Combinatorial Counting

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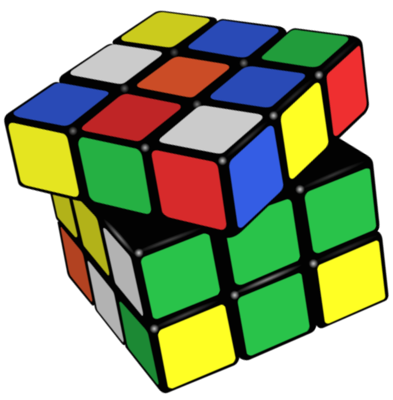

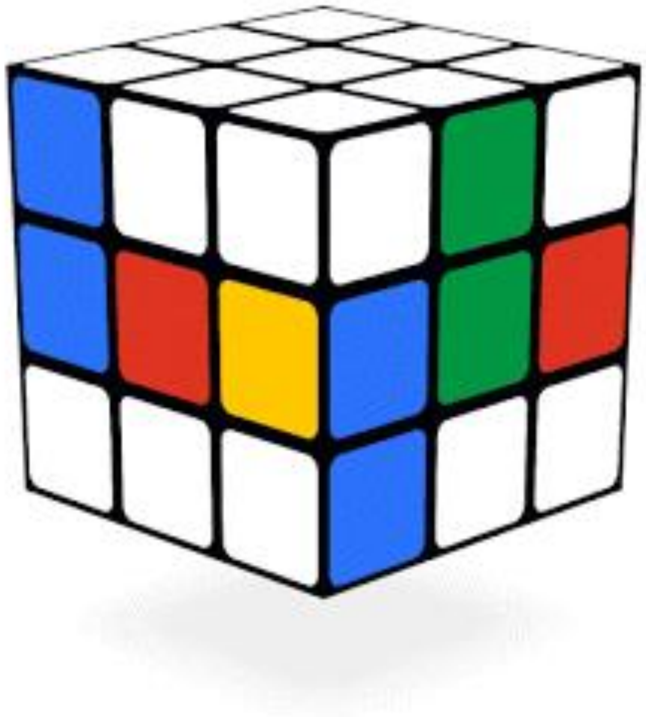

Let’s Count!

n balls are put into m bins

balls per bin	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins.			
n identical balls, m distinct bins.			
n distinct balls, m identical bins.			
n identical balls, m identical bins.			

n balls are put into m bins

balls per bin	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins.			
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n distinct balls, m identical bins.		$\begin{cases} 1 & n \leq m \\ 0 & n > m \end{cases}$	
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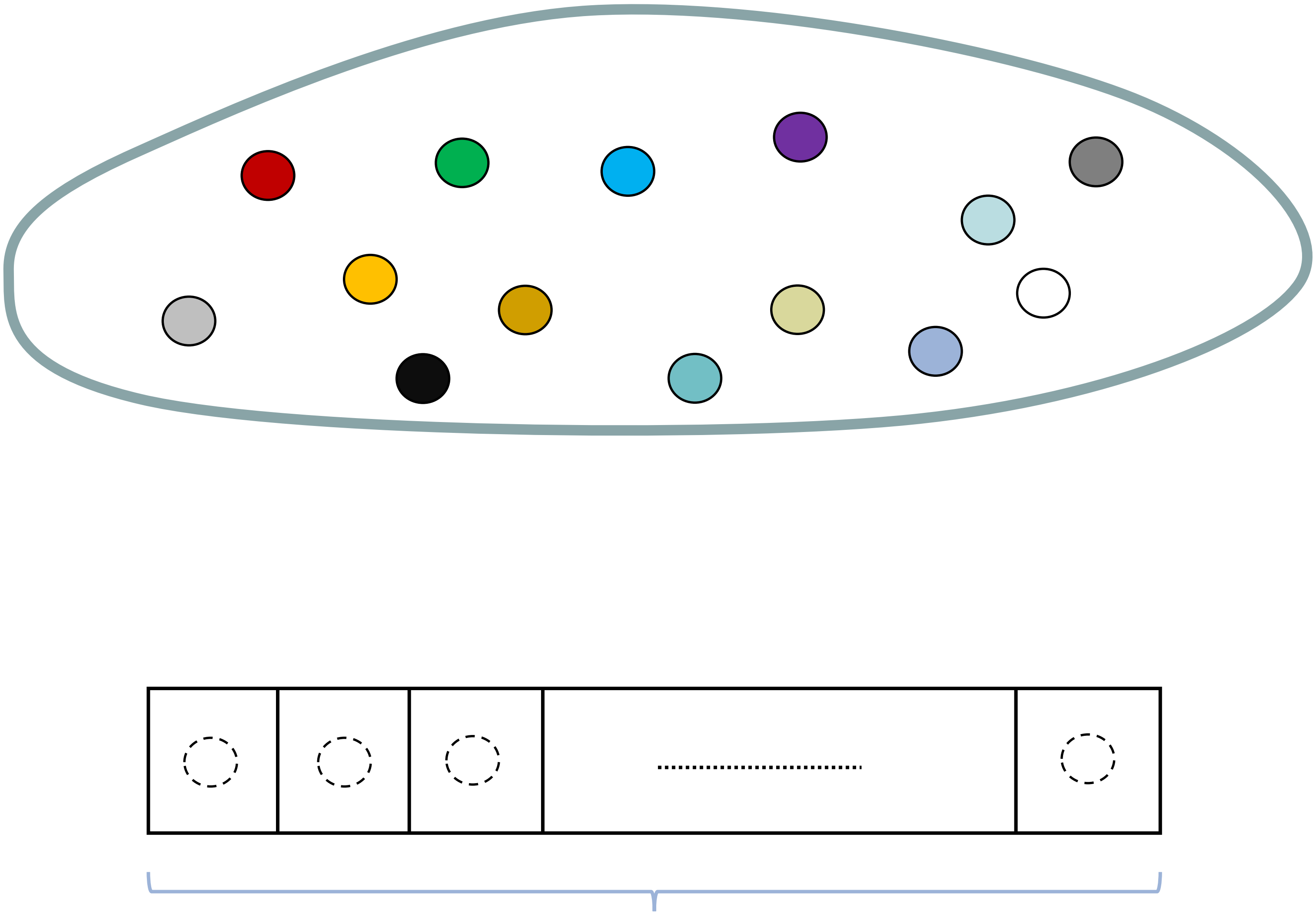
- Basic counting
- Binomial theorem
- Generalized Binomial theoremSome
- special numbers

We will start with counting the *ordered* objects.



Ordered sequence

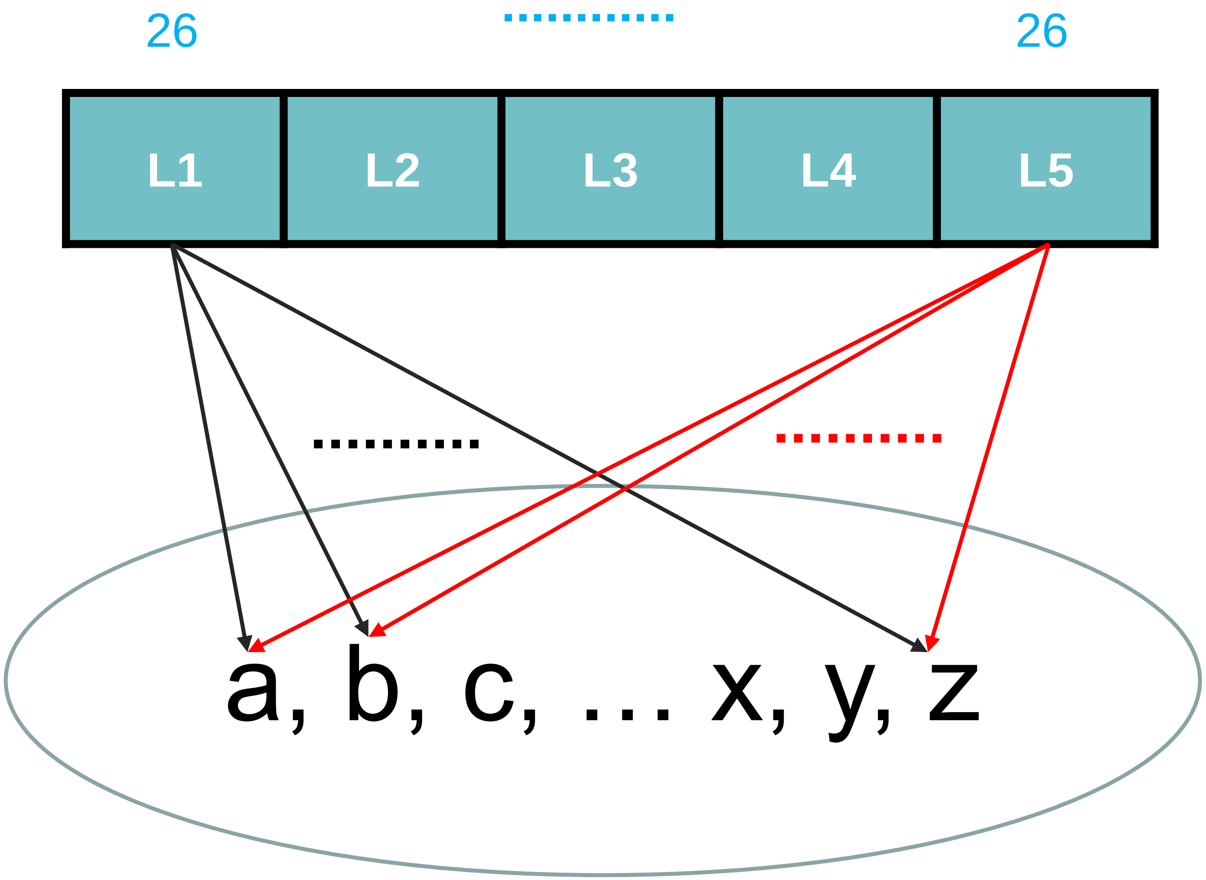
- **Problem1：** How many 5-letter words are there(using the 26-letter English alphabet)？

e. g. abcde, sssdd, …

- **Problem2：** How many distinct 5-letter words are there(using the 26-letter English alphabet) ？

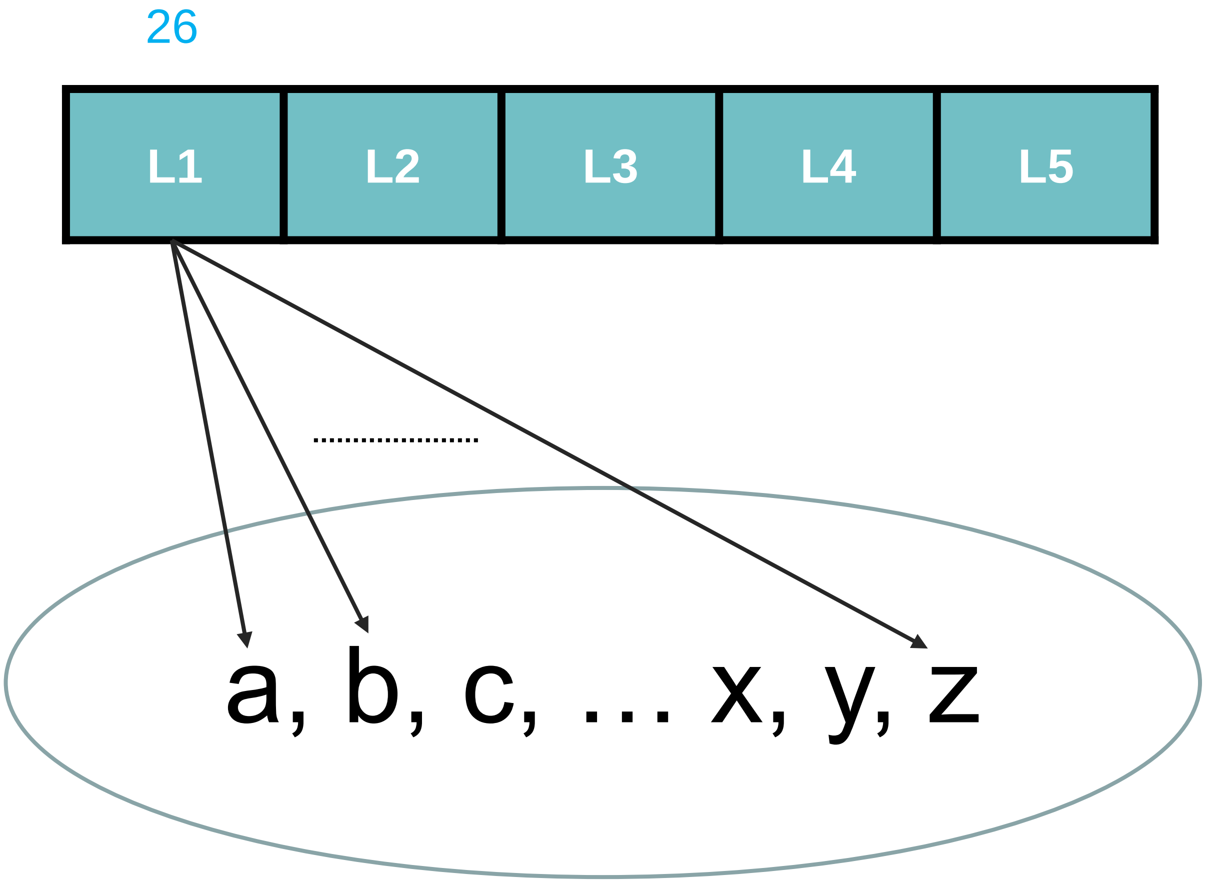
e. g. abcde, ~~sssdd~~, …

5-letter words

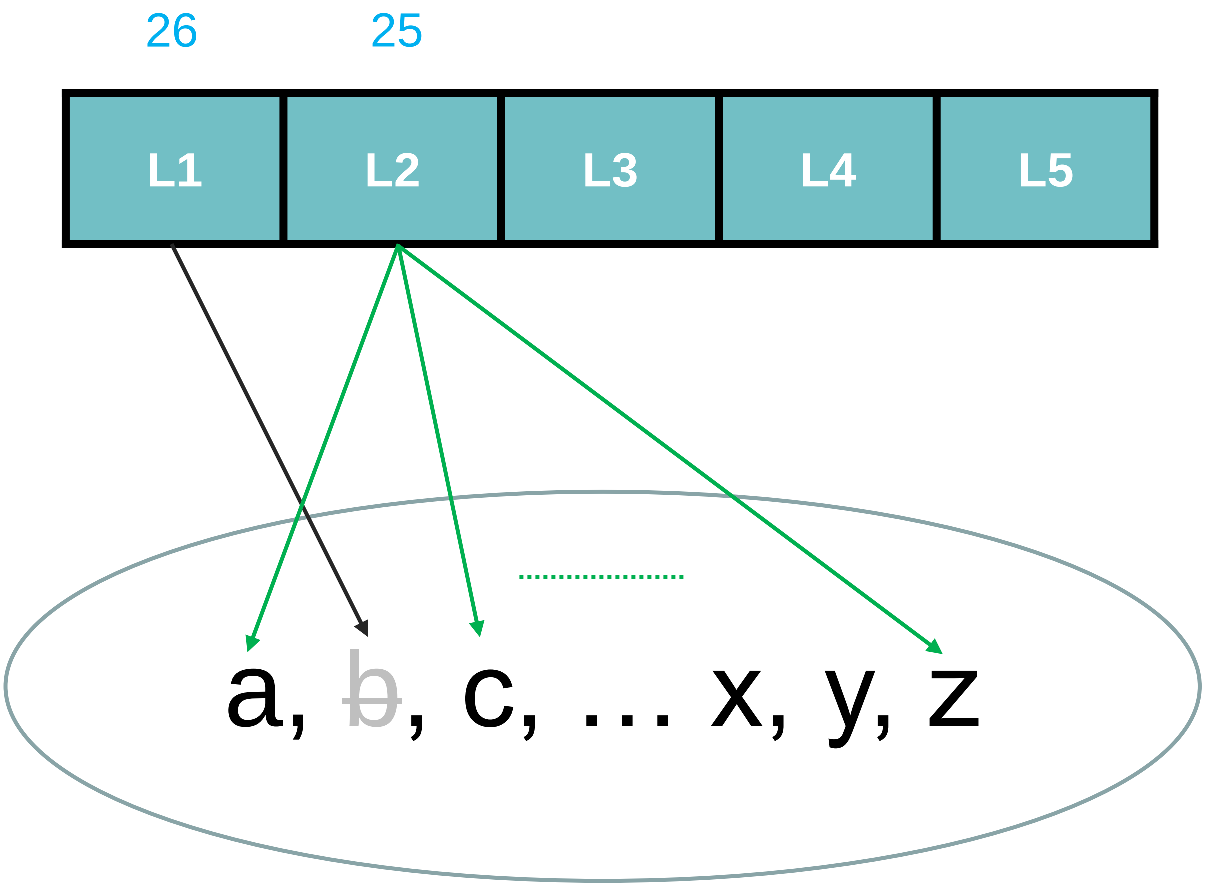


$$26 \times 26 \times 26 \times 26 \times 26 = 26^5$$

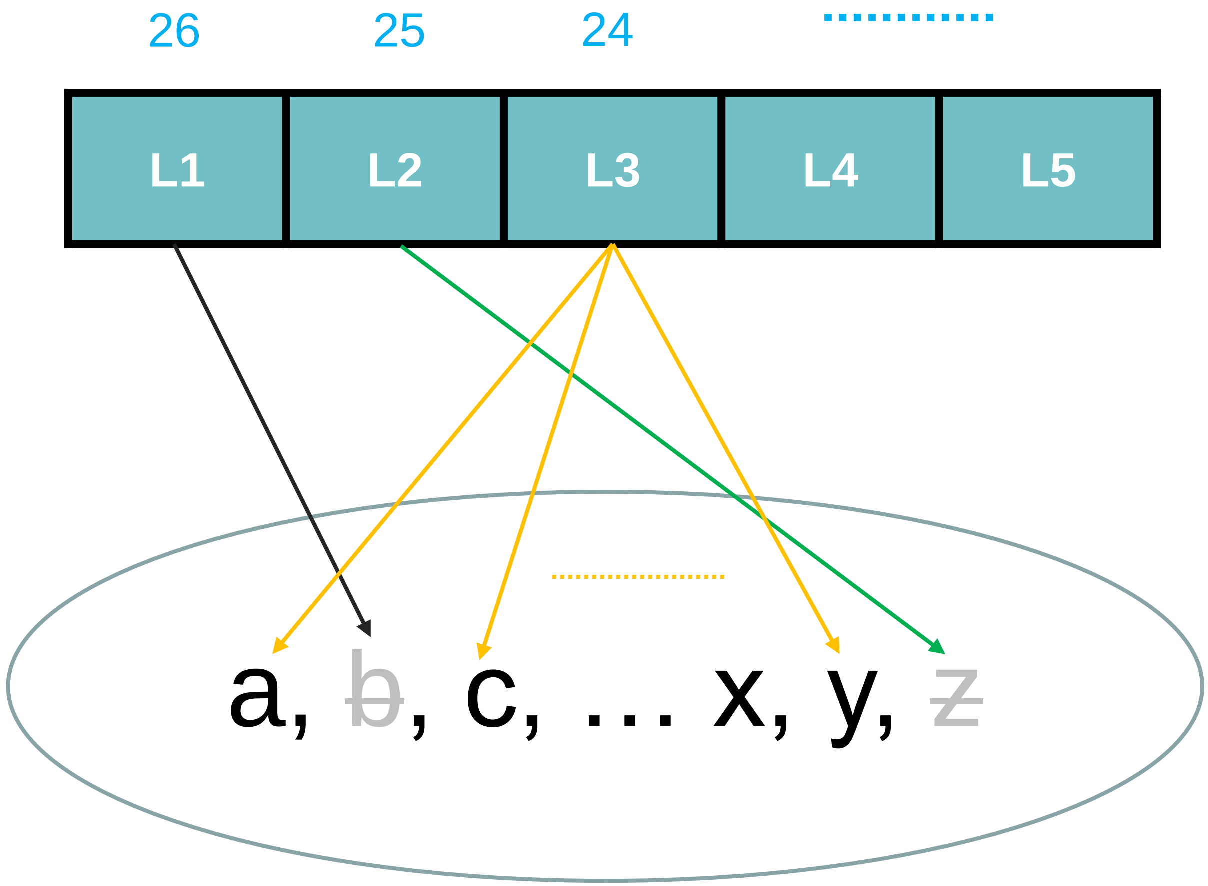
Distinct 5-letter words



Distinct 5-letter words



Distinct 5-letter words



$$26 \times 25 \times 24 \times 23 \times 22$$

Proof by induction

Goal: show that $P(x)$ is true for any $x \in \omega$

① Check that $P(0)$ is true;

② Suppose that $P(k)$ is true; // Induction hypothesis

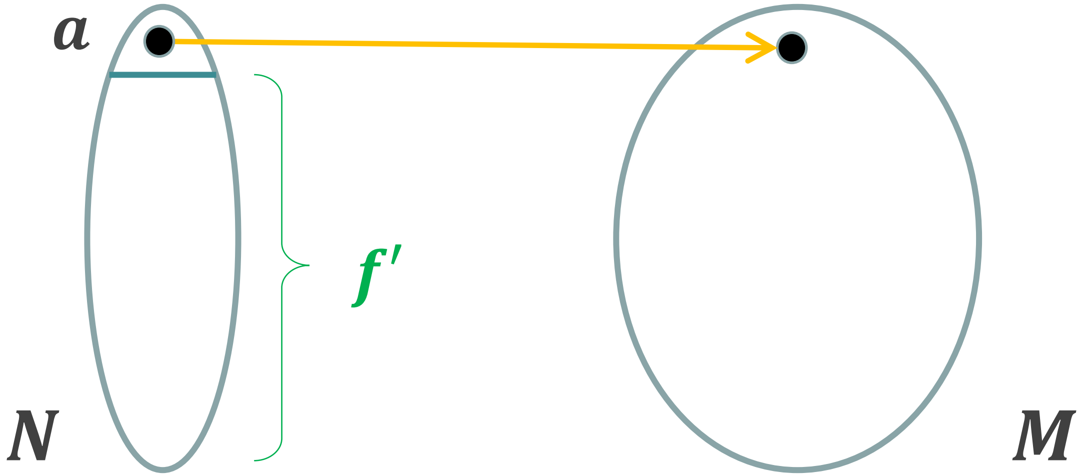
③ Prove that $P(k+1)$ is true.

The generalization of Problem 1

- Proposition1： Let N be an n-element set, and M be an m-element set, with $n \geq 0, m \geq 1$. Then the number of all possible mappings $f: N \to M$ is m^n.

- Proof: (By induction on n)
  - $n = 0$: $f = \emptyset$; $m^0 = 1$ 。
  - Suppose the results works for $n = k$;
  - If $n = k + 1$:

$n = k + 1$, take any $a \in N$:



$$m \cdot m^{n-1} = m^n$$

n balls are put into m bins

balls per bin	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins.	m^n		
n identical balls, m distinct bins.			
n distinct balls, m identical bins.		$\begin{cases} 1 & n \le m \\ 0 & n > m \end{cases}$	
n identical balls, m identical bins.		$\begin{cases} 1 & n \le m \\ 0 & n > m \end{cases}$	

The generalization of Problem 2

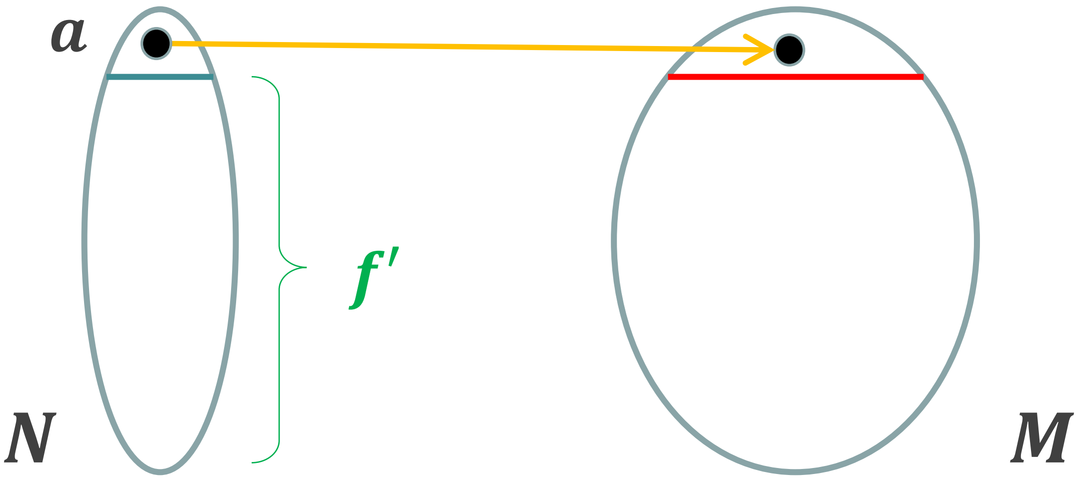
- Proposition2： Let N be an n-element set, and M be an m-element set, with $n, m \geq 0$. Then there exist exactly

$$m(m-1)\ldots(m-n+1)=\prod_{i=0}^{n-1}(m-i)$$

one-to-one mappings from N into M.

- Proof: (By induction on n)
  - $n=0$: $f=\emptyset$. The value of an empty product is defined as 1.
  - Suppose the results works for $n=k$;

– for $n = k + 1$, take any $a \in N$:



$$m(m-1)\ldots(m-n+1)$$

Falling factorial notation

$$
\begin{aligned}
&(x)_n \\
&= x^{\underline{n}} \\
&= x(x-1)\cdots(x-n+1)
\end{aligned}
$$

n balls are put into m bins

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Application 1: Counting the different subsets

Given set X , $|X| = n$, then X has exactly 2^n subsets ($n \geq 0$).

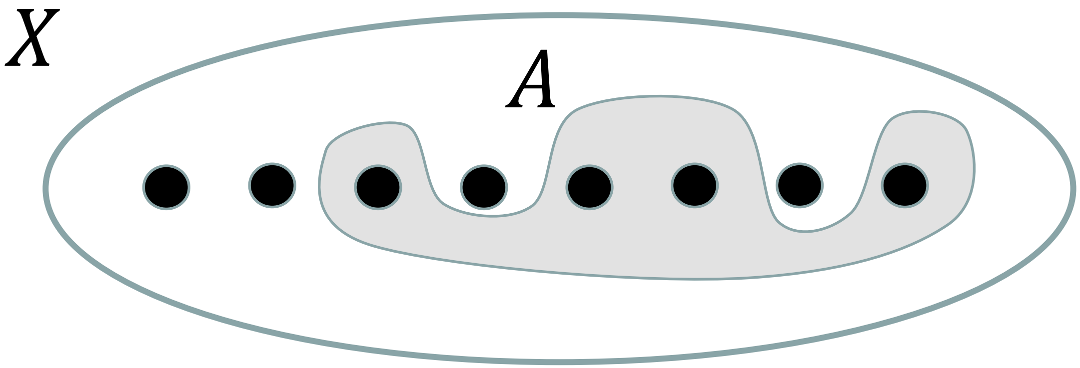
- Proof[1]: By induction on n. (Exercise)
- Proof[2]:

for any $A \subseteq X$, define $f_A: X \rightarrow \{0,1\}$ as

$$\boxed{f_A(x) = \begin{cases} 1 & if\ x \in A \\ 0 & if\ x \notin A \end{cases}}$$

Characteristic function

$$\boxed{f_A(x) = \begin{cases} 1 & if\ x \in A \\ 0 & if\ x \notin A \end{cases}}$$



f_A 0 0 1 0 1 1 0 1

There exists a bijective relation between the subsets of X and $f: X \to \{0,1\}$ (Recall: Equinumerous).

Application2：Counting the permutations

- ***Permutation***：A bijective mapping of a finite set X to itself is called a permutation of the set X.
- Recall：Bijective functions.

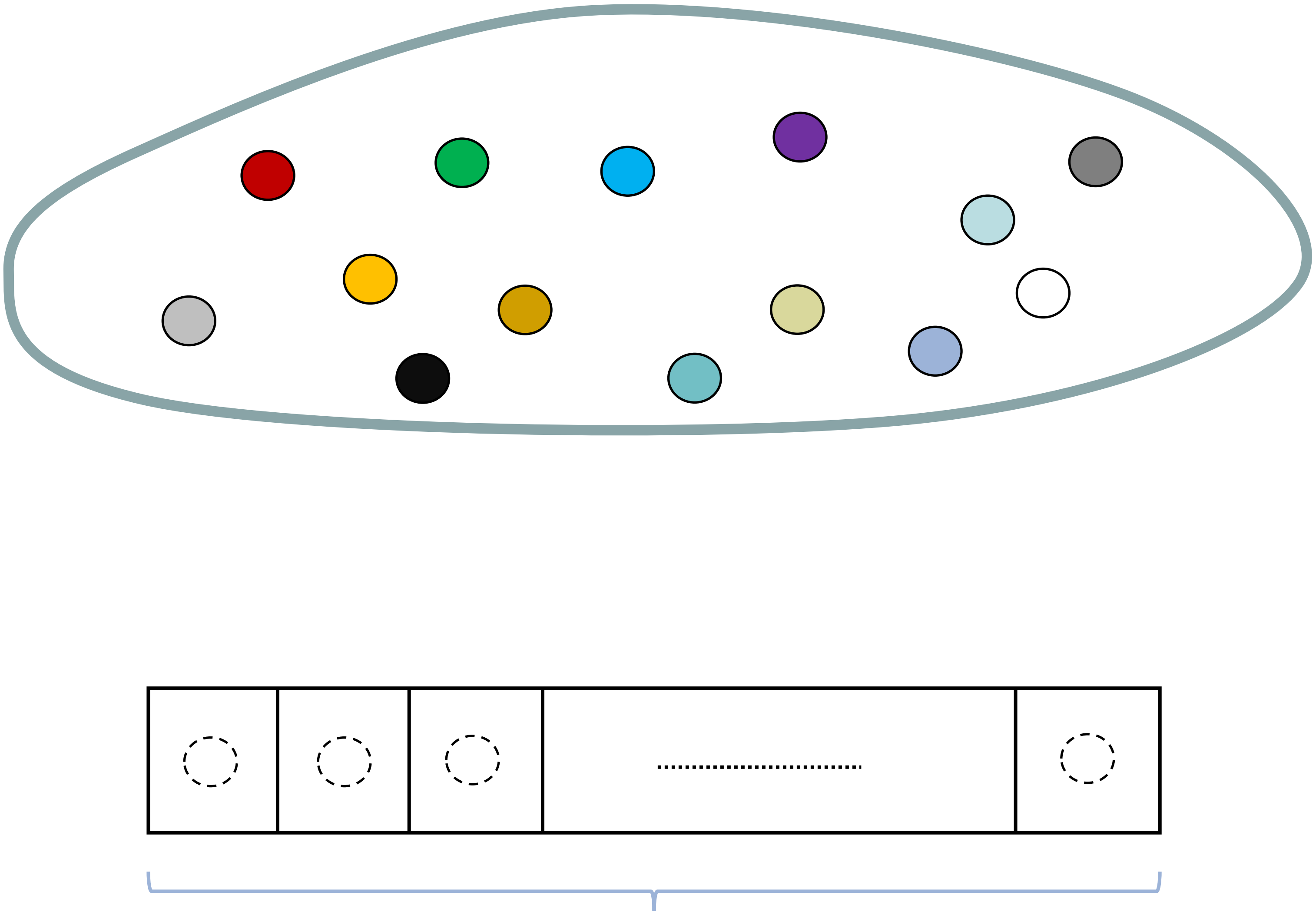
Counting permutations-Factorial

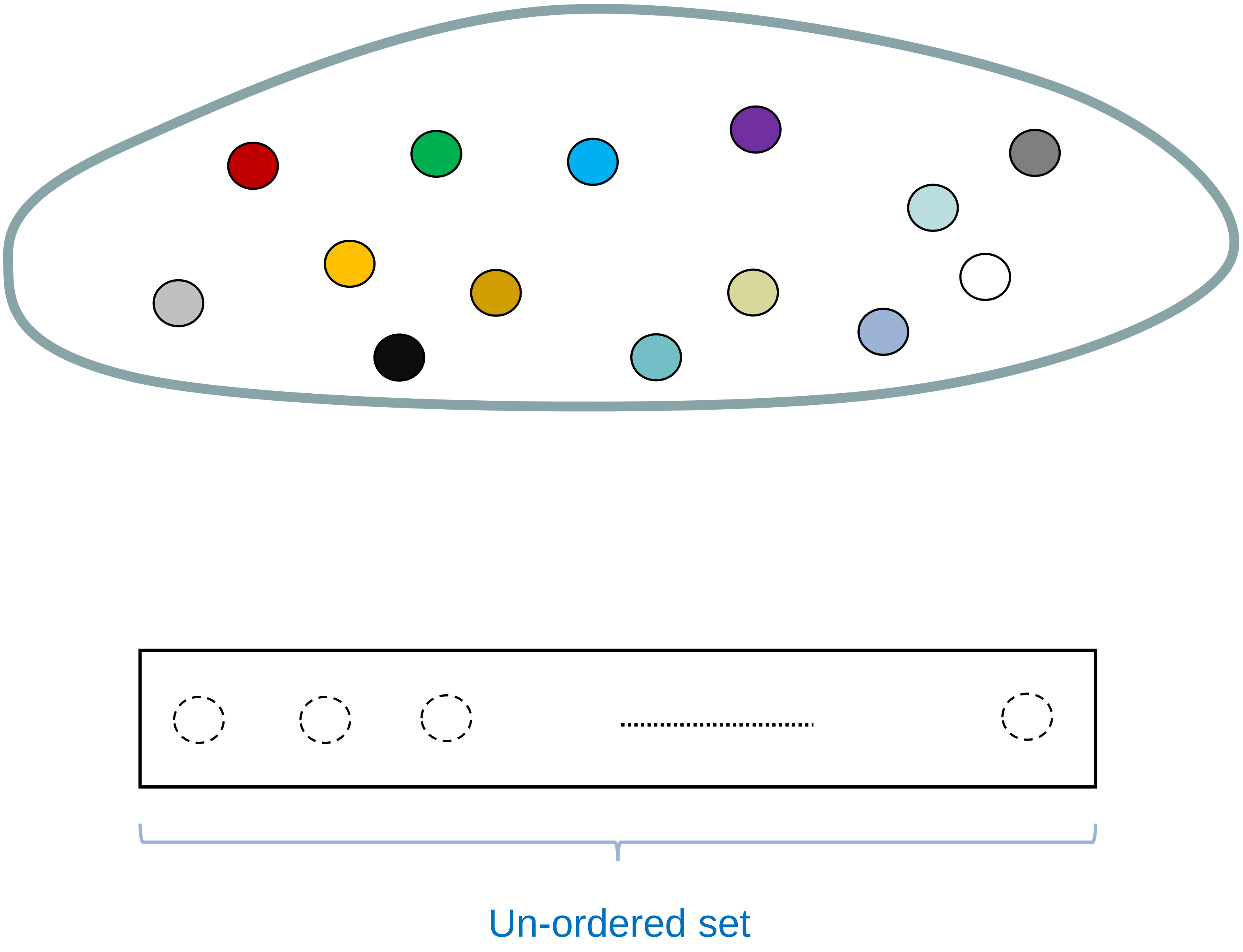
Given set X , $|X| = n$, then there are $n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$ different permutations on set X.

***n* factorial**:

$$n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1 = \prod_{i=1}^{n} i.$$

- So far, we considered ordered sequences.
- What about the un-ordered occasion?


Ordered sequence


Un-ordered set

Problem 3： counting k-element subsets

Given set X , $|X| = n$, $n \geq k \geq 0$ ， how many different subsets of X contains exactly k elements？

e. g. $X = \{a, b, c\}$ ， $k = 2$ 。

Then: $\{a, b\}$ ， $\{a, c\}$ ， $\{b, c\}$. Three 2-size subsets.

Convention: $\binom{X}{k}$ VS. $\left|\binom{X}{k}\right|$

e. g. $\binom{X}{k} = \{\{a, b\}, \{a, c\}, \{b, c\}\}$, $\left|\binom{X}{k}\right| = 3$.

- Proposition: For any finite set X with $|X| = n$, the number of all k-element subsets is

$$\left|\binom{X}{k}\right| = \frac{n(n-1)(n-2)\dots(n-k+1)}{k(k-1)\cdot\dots\cdot 2\cdot 1}.$$

- Proof: (Double counting!)

Binomial coefficients

- $\binom{n}{k} = \left|\binom{X}{k}\right| = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k(k-1)\cdot\ldots\cdot 2\cdot 1}$

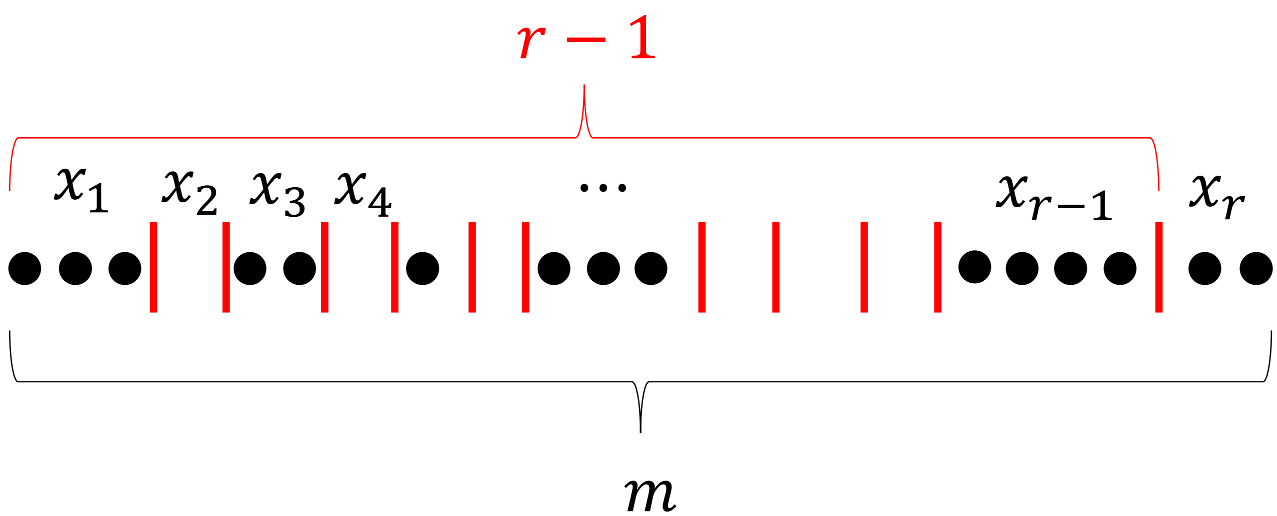
$$= \frac{\prod_{i=0}^{k-1}(n-i)}{k!}$$

$$= \frac{n(n-1)(n-2)\ldots(n-k+1)\cdot(n-k)\cdot\ldots\cdot 1}{k(k-1)\cdot\ldots\cdot 2\cdot 1\cdot(n-k)\cdot\ldots\cdot 1}$$

$$= \frac{n!}{k!\cdot(n-k)!}$$

Application: counting non-negative solutions.

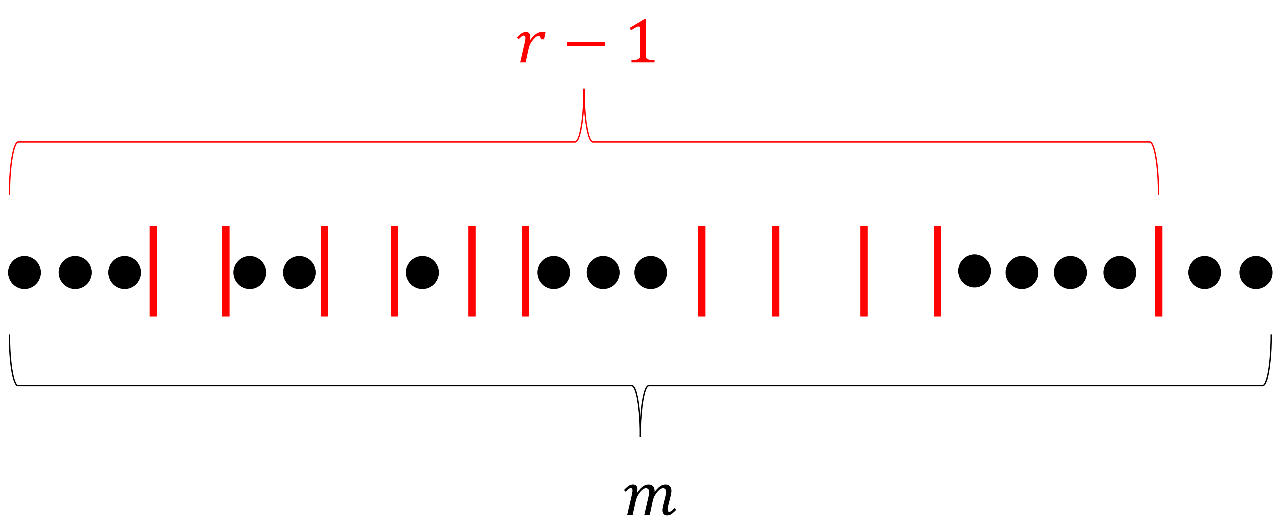
$m \geq r \geq 0$, the equation $x_1 + x_2 + \cdots + x_r = m$ has $\binom{m+r-1}{r-1}$ non-negative integers solutions of the form $(x_1, x_2, \ldots, x_r)$.



$\boldsymbol{x_1 = 3, x_2 = 0,\ x_3 = 2, x_4 = 0, \cdots,\ x_{r-1} = 4, x_r = 2}$

Application: counting non-negative solutions.

$m \geq r \geq 0$, the equation $x_1 + x_2 + \cdots + x_r = m$ has $\binom{m+r-1}{r-1}$ non-negative integers solutions of the form $(x_1, x_2, \ldots, x_r)$.



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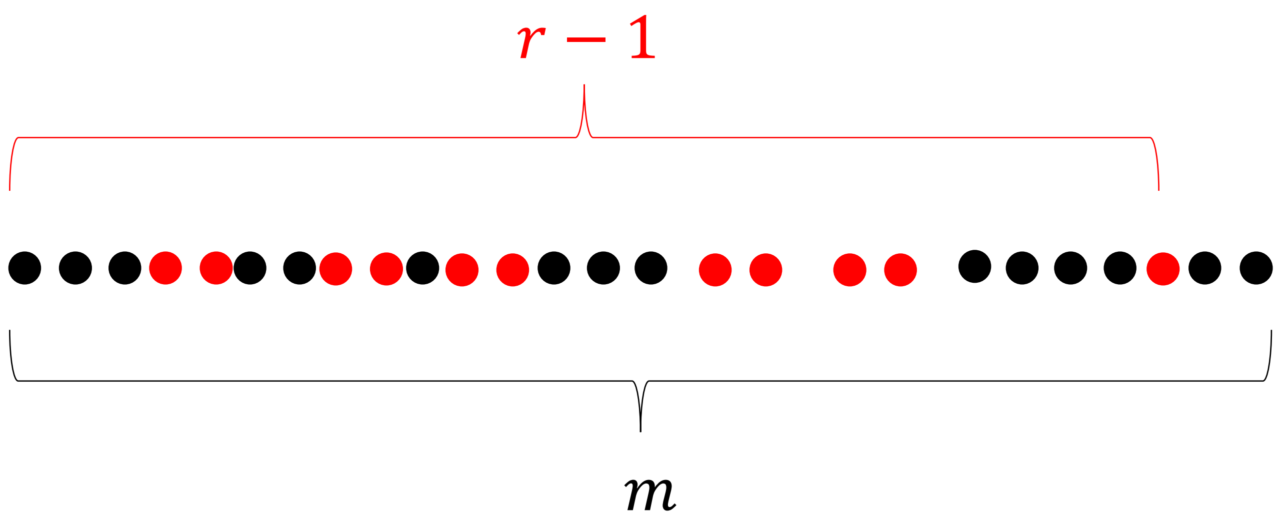
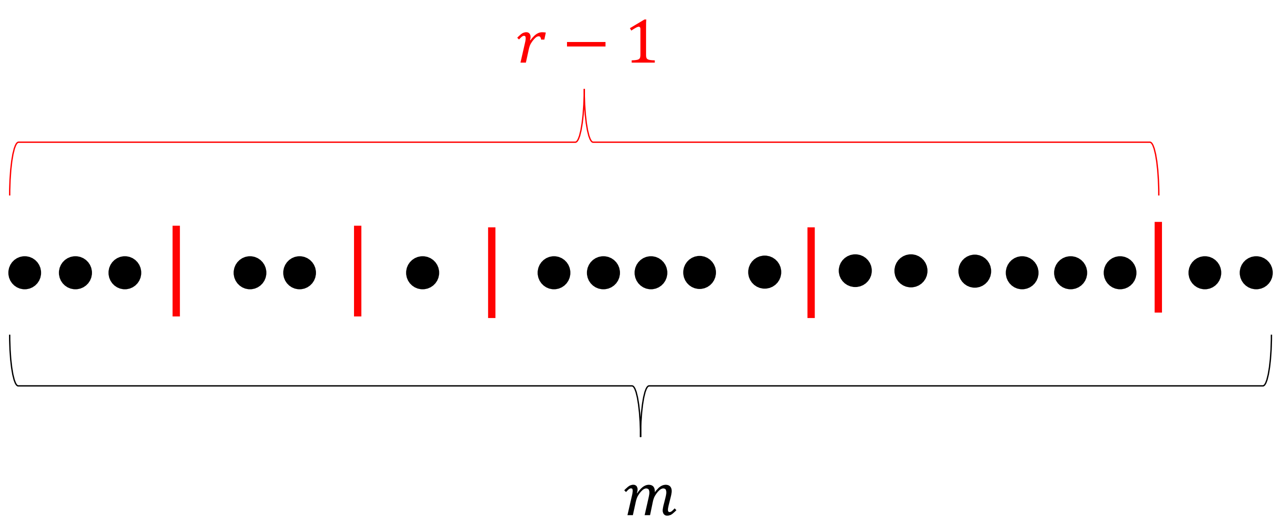


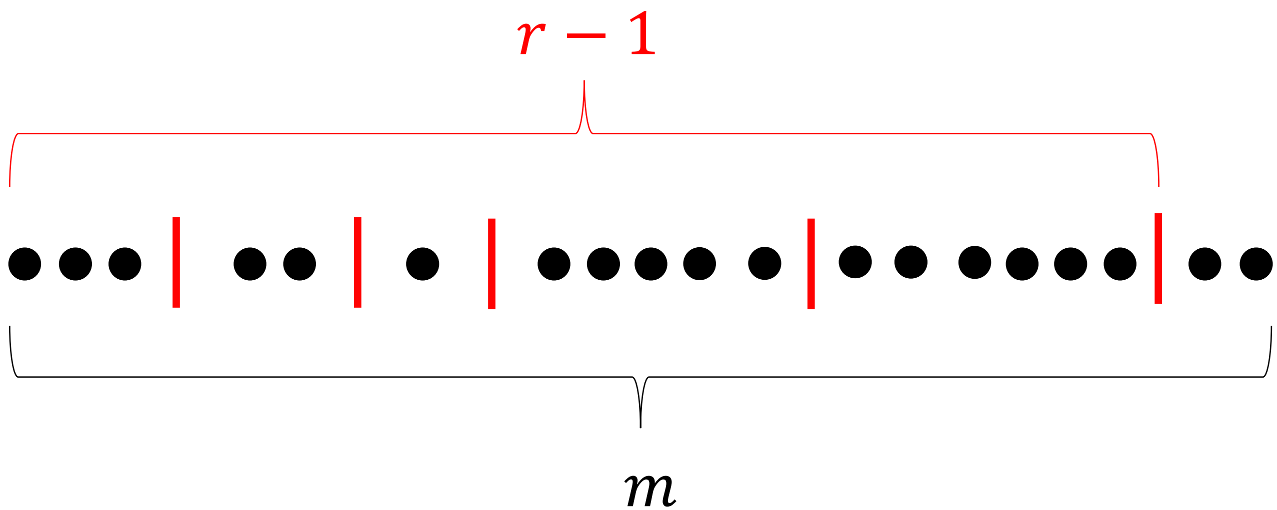
Question: counting positive solutions.

$m \geq r \geq 0$, the equation $x_1 + x_2 + \cdots + x_r = m$ has _____ positive integers solutions of the form $(x_1, x_2, \ldots, x_r)$.



Question: counting positive solutions.

$m \geq r \geq 0$, the equation $x_1 + x_2 + \cdots + x_r = m$ has $\binom{m-1}{r-1}$ positive integers solutions of the form $(x_1, x_2, \ldots, x_r)$.

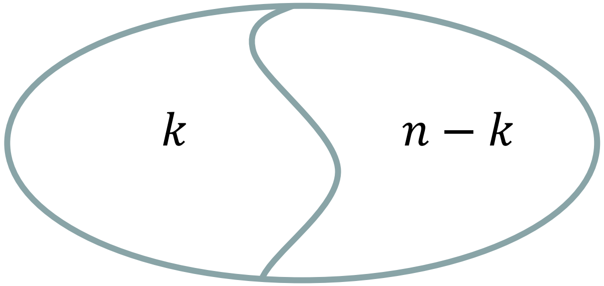


Basic Properties

$$\binom{n}{k} = \binom{n}{n-k}$$

- Proof[1]:

- Proof[2]:

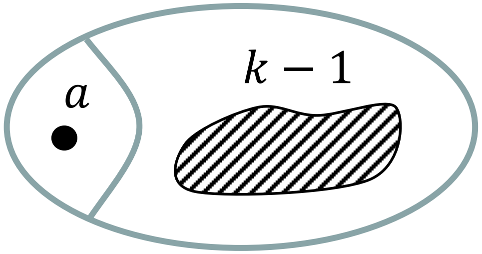


Pascal's Identity:

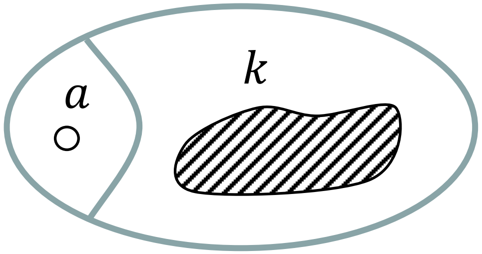
$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

- Proof:

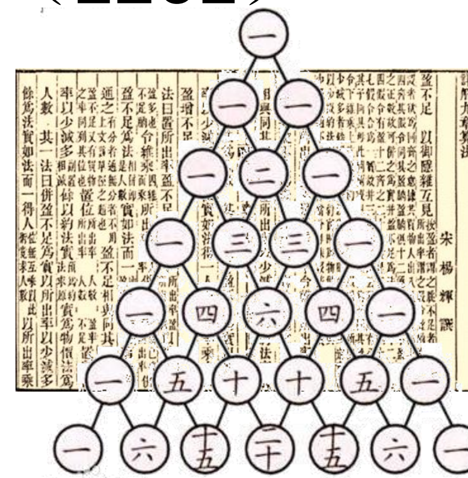
$$\binom{n-1}{k-1}$$



$$\binom{n-1}{k}$$



Pascal's Triangle （1654）/ 杨辉三角（1261）



$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

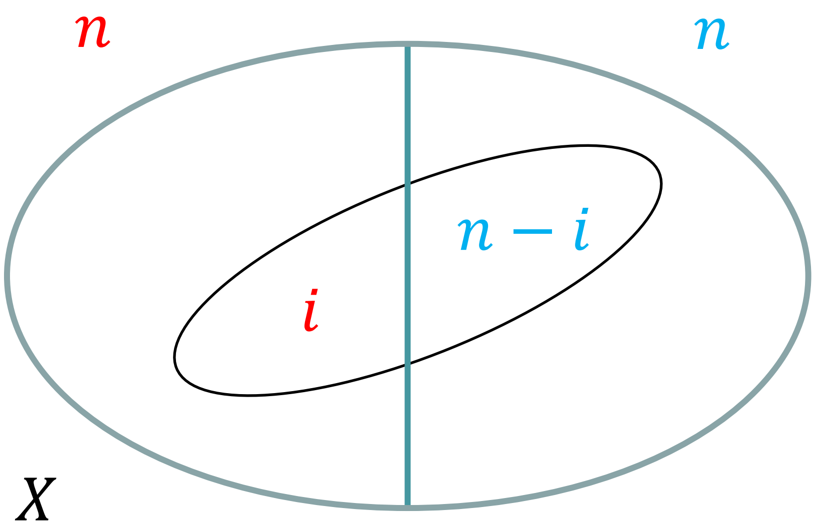
Exercise

$$\sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}$$

$$\sum_{k=0}^{n} \binom{m+k-1}{k} = \binom{n+m}{n}$$

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

- Proof: $\sum_{i=0}^{n} \binom{n}{i}^2 = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i}$



Vandermonde's identity/convolution

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

The general form

$$\binom{n_1 + \cdots + n_p}{m} = \sum_{k_1+\cdots+k_p=m} \binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_p}{k_p}$$

n balls are put into m bins

balls per bin	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins.	m^n	$(m)_n$	
n identical balls, m distinct bins.	$\binom{n+m-1}{m-1}$	$\binom{m}{n}$	$\binom{n-1}{m-1}$
n distinct balls, m identical bins.		$\begin{cases} 1 & n \leq m \\ 0 & n > m \end{cases}$	
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Multiset Coefficient

- The number of multisets of cardinality k, with elements taken from a finite set of cardinality n, is called the **multiset coefficient** or **multiset number**.
- $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{n-1} = \binom{n+k-1}{k}$

$$= \frac{n(n+1)(n+2)\cdots(n+k-1)}{k!} = \frac{n^{\overline{k}}}{k!}$$

n balls are put into m bins

balls per bin	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins.	m^n	$(m)_n$	
n identical balls, m distinct bins.	$\left(\!\binom{m}{n}\!\right)$	$\binom{m}{n}$	$\binom{n-1}{m-1}$
n distinct balls, m identical bins.		$\begin{cases} 1 & n \leq m \\ 0 & n > m \end{cases}$	
n identical balls, m identical bins.		$\begin{cases} 1 & n \leq m \\ 0 & n > m \end{cases}$	

- Basic counting
- Binomial theorem
- Generalized Binomial theorem
- Some special numbers

Binomial theorem

- ***Binomial Theorem*****:** for any non-negative integer n, we have

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

- Proof: Exercise
- Applications:
  - $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$ (take $x = 1$)
  - $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} \cdots = \sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0$
  - $2\left[\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots\right] = 2^n$

Pascal’s Triangle （1654）/ 杨辉三角（1261）

$$\binom{0}{0}$$

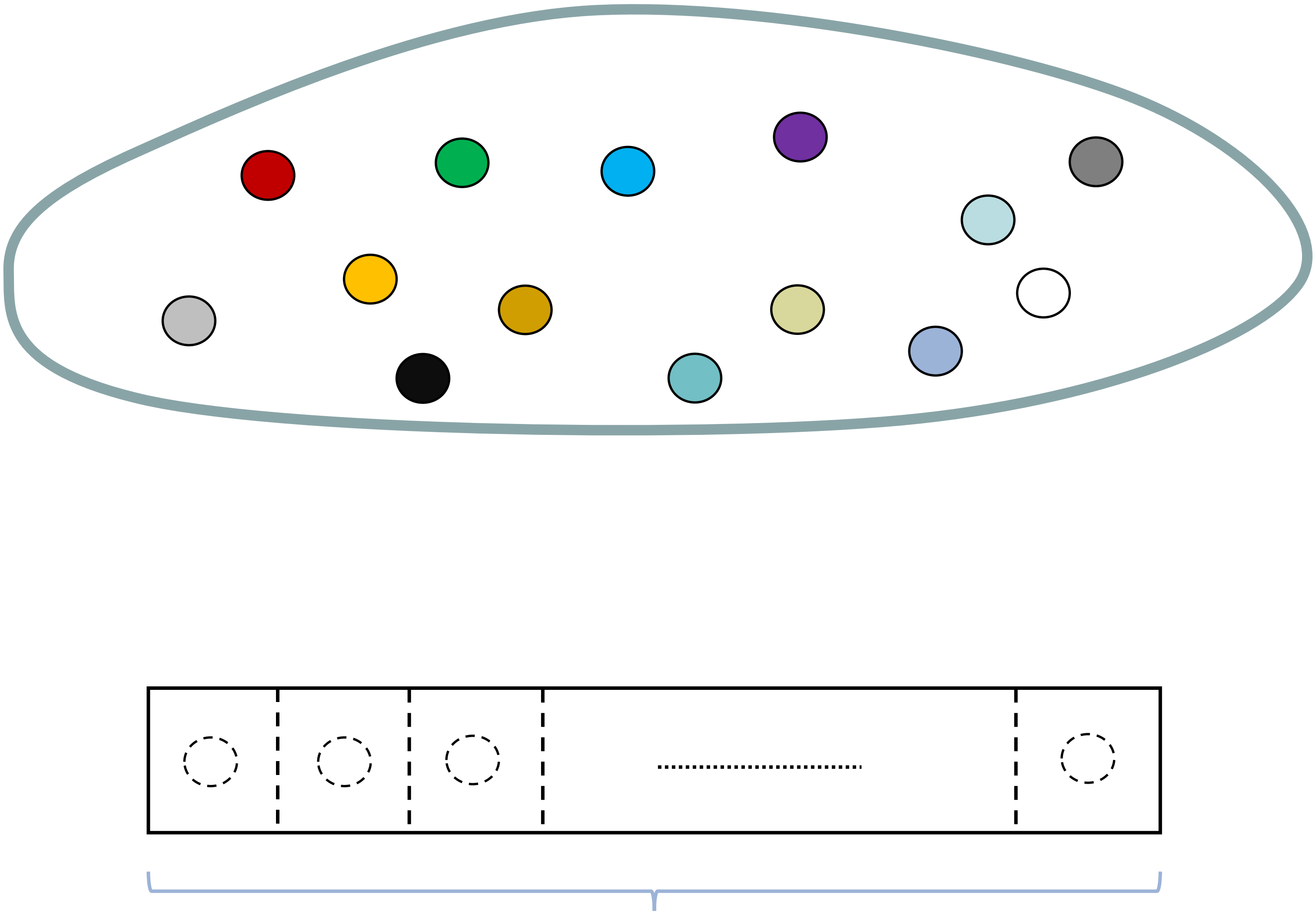
$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

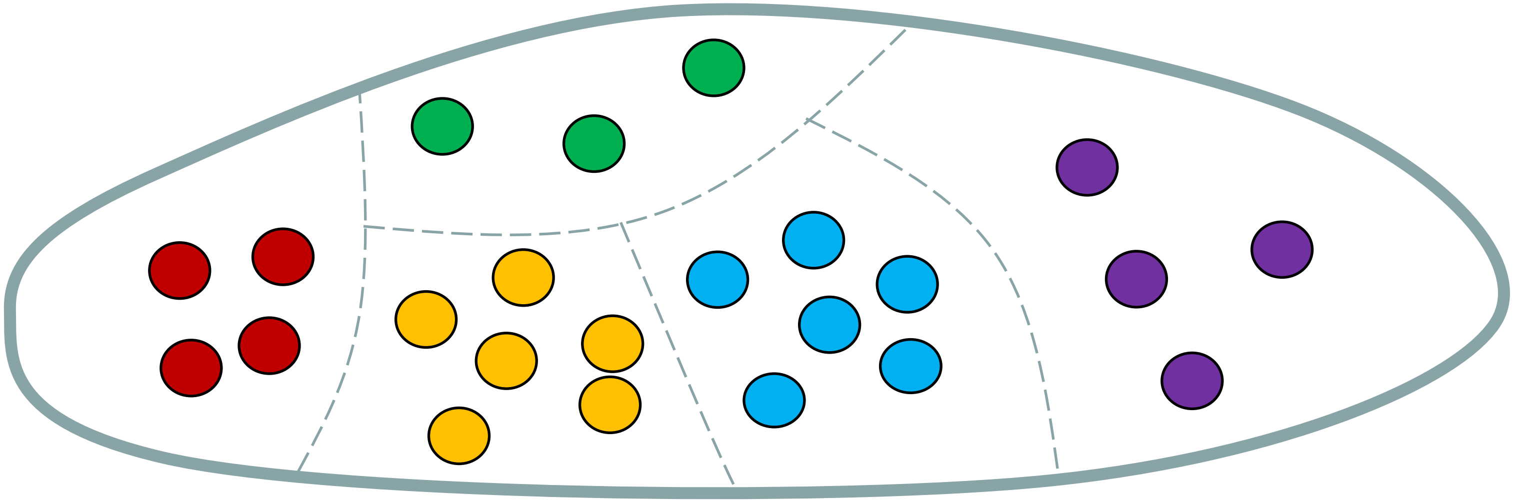
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

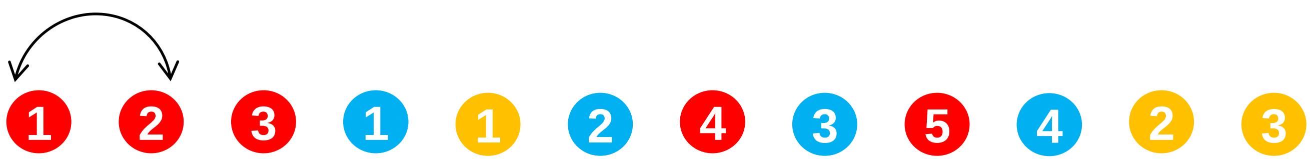


(Un-)Ordered sequence

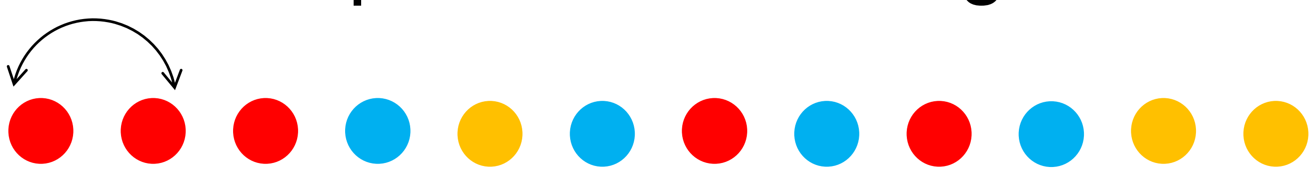


Ordered sequence

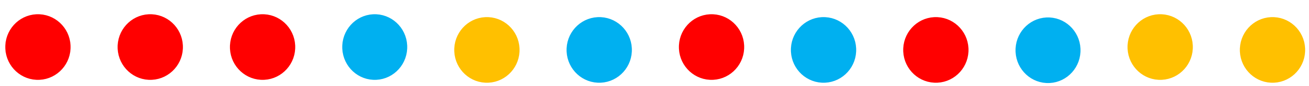
- With 5 different red balls, 3 different yellow balls, 4 different blue balls, we can get $(5+3+4)! = 12!$ different sequences.



- **Question:** With 5 equal red balls, 3 equal yellow balls, 4 equal blue balls, how many different sequences can we get?



- **Theorem：** if we have objects of m kinds, k_i indistinguishable objects of ith kind, where $k_1 + k_2 + \cdots + k_m = n$, then the number of distinct arrangements of the objects in a row is $\frac{n!}{k_1!k_2!\ldots k_m!}$. Usually written $\binom{n}{k_1, k_2, \ldots, k_m}$。



$\frac{12!}{5!3!4!}$ 种

- ***Multinomial Theorem*:** For arbitrary real number $x_1, x_2, \dots, x_m$ and any natural number $n \geq 1$, the following equality holds:

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{\substack{k_1 + \cdots k_m = n \\ k_1, \cdots, k_m \geq 0}} \binom{n}{k_1, k_2, \dots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}.$$

- e. g. In $(x + y + z)^{10}$ the coefficient of $x^2 y^3 z^5$ is $\binom{10}{2,3,5} = 2520$.

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- Some special numbers

Newton(1665)'s generalized binomial theorem

Let $\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!} = \frac{(r)_k}{k!}$ where r is arbitrary, $k > 0$ is an integer

If x and y are real numbers with $|x| > |y|$

$$(x+y)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^{r-k} y^k$$

$$= x^r + r x^{r-1} y + \frac{r(r-1)}{2!} x^{r-2} y^2 + \frac{r(r-1)(r-2)}{3!} x^{r-3} y^3 + \cdots$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \cdots$$

$$(1+x)^{-1} = \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

Generally: $r = -s$

$$\frac{1}{(1-x)^s} = \sum_{k=0}^{\infty} \binom{s+k-1}{k} x^k$$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4 - \frac{63}{256}x^5 + \cdots$$

- Basic counting
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- Some special numbers

Stirling subset numbers

- The second Stirling Numbers $\left\{ {n \atop k} \right\}$: The number of ways to partition a set of n things into k nonempty subsets.
- e.g. $\left\{ {4 \atop 2} \right\} = 7$

$N = \{1, 2, 3, 4\}$

$\{1, 2\}\ \{3, 4\}$, $\{1\}\ \{2, 3, 4\}$,
$\{1, 3\}\ \{2, 4\}$, $\{2\}\ \{1, 3, 4\}$,
$\{1, 4\}\ \{2, 3\}$, $\{3\}\ \{1, 2, 4\}$;
$\{4\}\ \{1, 2, 3\}$.

Stirling subset numbers

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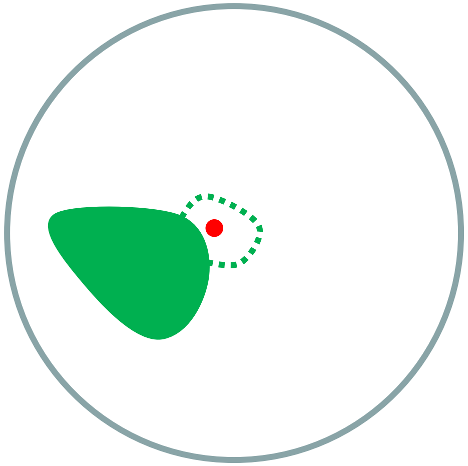
- e.g. $\left\{ {4 \atop 2} \right\} = 7$
- $\left\{ {n \atop 2} \right\} = 2^{n-1} - 1$ why?



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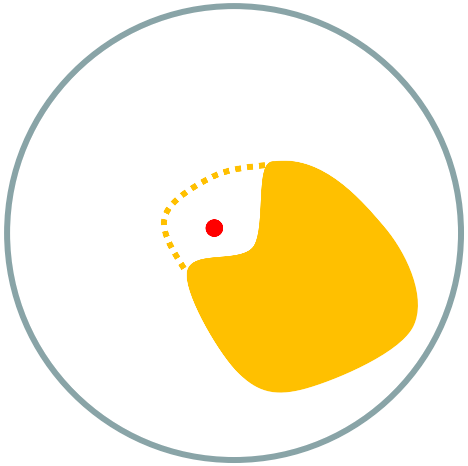

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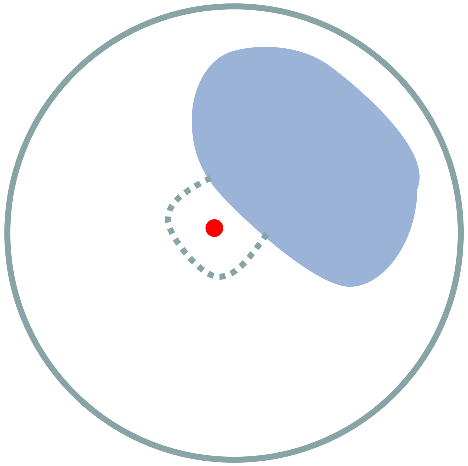
- $\left\{ {n \atop 2} \right\} = 2^{n-1} - 1$ why?



Stirling subset numbers

- The second Stirling Numbers $\left\{ {n \atop k} \right\}$: The number of ways to partition a set of n things into k nonempty subsets.
- e.g. $\left\{ {4 \atop 2} \right\} = 7$
- $\left\{ {n \atop k} \right\} = k \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}$

Stirling cycle numbers

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- $\left[{n \atop k}\right] \geq \left\{{n \atop k}\right\}$,

Stirling cycle numbers

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- $\begin{bmatrix} n \\ k \end{bmatrix} \geq \begin{Bmatrix} n \\ k \end{Bmatrix}$, e.g. $\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 11$
- $\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!$
- $\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} = n!$ where $n \in Z^+$.
- $\begin{bmatrix} n \\ k \end{bmatrix} = (n-1) \cdot \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ Why?

- $\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} = n!$ where $n \in Z^{+}$.

n balls are put into m bins

balls per bin	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins.	m^n	$(m)_n$	$m!\left\{ {n \atop m} \right\}$
n identical balls, m distinct bins.	$\binom{n+m-1}{m-1}$	$\binom{m}{n}$	$\binom{n-1}{m-1}$
n distinct balls, m identical bins.	$\sum_{k=1}^{m} \left\{ {n \atop k} \right\}$	$\begin{cases} 1 & n \leq m \\ 0 & n > m \end{cases}$	$\left\{ {n \atop m} \right\}$
n identical balls, m identical bins.		$\begin{cases} 1 & n \leq m \\ 0 & n > m \end{cases}$	

Partition of a number

- $P_k(n)$： number of partition the positive integer n into k parts.
- e.g. $P_2(7) = 3 \quad \{\{1,6\},\{2,5\},\{3,4\}\}$
  $P_6(7) = 1 \quad \{\{1,1,1,1,1,2\}\}$
- Number of integral solutions to

$$\begin{cases} x_1 + x_2 + \cdots + x_k = n \\ x_1 \geq x_2 \geq \cdots \geq x_k \geq 1 \end{cases}$$

- $P_k(n) = P_{k-1}(n-1) + P_k(n-k)$ why?

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n identical balls, m identical bins.	$\sum_{k=1}^{m} p_k(n)$	$\begin{cases} 1 & n \le m \\ 0 & n > m \end{cases}$	$p_m(n)$

Partition of a number

- $P_k(n)$： number of partition the positive integer n into k parts.

- $\sum_{k=1}^{m} p_k(n) = p_m(n+m)$ why?

Twelvefold way

The twelve combinatorial objects and their enumeration formulas.

***f*-class**	**Any *f***	**Injective *f***	**Surjective *f***
f	*n*-sequence in *X* x^n	*n*-permutation in *X* $x^{\underline{n}}$	composition of *N* with *x* subsets $x!\{{n \atop x}\}$
***f* ∘ S$_n$**	*n*-multisubset of *X* $\binom{x+n-1}{n}$	*n*-subset of *X* $\binom{x}{n}$	composition of *n* with *x* terms $\binom{n-1}{n-x}$
S$_x$ ∘ *f*	partition of *N* into ≤ *x* subsets $\sum_{k=0}^{x} \left\{ {n \atop k} \right\}$	partition of *N* into ≤ *x* elements $[n \le x]$	partition of *N* into *x* subsets $\left\{ {n \atop x} \right\}$
S$_x$ ∘ *f* ∘ S$_n$	partition of *n* into *x* non-negative parts $p_x(n+x)$	partition of *n* into ≤ *x* parts 1 $[n \le x]$	partition of *n* into *x* parts $p_x(n)$

https://en.wikipedia.org/wiki/Twelvefold_way