$;
  else
    $f(e^{\text{reverse}}) \leftarrow f(e^{\text{reverse}}) - \delta$
  end
end
RETURN f;

A Flow Example



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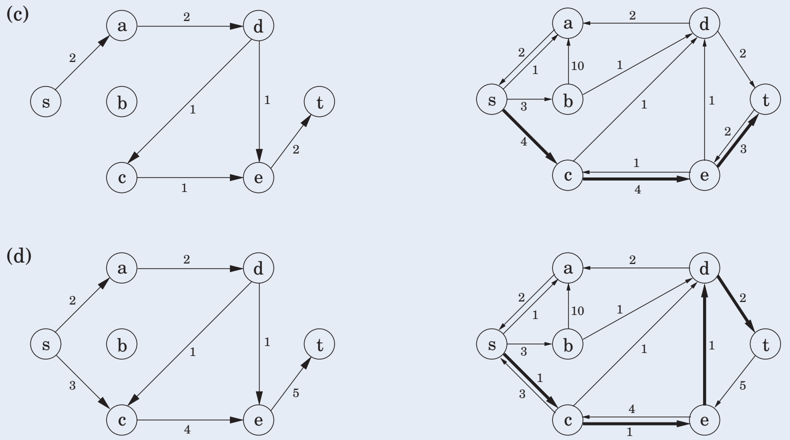


Ford–Fulkerson Algorithm

Ford–Fulkerson augmenting path algorithm.

- Start with $f(e) = 0$ for each edge $e \in E$.
- Find an $s \rightsquigarrow t$ path P in the residual network G_f.
- Augment flow along path P.
- Repeat until you get stuck.

Ford–Fulkerson Algorithm

FORD–FULKERSON(G)

for *each edge* $e \in E$ **do**
    $f(e) \leftarrow 0$
end
$G_f \leftarrow$ residual network of G with respect to flow f;
while *there exists an* $s \rightsquigarrow t$ *path* P *in* G_f **do**
    $f \leftarrow$ AUGMENT(f,P) ;
    UPDATE(G_f) ;
end
RETURN f;

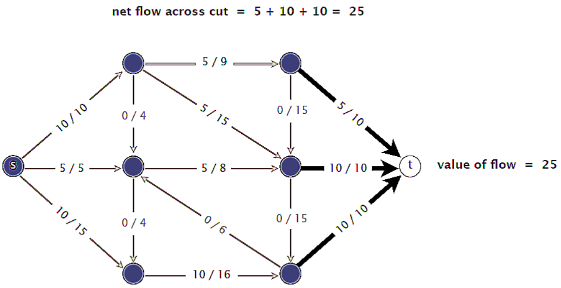
Max-Flow Min-Cut Theorem

Flows and Cuts

Lemma

Let f be any flow and let (A, B) be any cut. Then, the value of the flow f equals the net flow across the cut (A, B).

$$val(f) = \sum_{\text{out of } A} f(e) - \sum_{e \text{ in to } A} f(e)$$



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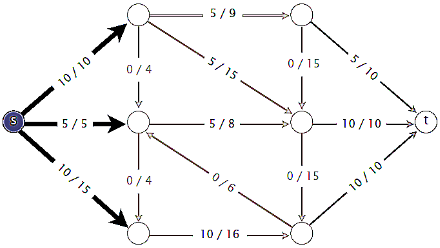
net flow across cut = 10 + 5 + 10 = 25



value of flow = 25

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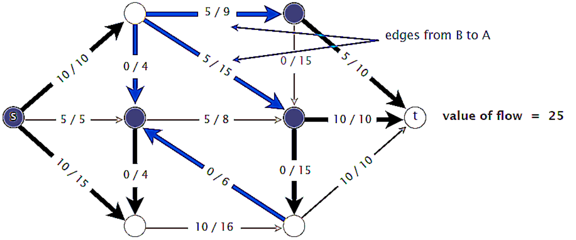
net flow across cut = (10 + 10 + 5 + 10 + 0 + 0) – (5 + 5 + 0 + 0) = 25



Quiz 3

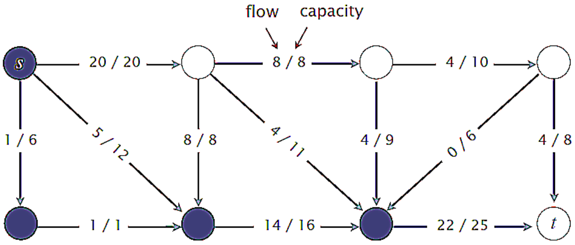
Which is the net flow across the given cut?

A. 11 $(20 + 25 - 8 - 11 - 9 - 6)$

B. 26 $(20 + 22 - 8 - 4 - 4)$

C. 42 $(20 + 22)$

D. 45 $(20 + 25)$



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$$val(f) = \sum_{\text{out of } A} f(e) - \sum_{e \text{ in to } A} f(e)$$

Proof.

$$\begin{aligned} val(f) &= \sum_{e \text{ out of } s} f(e) - \sum_{\text{e in to } s} f(e) \\ &= \sum_{v \in A} \left(\sum_{e \text{ out of } v} f(e) - \sum_{\text{e in to } v} f(e) \right) \\ &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e). \end{aligned}$$

Flows and Cuts

Theorem (Weak Duality)

Let f be any flow and (A, B) be any cut. Then, $val(f) \leq \mathrm{cap}(A, B)$.

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Theorem (Weak Duality)

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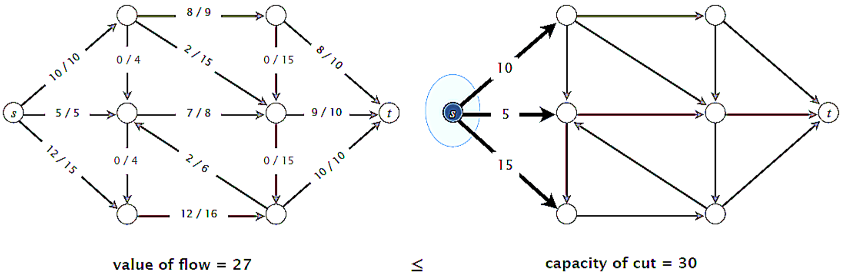
Proof.

$$val(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \leq \sum_{e \text{ out of } A} f(e) \leq \sum_{e \text{ out of } A} c(e)$$
$$= \text{cap}(A, B)$$



Certificate of Optimality

Corollary

Let f be a flow and let (A, B) be any cut. If $val(f) = \mathrm{cap}(A, B)$, then f is a max flow and (A, B) is a min cut.

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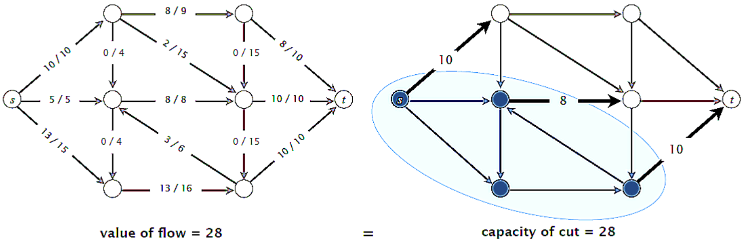
Proof.

- For any flow f' : $\quad val\,(f') \le \text{cap}(A, B) = val(f)$.
- For any cut (A', B') : $\text{cap}\,(A', B') \ge val(f) = \text{cap}(A, B)$



Max-Flow Min-Cut Theorem

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Value of a max flow = Capacity of a min cut.

MAXIMAL FLOW THROUGH A NETWORK

L. R. FORD, JR. AND D. R. FULKERSON

Introduction. The problem discussed in this paper was formulated by T. Harris as follows:

"Consider a rail network connecting two cities by way of a number of intermediate cities, where each link of the network has a number assigned to it representing its capacity. Assuming a steady state condition, find a maximal flow from one given city to the other."

ON THE MAX FLOW MIN CUT THEOREM OF NETWORKS

G. B. Dantzig

D. R. Fulkerson

P-826

April 15, 1955

A Note on the Maximum Flow Through a Network*

P. ELIAS†, A. FEINSTEIN‡, AND C. E. SHANNON§

Summary—This note discusses the problem of maximizing the rate of flow from one terminal to another, through a network which consists of a number of branches, each of which has a limited capacity. The main result is a theorem: The maximum possible flow from left to right through a network is equal to the minimum value among all simple cut-sets. This theorem is applied to solve a more general problem, in which a number of input nodes and a number of output nodes are used.

from one terminal to the other in the original network passes through at least one branch in the cut-set. In the network above, some examples of cut-sets are (d, e, f), and (b, c, e, g, h), (d, g, h, i). By a *simple cut-set* we will mean a cut-set such that if any branch is omitted it is no longer a cut-set. Thus (d, e, f) and (b, c, e, g, h) are simple

Max-Flow Min-Cut Theorem

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Augmenting Path Theorem

A flow f is a max flow iff no augmenting paths.

Proof. The following three conditions are equivalent for any flow f:

i. There exists a cut (A, B) such that $\text{cap}(A, B) = val(f)$.
ii. f is a max flow.
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[i$\Rightarrow$ ii] This is the weak duality corollary.

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iii. There is no augmenting path with respect to f.

[ii⇒ iii] We prove contrapositive: ¬ iii ⇒ ¬ii.

Max-Flow Min-Cut Theorem

Max-Flow Min-Cut Theorem

Value of a max flow = capacity of a min cut.

Augmenting Path Theorem

A flow f is a max flow iff no augmenting paths.

Proof. The following three conditions are equivalent for any flow f:

i. There exists a cut (A, B) such that $\text{cap}(A, B) = val(f)$.
ii. f is a max flow.
iii. There is no augmenting path with respect to f.

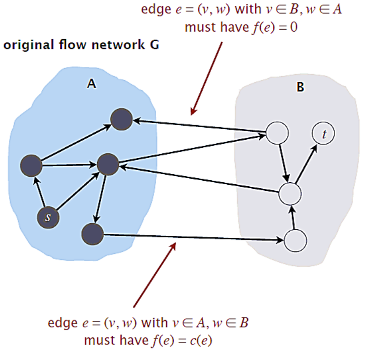
[ii$\Rightarrow$ iii] We prove contrapositive: $\neg$ iii $\Rightarrow$ $\neg$ii.

- Suppose that there is an augmenting path with respect to f.
- Can improve flow f by sending flow along this path.
- Thus, f is not a max flow.

[iii $\Rightarrow$ i]

- Let f be a flow with no augmenting paths.
- Let A be set of nodes reachable from s in residual network G_f.
- By definition of A : $s \in A$.
- By definition of flow f : $t \notin A$.

$$
\begin{aligned}
val(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\
&= \sum_{e \text{ out of } A} c(e) - 0 \\
&= \mathrm{cap}(A, B)
\end{aligned}
$$



Analysis of the Algorithm

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Assumption. Every edge capacity $c(e)$ is an integer between 1 and C.

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Theorem

Ford–Fulkerson terminates after at most $\mathrm{val}(f^*) \leq |V| \cdot C$ *augmenting paths, where* f^* *is a max flow.*

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Theorem

Ford–Fulkerson terminates after at most $\mathrm{val}(f^*) \leq |V| \cdot C$ augmenting paths, where f^* is a max flow.

Proof. Each augmentation increases the value of the flow by at least 1.

Analysis of the Algorithm

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Corollary

The running time of Ford–Fulkerson is $O(|V| \cdot |E| \cdot C)$.

Analysis of the Algorithm

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Proof. Can use either BFS or DFS to find an augmenting path in $O(|E|)$ time.

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Integrality Theorem

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Proof. Since Ford–Fulkerson terminates, theorem follows from integrality invariant.

Exponential Example

Q. Is generic Ford–Fulkerson algorithm poly-time in input size?

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A. No. If max capacity is C, then algorithm can take $\geq C$ iterations.

Exponential Example

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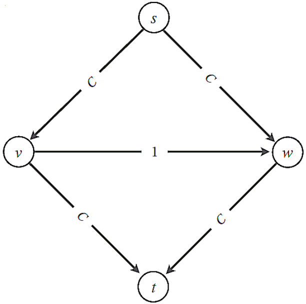
Q. Is generic Ford–Fulkerson algorithm poly-time in input size?

A. No. If max capacity is C, then algorithm can take $\geq C$ iterations.

- $s \to v \to w \to t$
- $s \to w \to v \to t$
- $s \to v \to w \to t$
- $s \to w \to v \to t$
- $\ldots$
- $s \to v \to w \to t$
- $s \to w \to v \to t$



Quiz 4

The Ford–Fulkerson algorithm is guaranteed to terminate if the edge capacities are . . .

A. Rational numbers.

B. Real numbers.

C. Both A and B.

D. Neither A nor B.

Choosing Good Augmenting Paths

Use care when selecting augmenting paths.

- Some choices lead to exponential algorithms.
- Clever choices lead to polynomial algorithms.

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Choosing Good Augmenting Paths

Use care when selecting augmenting paths.

- Some choices lead to exponential algorithms.
- Clever choices lead to polynomial algorithms.

Pathology. When edge capacities can be irrational, no guarantee that Ford–Fulkerson terminates (or converges to a maximum flow)!

Goal. Choose augmenting paths so that:

- Can find augmenting paths efficiently.
- Few iterations.

Choosing Good Augmenting Paths

Choose augmenting paths with:

Choosing Good Augmenting Paths

Choose augmenting paths with:

- Max bottleneck capacity ("fattest").

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Choose augmenting paths with:

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Choosing Good Augmenting Paths

Choose augmenting paths with:

- Max bottleneck capacity ("fattest").
- Sufficiently large bottleneck capacity.
- Fewest edges.

Referred Materials

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- Content of this lecture comes from Section 7.1-7.2 in [KT05].