Algorithm Design V

Divide and Conquer II

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$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

- length: $r = \sqrt{a^2 + b^2}$.
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Basic arithmetic:

- $-z = (r, \theta + \pi)$.
- $(r_1, \theta_1) \times (r_2, \theta_2) = (r_1 r_2, \theta_1 + \theta_2)$.
- If z is on the unit circle (i.e., $r = 1$), then $z^n = (1, n\theta)$.

The n-th Complex Roots of Unity

Solutions to the equation $z^n = 1$

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For n is even:

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The angle between two vectors $u = (u_0, \ldots, u_{n-1})$ and $v(v_0, \ldots, v_{n-1})$ in $\mathbb{C}^n$ is just a scaling factor times their inner product

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The above quantity is maximized when the vectors lie in the same direction and is zero when the vectors are orthogonal to each other.

The Fast Fourier Transform

Polynomial multiplication

If $A(x) = a_0 + a_1x + \ldots + a_dx^d$ and $B(x) = b_0 + b_1x + \ldots + b_dx^d$, their product

$$C(x) = c_0 + c_1x + \ldots + c_{2d}x^{2d}$$

has coefficients

$$c_k = a_0b_k + a_1b_{k-1} + \ldots + a_kb_0 = \sum_{i=0}^{k} a_ib_{k-i}$$

where for $i > d$, take a_i and b_i to be zero.

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Q: Can we do better?

An alternative representation

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coefficient representation —evaluation→ value representation

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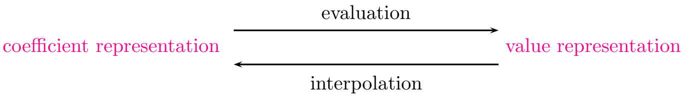

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Therefore, polynomial multiplication takes linear time in the value representation.

The algorithm

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Interpolation
Recover $C(x) = c_0 + c_1 x + \ldots + c_{2d} x^{2d}$

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Evaluating a polynomial of degree $d \leq n$ at a single point takes $O(n)$, and so the baseline for n points is $\Theta(n^2)$.

The Fast Fourier Transform (FFT) does it in just $O(n \log n)$ time, for a particularly clever choice of $x_0, \ldots, x_{n-1}$.

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First idea, we pick the n points,

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then the computations required for each $A(x_i)$ and $A(-x_i)$ overlap a lot, because the even power of x_i coincide with those of $-x_i$.

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We need to split $A(x)$ into its odd and even powers, for instance

$$3 + 4x + 6x^2 + 2x^3 + x^4 + 10x^5 = (3 + 6x^2 + x^4) + x(4 + 2x^2 + 10x^4)$$

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More generally

$$A(x) = A_e(x^2) + xA_o(x^2)$$

where $A_e(\cdot)$, with the even-numbered coefficients, and $A_o(\cdot)$, with the odd-numbered coefficients, are polynomials of degree $\leq n/2 - 1$.

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Given paired points $\pm x_i$, the calculations needed for $A(x_i)$ can be recycled toward computing $A(-x_i)$:

$$\begin{aligned} A(x_i) &= A_e(x_i^2) + x_i A_o(x_i^2) \\ A(-x_i) &= A_e(x_i^2) - x_i A_o(x_i^2) \end{aligned}$$

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If we could recurse, we would get a divide-and-conquer procedure with running time

$$T(n) = 2T(n/2) + O(n) = O(n \log n)$$

How to choose n points?

Aim: To recurse at the next level, we need the $n/2$ evaluation points $x_0^2, x_1^2, \ldots, x_{n/2-1}^2$ to be themselves plus-minus pairs.

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By continuing in this manner, we eventually reach the initial set of n points: the complex $n\,th$ roots of unity, that is the n complex solutions of the equation

$$z^n = 1$$

The n-th complex roots of unity

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The FFT algorithm


```
FFT(A, ω)
```

input : coefficient reprentation of a polynomial $A(x)$ of degree $\leq n-1$, where n is a power of 2;
ω, an n-th root of unity

output: value representation $A(\omega^0), \ldots, A(\omega^{n-1})$

if $\omega = 1$ **then** `return` $A(1)$;
express $A(x)$ in the form $A_e(x^2) + xA_o(x^2)$;
call `FFT(`A_e,ω^2`)` to evaluate A_e at even powers of ω;
call `FFT(`A_o,ω^2`)` to evaluate A_o at even powers of ω;
for $j = 0$ *to* $n-1$ **do**
  compute $A(\omega^j) = A_e(\omega^{2j}) + \omega^j A_o(\omega^{2j})$;
end
`return(`$A(\omega^0), \ldots, A(\omega^{n-1})$`);`

Interpolation

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We will see that the interpolation can be computed by

$$\langle coefficients \rangle = \frac{1}{n}\texttt{FFT}(\langle values \rangle, \omega^{-1})$$

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- If $x_0, x_1, \dots, x_{n-1}$ are distinct numbers, then M is invertible.
- evaluation is multiplication by M, while interpolation is multiplication by M^{-1}.

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However, using this for interpolation would still not be fast enough for us..

Interpolation resolved

In linear algebra terms, the `FFT` multiplies an arbitrary n-dimensional vector, which we have been calling the coefficient representation, by the $n \times n$ matrix.

$$M_n(\omega) = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ & & \vdots & & \\ 1 & \omega^j & \omega^{2j} & \dots & \omega^{(n-1)j} \\ & & \vdots & & \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \dots & x^{(n-1)(n-1)} \end{bmatrix}$$

Its (j, k)-th entry (starting row- and column-count at zero) is ω^{jk}

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Inversion formula

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Proof.

- Take the inner product of of any columns j and k of matrix M,

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This is a geometric series with first term 1, last term $\omega^{(n-1)(j-k)}$, and ratio ω^{j-k}.

Interpolation resolved

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- If $j = k$, then it evaluates to n.

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Corollary

$MM^* = nI$, *i.e.*,

$$M_n^{-1} = \frac{1}{n} M_n^*$$

The definitive FFT algorithm

The FFT takes as input a vector $a = (a_0, \ldots, a_{n-1})$ and a complex number ω whose powers $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are the complex n-th roots of unity.

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This divide-and-conquer strategy leads to the definitive FFT algorithm, whose running time is $T(n) = 2T(n/2) + O(n) = O(nlogn)$.

The general FFT algorithm

`FFT` (a, ω)
input : An array $a = (a_0, a_1, \ldots, a_{n-1})$ for n is a power of 2; ω, an n-th root of unity
output: $M_n(\omega)a$

if $\omega = 1$ **then** `return` a;
$(s_0, s_1, \ldots, s_{n/2-1})$=`FFT` $((a_0, a_2, \ldots, a_{n-2}), \omega^2)$;
$(s'_0, s'_1, \ldots, s'_{n/2-1})$=`FFT` $((a_1, a_3, \ldots, a_{n-1}), \omega^2)$;
for $j = 0$ *to* $n/2 - 1$ **do**
    r_j=$s_j + \omega^j s'_j$;
    $r_{j+n/2}$=$s_j - \omega^j s'_j$;
end
`return` $(r_0, r_1, \ldots, r_{n-1})$;

Top 10 algorithms of the 20th century

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1946: The Metropolis Algorithm

1947: Simplex Method

1950: Krylov Subspace Method

1951: The Decompositional Approach to Matrix Computations

1957: The Fortran Optimizing Compiler

1959: QR Algorithm

1962: Quicksort

1965: Fast Fourier Transform

1977: Integer Relation Detection

1987: Fast Multipole Method

Homework

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- Assignment 2 (1 week). Exercises 2.13, 2.19, 2.22, and 2.28.