Algorithm Design IV

Divide and Conquer I

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Divide-and-Conquer

The divide-and-conquer strategy solves a problem by:

1. Breaking it into subproblems that are themselves smaller instances of the same type of problem.
2. Recursively solving these subproblems.
3. Appropriately combining their answers.

Multiplication

Product of Complex Numbers

Carl Friedrich Gauss(1777-1855) noticed that although the product of two complex numbers

$$(a+bi)(c+di) = ac - bd + (bc + ad)i$$

seems to involve four real-number multiplications, it can in fact be done with just three: ac, bd, and $(a+b)(c+d)$, since

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- In big O way of thinking, reducing the number of multiplications from four to three seems wasted ingenuity.
- But this modest improvement becomes very significant when applied recursively.

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As a first step toward multiplying x and y, we split each of them into their left and right halves, which are $n/2$ bits long

$$x = \boxed{x_L}\ \boxed{x_R} = 2^{n/2}x_L + x_R$$

$$y = \boxed{y_L}\ \boxed{y_R} = 2^{n/2}y_L + y_R$$

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$$xy = (2^{n/2}x_L + x_R)(2^{n/2}y_L + y_R) = 2^n x_L y_L + 2^{n/2}(x_L y_R + x_R y_L) + x_R y_R$$

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Additions and multiplications by powers of 2 take linear time.

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By Gauss's trick, three multiplications $x_L y_L$, $x_R y_R$, and $(x_L + x_R)(y_L + y_R)$ suffice.

Algorithm for Integer Multiplication

MULTIPLY(x, y)

Two positive integers x *and* y*, in binary*;

n=max (size of x, size of y) rounded as a power of 2;

if $n = 1$ **then** return(xy);

x_L, x_R= leftmost $n/2$, rightmost $n/2$ bits of x;

y_L, y_R= leftmost $n/2$, rightmost $n/2$ bits of y;

$P1$=MULTIPLY(x_L, y_L);

$P2$=MULTIPLY(x_R, y_R);

$P3$=MULTIPLY($x_L + x_R, y_L + y_R$);

return($P_1 \times 2^n + (P_3 - P_1 - P_2) \times 2^{n/2} + P_2$)

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The recurrence relation

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The branch factor is 3: each problem produces three smaller ones, with the result that at depth k there are 3^k subproblems, each of size $n/2^k$.

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For each subproblem, a linear amount of work is done in combining their answers.

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$$3^k \times O(\frac{n}{2^k}) = (\frac{3}{2})^k \times O(n)$$

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The sum of any increasing geometric series is, within a constant factor, the last term of the series.

Therefore, the overall running time is

$$O(n^{\log_2 3}) \approx O(n^{1.59})$$

Time Analysis

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Q: Can we do better?

Time Analysis

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- Yes!

Recurrence Relations

Master Theorem

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If $T(n) = aT(\lceil n/b \rceil) + O(n^d)$ for some constants $a > 0$, $b > 1$ and $d \geq 0$, then

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$

The Proof of the Theorem

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Its branching factor is a, so the k-th level of the tree is made up of a^k subproblems, each of size n/b^k.

$$a^k \times O(\frac{n}{b^k})^d = O(n^d) \times (\frac{a}{b^d})^k$$

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$$a^k \times O(\frac{n}{b^k})^d = O(n^d) \times (\frac{a}{b^d})^k$$

k goes from 0 to $\log_b n$, these numbers form a geometric series with ratio a/b^d, comes down to three cases.

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The ratio is less than 1.

Then the series is decreasing, and its sum is just given by its first term, $O(n^d)$.

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The ratio is exactly 1.

In this case all $O(\log n)$ terms of the series are equal to $O(n^d)$.

Merge Sort

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The Algorithm

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```
MERGESORT(a[1...n])
An array of numbers a[1...n];
if n > 1 then
    return(MERGE(MERGESORT(a[1...⌊n/2⌋]),
     MERGESORT(a[⌊n/2⌋ + 1...,n])));
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```
MERGE(x[1...k], y[1...l])
if k = 0 then return y[1...l];
if l = 0 then return x[1...k];
if x[1] ≤ y[1] then return( x[1]∘MERGE(x[2...k], y[1...l]));
else return( y[1]∘MERGE(x[1...k], y[2...l]));
```

An Iterative Version

```
ITERTIVE-MERGESORT(a[1...n])
An array of numbers a[1...n];

Q = [  ] empty queue;
for i = 1 to n do
    Inject(Q, [a[i]]);
end
while |Q| > 1 do
    Inject (Q,MERGE (Eject (Q),Eject (Q)));
end
return(Eject (Q));
```

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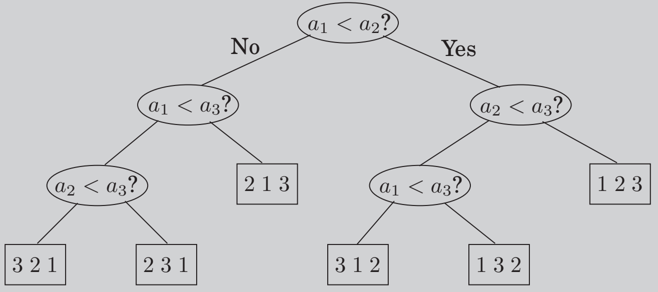
Q: Can we do better?

Sorting



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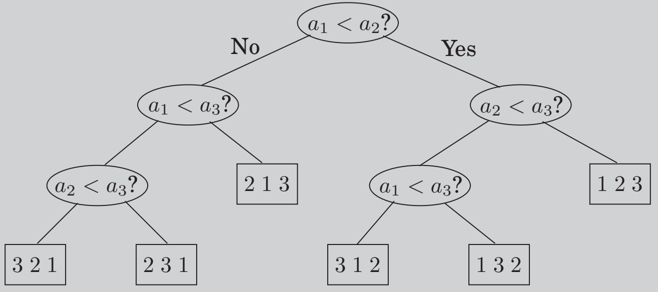

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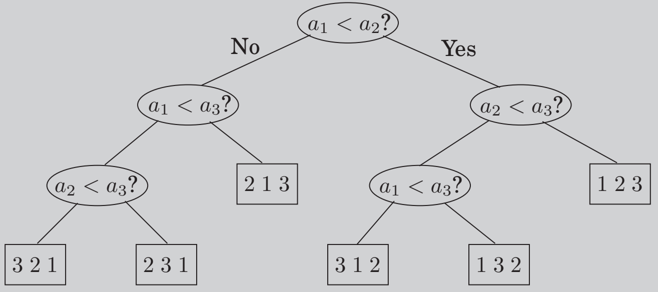

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Assume n elements. Each of its leaves is labeled by a permutation of $\{1, 2, \dots, n\}$.

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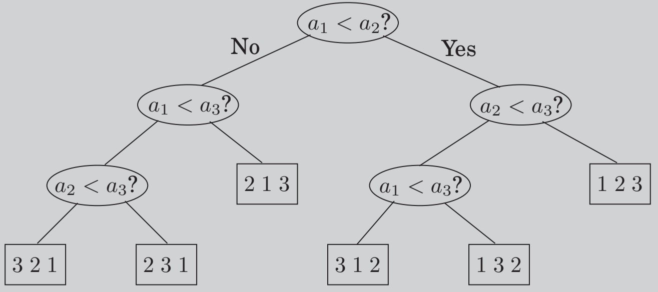

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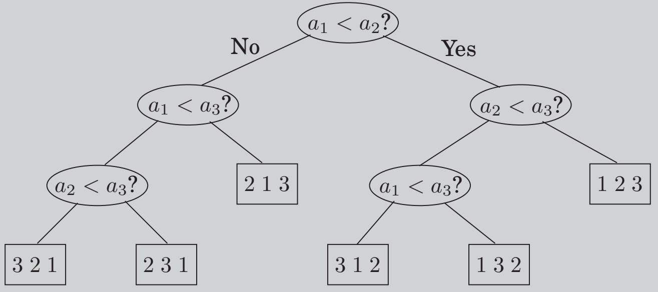

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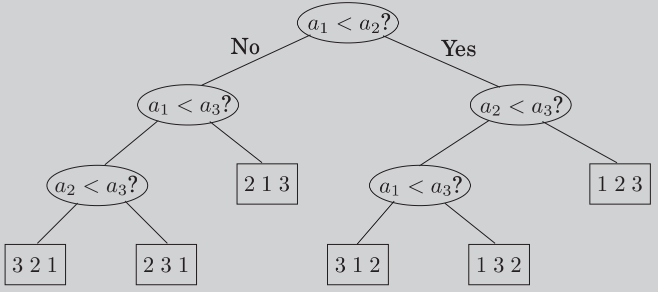

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So, the depth of the tree - and the complexity of the algorithm - must be at least

$$\log(n!) \approx \log(\sqrt{\pi(2n + 1/3)} \cdot n^n \cdot e^{-n}) = \Omega(n \log n)$$

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Selection

Input: A list of number S; an integer k.
Output: The $k\,th$ smallest element of S.

A Randomized Selection

For any number v, imagine splitting list S into three categories:

- elements smaller than v, i.e., S_L;
- those equal to v, i.e., S_v (there might be duplicates);
- and those greater than v, i.e., S_R; respectively.

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$$selection(S,k) = \begin{cases} selection(S_L, k) & \text{if } k \le |S_L| \\ v & \text{if } |S_L| < k \le |S_L| + |S_v| \\ selection(S_R, k - |S_L| - |S_v|) & \text{if } k > |S_L| + |S_v| \end{cases}$$

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Instead, we pick v randomly from S!

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Worst-case scenario would force our selection algorithm to perform

$$n + (n-1) + (n-2) + \ldots + \frac{n}{2} = \Theta(n^2)$$

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Best-case scenario $O(n)$

The Efficiency Analysis

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Proof:

Let E be the expected number of tosses before heads is seen.

$$E = 1 + \frac{1}{2}E$$

Therefore, $E = 2$.

The Efficiency Analysis

Let $T(n)$ be the expected running time on the array of size n, we get

$$T(n) \leq T(3n/4) + O(n) = O(n)$$

Matrix Multiplication

Matrix

The product of two $n \times n$ matrices X and Y is a $n \times n$ matrix $Z = XY$, with which $(i, j)th$ entry

$$Z_{ij} = \sum_{i=1}^{n} X_{ik} Y_{kj}$$

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The running time for matrix multiplication is $O(n^3)$

- There are n^2 entries to be computed, and each takes $O(n)$ time.

Divide-and-Conquer

Matrix multiplication can be performed blockwise.

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad Y = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

$$XY = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

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$$T(n) = 8T(n/2) + O(n^2)$$

$$T(n) = O(n^3)$$

Strassen Algorithm

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$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad Y = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

$$XY = \begin{bmatrix} P_5 + P_4 - P_2 + P_6 & P_1 + P_2 \\ P_3 + P_4 & P_1 + P_5 - P_3 - P_7 \end{bmatrix}$$

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$$\begin{aligned} P_1 &= A(F - H) & P_5 &= (A + D)(E + H) \\ P_2 &= (A + B)H & P_6 &= (B - D)(G + H) \\ P_3 &= (C + D)E & P_7 &= (A - C)(E + F) \\ P_4 &= D(G - E) \end{aligned}$$

Strassen Algorithm

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad Y = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

$$XY = \begin{bmatrix} P_5 + P_4 - P_2 + P_6 & P_1 + P_2 \\ P_3 + P_4 & P_1 + P_5 - P_3 - P_7 \end{bmatrix}$$

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$$T(n) = 7T(n/2) + O(n^2)$$

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$$T(n) = 7T(n/2) + O(n^2)$$

$$T(n) = O(n^{\log_2 7}) \approx O(n^{2.81})$$