Algorithm Design VIII

Greedy Algorithms

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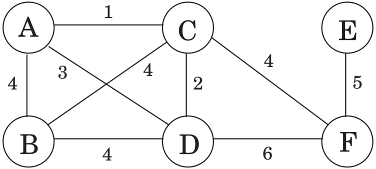
Minimum Spanning Trees

Build a Network



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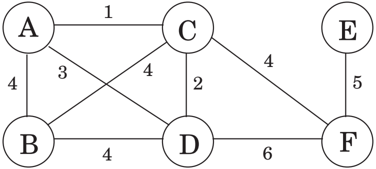

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This translates into a graph problem in which

- nodes are computers,
- undirected edges are potential links, each with a maintenance cost.

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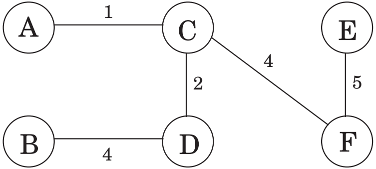

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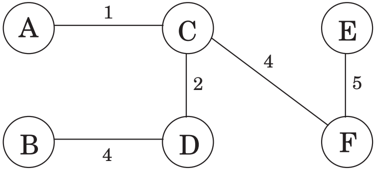

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- the total maintenance cost is minimum.

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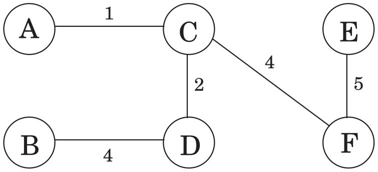

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- pick enough of these edges that the nodes are connected,
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One immediate observation is that the optimal set of edges cannot contain a cycle.

Properties of the Optimal Solutions

Lemma (1)

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Input: An undirected graph $G = (V, E)$; edge weights w_e

Output: A tree $T = (V, E')$ with $E' \subseteq E$ that minimizes

$$\texttt{weight}(T) = \sum_{e \in E'} w_e$$

Trees

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When a particular edge (u,v) comes up, we can be sure that u and v lie in separate connected components, for otherwise there would already be a path between them and this edge would create a cycle.

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The process terminates with some graph $G' = (V, E'), E' \subseteq E$, which is acyclic and, by Lemma (1), is also connected.

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Therefore G' is a tree, whereupon $|E'| = |V| - 1$ by Lemma (2). So $E' = E$, no edges were removed, and G was acyclic to start with.

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In a tree, any two nodes can only have one path between them; for if there were two paths, the union of these paths would contain a cycle.

On the other hand, if a graph has a path between any two nodes, then it is connected. If these paths are unique, then the graph is also acyclic.

A Greedy Approach

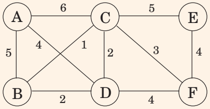
Kruskal's minimum spanning tree algorithm starts with the empty graph and then selects edges from E according to the following rule.

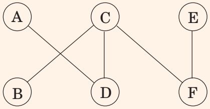
Repeatedly add the next lightest edge that doesn't produce a cycle.

Example

Starting with an empty graph and then attempt to add edges in increasing order of weight

$$B - C; C - D; B - D; C - F; D - F; E - F; A - D; A - B; C - E; A - C$$





The Cut Property

Lemma

Suppose edges X are part of a MST of $G = (V, E)$. Pick any subset of nodes S for which X does not cross between S and $V \setminus S$, and let e be the lightest edge across this partition. Then

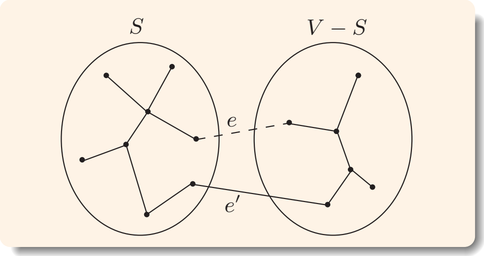
$$X \cup \{e\}$$

is part of some MST.

The Cut Property

A cut is any partition of the vertices into two groups, S and $V \backslash S$.

It is safe to add the lightest edge across any cut, provided X has no edges across the cut.



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Add edge e to T.

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which we will show to be a tree.

T' is connected by Lemma (1), since e' is a cycle edge. And it has the same number of edges as T; so by Lemma (2) and Lemma (3), it is also a tree.

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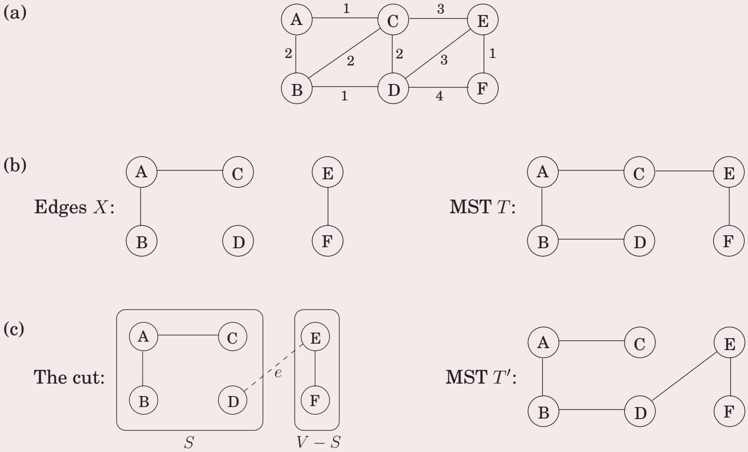
Since T is an MST, it must be the case that $weight(T') = weight(T)$ and that T' is also an MST.

An Example of Cut Property



Kruskal's Algorithm


```
KRUSKAL(G, w)
input  : A connected undirected graph G = (V, E), with edge weight w_e
output: A minimum spanning tree defined by the edges X

for all u ∈ V do
    makeset (u);
end
X = { };
Sort the edges E by weight;
for all (u, v) ∈ E in increasing order of weight do
    if find (u) ≠ find (v) then
        add (u, v) to X;
        union (u,v)
    end
end
```

Data Structure Retailer: Disjoint Sets

$\texttt{makeset}(x)$	create a singleton set containing x	$\lvert V\rvert$
$\texttt{find}(x)$	find the set that x belong to	$2 \cdot \lvert E\rvert$
$\texttt{union}(x, y)$	merge the sets containing x and y	$\lvert V\rvert - 1$

Prim’s Algorithm

A General Kruskal's Algorithm

$X = \{\ \}$;
repeat until $|X| = |V| - 1$;
    pick a set $S \subset V$ for which X has no edges between S and $V - S$;
    let $e \in E$ be the minimum-weight edge between S and $V - S$;
    $X = X \cup \{e\}$;

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A popular alternative to Kruskal's algorithm is Prim's, in which the intermediate set of edges X always forms a subtree, and S is chosen to be the set of this tree's vertices.

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$$\texttt{cost}(v) = \min_{u \in S} w(u, v)$$

The Algorithm


```
PRIM(G, w)
input  : A connected undirected graph G = (V, E), with edge weights w_e
output: A minimum spanning tree defined by the array prev

for all u ∈ V do
    cost(u) = ∞;
    prev(u) = nil;
end
pick any initial node u_0;
cost(u_0) = 0;
H =makequeue(V)\\ using cost-values as keys;
while H is not empty do
    v=deletemin(H);
    for each (v, z) ∈ E do
        if cost(z) > w(v, z) then
            cost(v) = w(v, z); prev(z) = v;
            decreasekey (H,z);
        end
    end
end
```

Dijkstra's Algorithm

DIJKSTRA(G, l, s)

input : Graph $G = (V, E)$, directed or undirected; positive edge length $\{l_e \mid e \in E\}$; Vertex $s \in V$

output: For all vertices u reachable from s, $dist(u)$ is the set to the distance from s to u

```
for all u ∈ V do
    dist(u) = ∞;
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end
dist(s) = 0;
H =makequeue(V) \\ using dist-values as keys;
while H is not empty do
    u=deletemin(H);
    for all edge (u, v) ∈ E do
        if dist(v) > dist(u) + l(u, v) then
            dist(v) = dist(u) + l(u, v);  prev(v) = u;
            decreasekey (H,v);
        end
    end
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Set Cover

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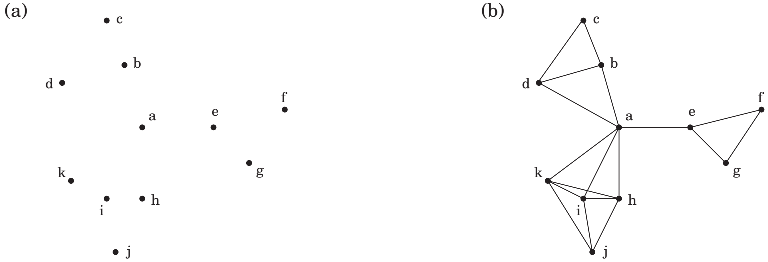
Q: What is the minimum number of schools needed?



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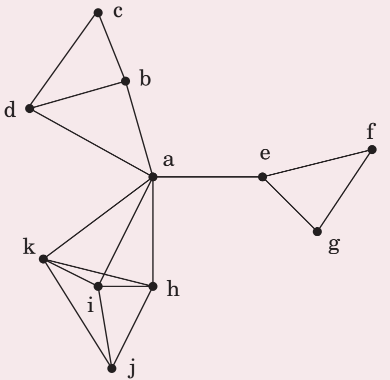
- For each town x, let S_x be the set of towns within 30 miles of it.
- A school at x will essentially "cover" these other towns.
- The question is then, how many sets S_x must be picked in order to cover all the towns in the county?

Set Cover Problem

SET COVER

- Input: A set of elements B, sets $S_1, \ldots, S_m \subseteq B$
- Output: A selection of the S_i whose union is B.
- Cost: Number of sets picked.

The Example



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Therefore, the greedy strategy will ensure that

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which by repeated application implies

$$n_t \leq n_0(1 - \frac{1}{OPT})^t$$

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At $t = \ln n \cdot OPT$, therefore, n_t is strictly less than $ne^{-\ln n} = 1$, which means no elements remain to be covered.