The number of spanning trees

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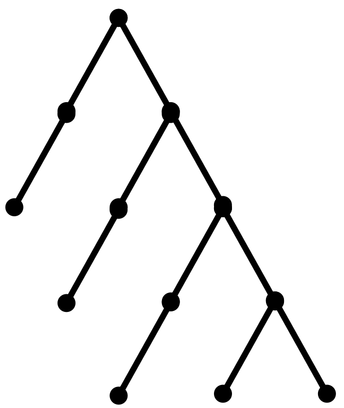
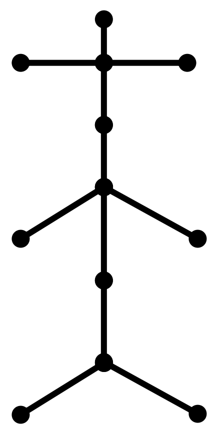
树的刻画

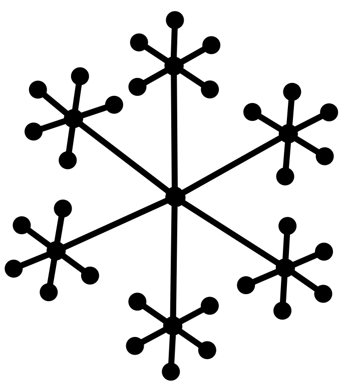
- 树(Tree)：连通无环图。
- 树的例子：



T_1

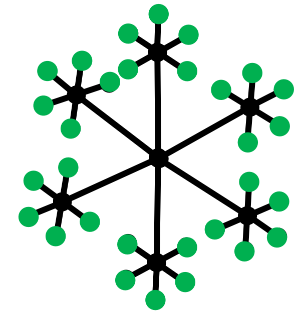


T_2

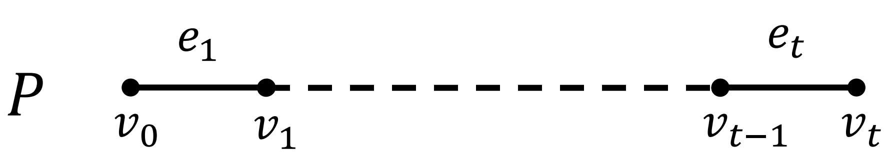


T_3

叶子(leaf)



- 叶子(leaf)：图G中度数为1的顶点被称为叶子或终点(end-vertex)。
- 引理：对任意树T，如果$|T| \geq 2$，则T必含有至少两个终点。
- 证明：取T中的一条*极长*路径P



$$\deg_T(v_0) = \deg_T(v_t) = 1$$

树的基本性质

- 树生长引理(Tree-growing lemma)：对图G及图G上的叶子结点v而言，如下命题等价
  I. 图G是树。
  II. 图$G - v$是树。

- 证明：

树生长引理的意义：在归纳证明中的应用。

树的等价刻画

- 对图$G = (V, E)$而言，以下陈述等价
  I. 图G是树。
  II. 路径唯一：对任意两点$u, v \in V$，存在从u到v的唯一路径。
  III. 最小连通图：G是连通图，且去掉任意一条边后都成为非连通图。
  IV. 最大无环图：G不含环，但增加任何一条边所得到的图$G + e$（其中$e \in \binom{V}{2} \setminus \mathrm{E}$）中含有一个环。
  V. Euler方程：G是连通图，且$|V| = |E| + 1$。

树的等价刻画

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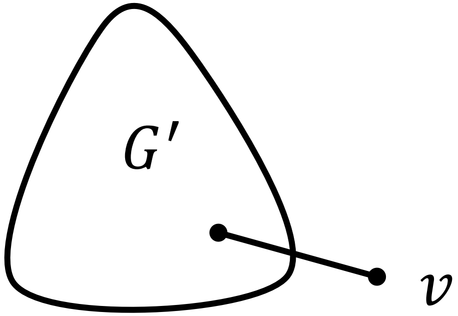
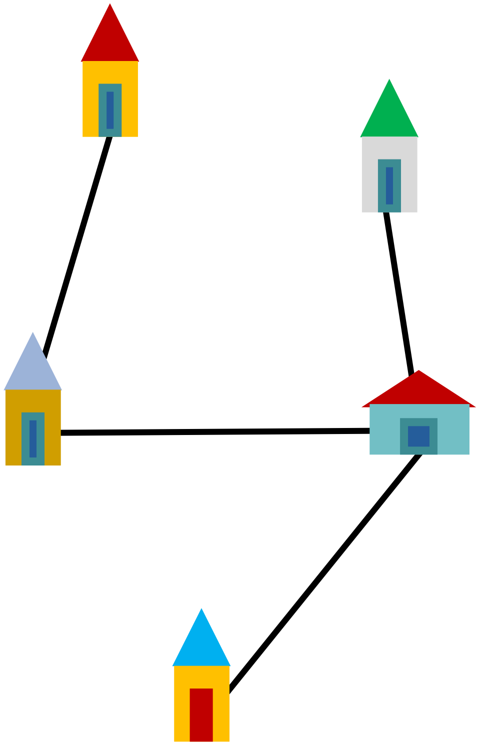
树的等价刻画

- 对图$G = (V, E)$而言，以下陈述等价
  I. 图G是树。
  V. Euler方程：G是连通图，且$|V| = |E| + 1$。
- 证明：(I.⇔V.)
  - 充分性：归纳法（用树生长引理，对顶点个数做归纳）。
  - 必要性：（归纳法）考虑连通图G满足$|V| = |E| + 1 \geq 2$。
    - 由握手定理，图G中顶点度数之和为$2|V| - 2$。故图G中必存在度数小于2的顶点，且图G是连通图，任何点度数非0，故存在度数为1的点，设为v。
    - 考虑$G' = G - v$。易验证归纳假设条件成立，根据归纳假设G'是树。
    - 显然，G'是树蕴含G是树。

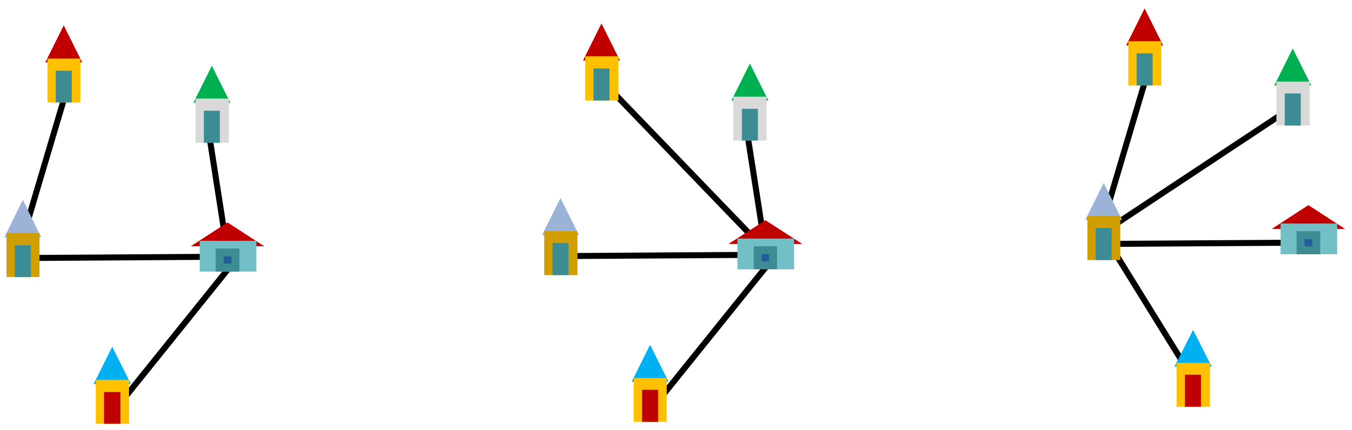




- **问题**：某小镇一共有n座房子，某天小镇上的人计划在房子下面修建紧急逃生地道，使得所有房子从地下最后是连通的，同时出于安全考虑地道图中不允许有环。问有多少种挖掘地道的方案？

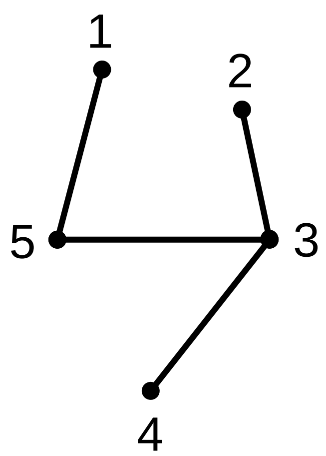
树的计数

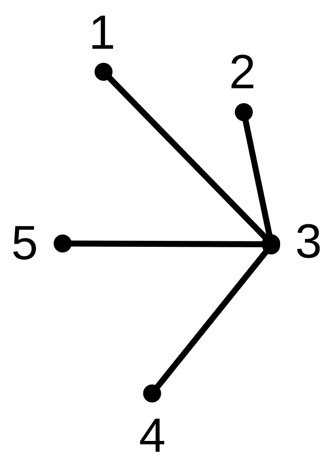
- 问题抽象：n个不同顶点所能构成的树的个数。

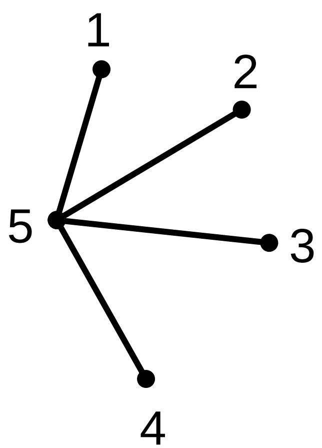


树的计数

- 问题抽象：n个不同顶点所能构成的树的个数。
- 两棵树T, T'是“相等”的当且仅当树T的边集与树T'的边集相等。





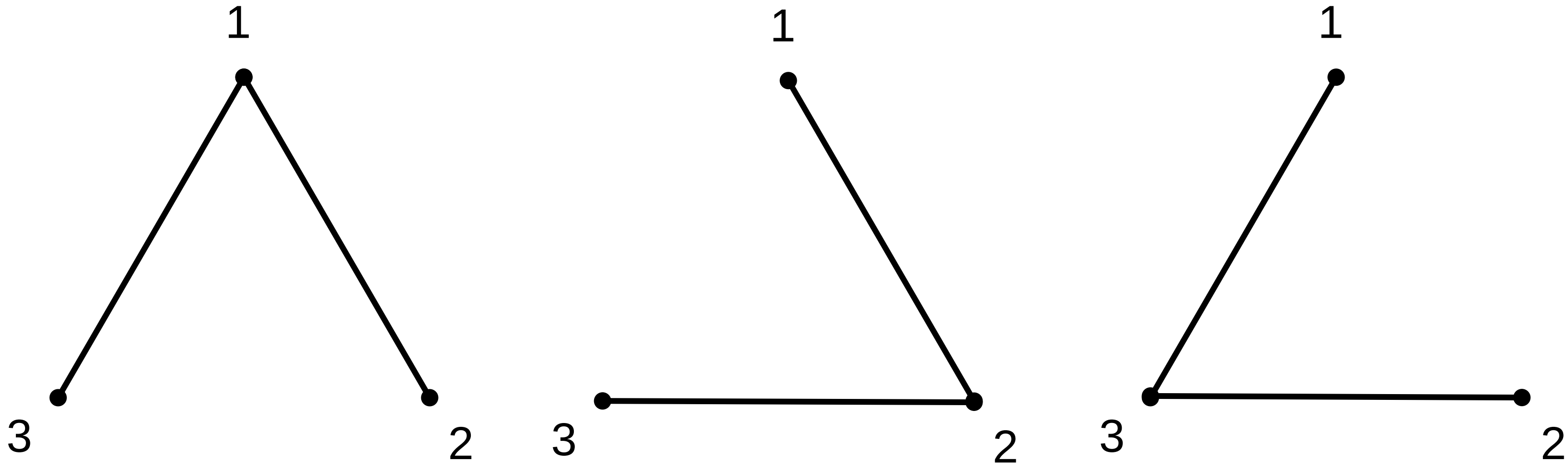


生成树

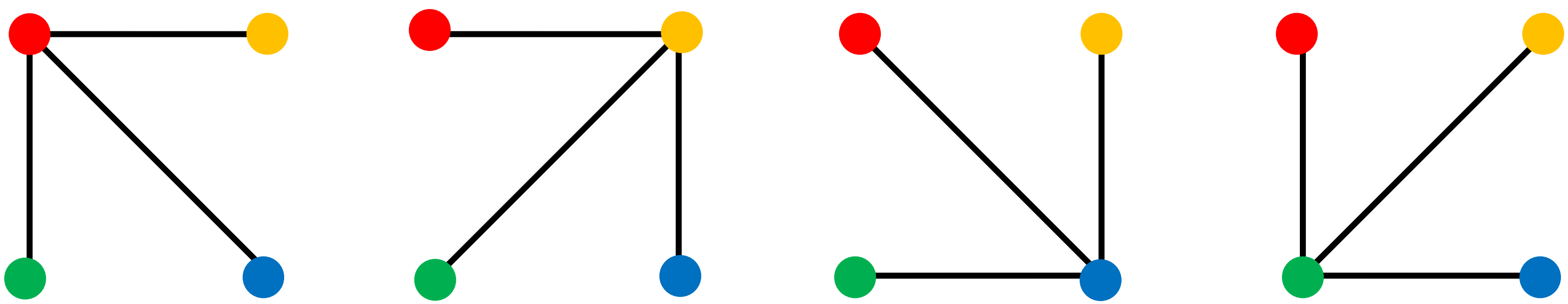
- **生成树(Spanning tree)**：对连通图$G = (V, E)$，生成树是包含G的所有顶点且为树的子图。
- 上述问题最终抽象为：设$V = \{1,2,\ldots,n\}$，$n \geq 2$，问K_n的生成树一共有多少种？

- $n = 2$： 1种

- $n = 3$： 3种

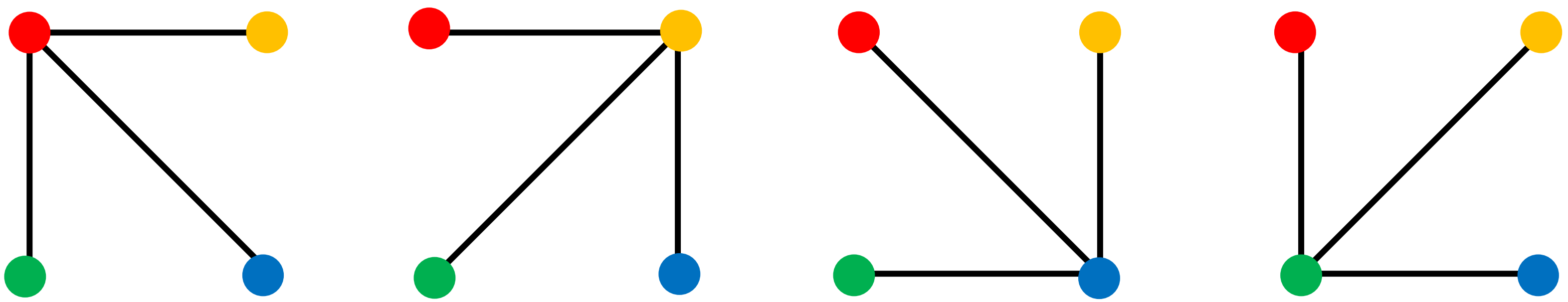


- $n = 4$：?种
  - 星形：4种



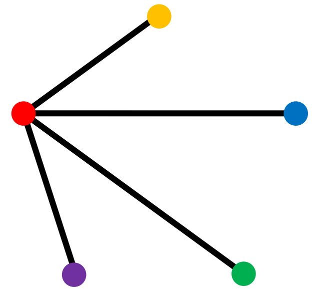
  - 路径：$\frac{4!}{2} = 12$ 种

- $n = 4$： 16种
  - 星形：4种

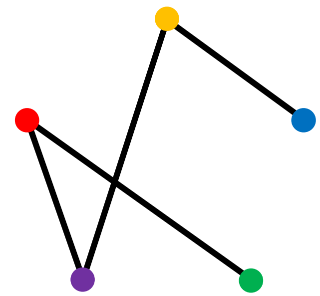


  - 路径：$\frac{4!}{2} = 12$ 种

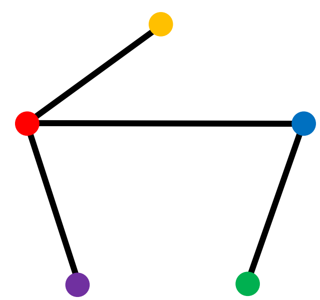
- $n = 5$：?种
  - 星形： 5 种



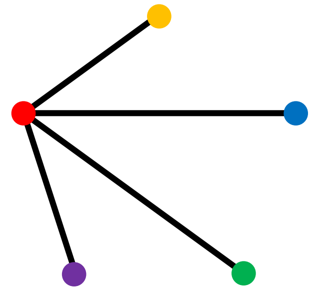
  - 路径： $\frac{5!}{2} = 60$ 种



  - T形：$5 \cdot 4 \cdot 3 = 60$种



- $n = 5$： 125种
  - 星形： 5 种



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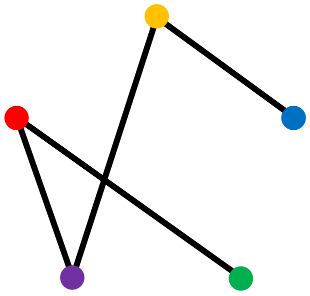

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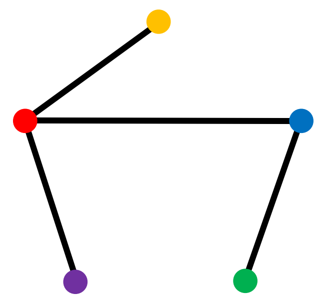

- $n = 6$：?种

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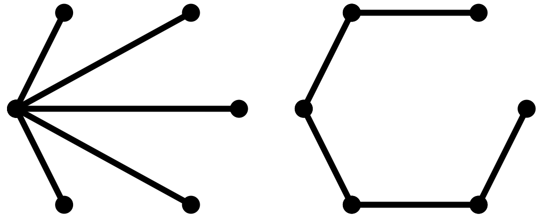
  - 星形：6种，路径：$\frac{6!}{2} = 360$ 种



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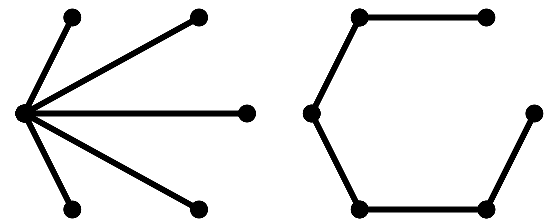
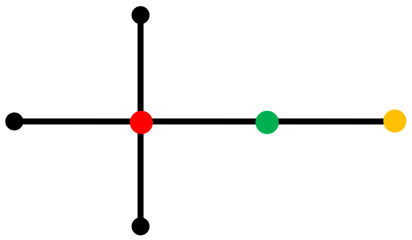


  - 十字架形：$6 \times 5 \times 4 = 120$ 种



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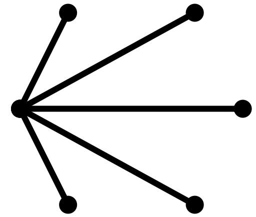
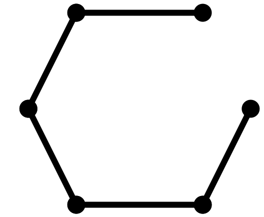

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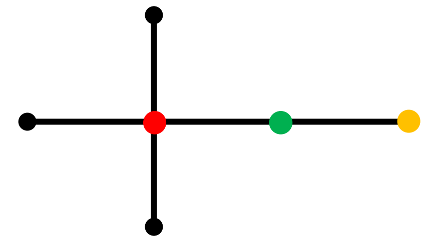
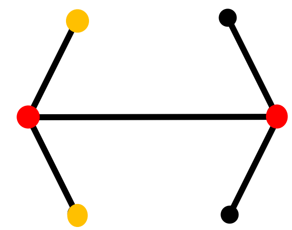


  - 双箭头形：$\binom{6}{2} \times \binom{4}{2} = 90$ 种



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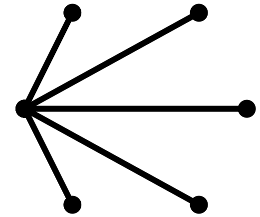

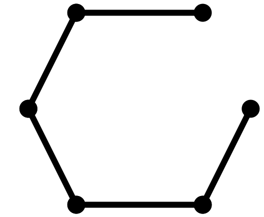

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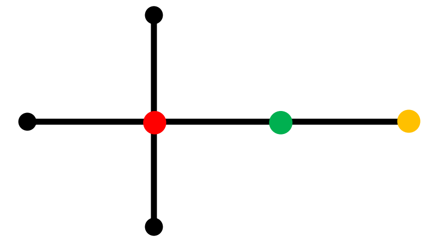

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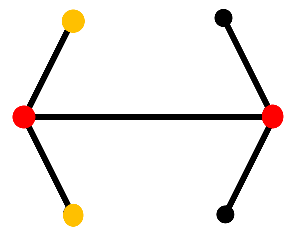

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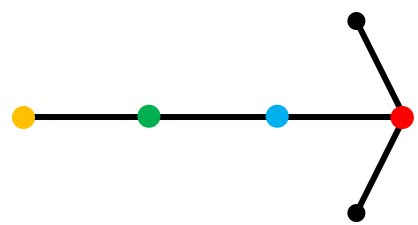

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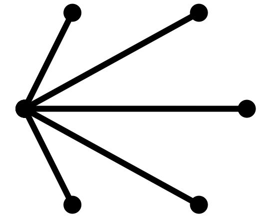

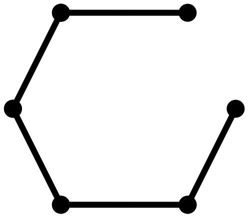

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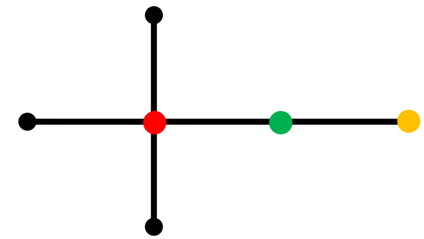

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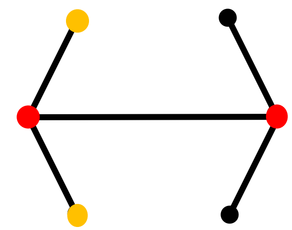

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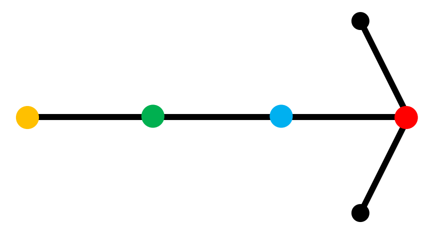
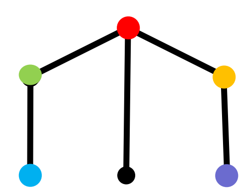
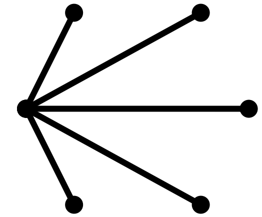
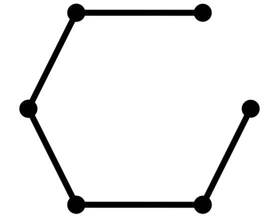


  - 雨棚形：$\frac{6\times5\times4\times3\times2}{2} = 360$ 种



- $n = 6$：1296种
  - 星形：6种，路径：$\frac{6!}{2} = 360$ 种





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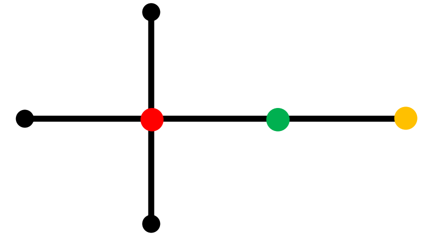

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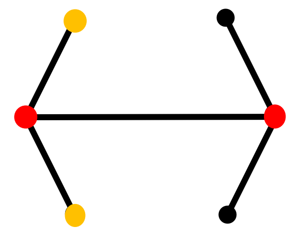

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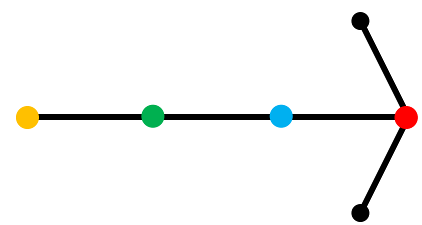
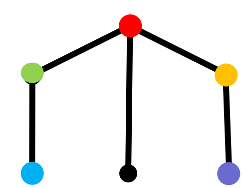


  - 雨棚形：$\frac{6\times5\times4\times3\times2}{2} = 360$ 种



含$\boldsymbol{n}$个顶点的树

顶点个数	树的种数
1	1
2	1
3	3
4	16
5	125
6	1296

含n个顶点的树

顶点个数	树的种数
1	1$= 1^{-1}$
2	1$= 2^0$
3	3 $= 3^1$
4	16 $= 4^2$
5	125 $= 5^3$
6	1296 $= 6^4$

- Caley 定理(Caley's formula)： n个顶点能构成的不同树一共有n^{n-2}种。

A proof via score

Proposition. Let $d_1, d_2, \ldots, d_n$ be positive integers summing up to $2n - 2$. Then the number of spanning trees of the graph K_n in which the vertex i has degree exactly d_i for all $i = 1,2,\ldots,n$ equals

$$\frac{(n-2)!}{(d_1-1)!\,(d_2-1)!\cdots(d_n-1)!}.$$

Proof. (By induction on n) T: the set of STs of K_n with given degrees.

- $n = 1,2$, the proposition holds trivially.
- $n > 2$: there must exist an i with $d_i = 1$. w.l.o.g. $d_n = 1$.
- For $1 \le i \le n-1$, $T_i \subseteq T$, where T_i is the STs with $\{i, n\} \in E$
- Delete v_n from each tree in T_i to get T_i': STs of K_{n-1} with degrees $d_1, d_2, \ldots d_{i-1}, d_i - 1, d_{i+1}, \ldots, d_{n-1}$.

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$$\frac{(n-2)!}{(d_1-1)!\,(d_2-1)!\cdots(d_n-1)!}.$$

Proof. (Continue)

- $$|T_i| = |T_i'| = \frac{(n-3)!}{(d_1-1)!\cdots(d_{i-1}-1)!(d_i-2)!(d_{i+1}-1)!\cdots(d_{n-1}-1)!}$$

$$= \frac{(n-3)!(d_i-1)}{(d_1-1)!(d_2-1)!\cdots(d_{n-1}-1)!}$$

$$|T| = \sum_{i=1}^{n} |T_i| = \sum_{i=1}^{n-1} \frac{(n-3)!\,(d_i-1)}{(d_1-1)!\,(d_2-1)!\cdots(d_{n-1}-1)!} = \cdots$$

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$$\frac{(n-2)!}{(d_1-1)!\,(d_2-1)!\cdots(d_n-1)!}.$$

Finally

$$|T(K_n)| = \sum_{\substack{d_1,d_2,\dots,d_n \geq 1 \\ d_1+d_2+\cdots+d_n=2n-2}} \frac{(n-2)!}{(d_1-1)!\,(d_2-1)!\cdots(d_n-1)!}$$

$$= \sum_{\substack{k_1+k_2+\cdots+k_n=n-2 \\ k_1,\dots,k_n \geq 0}} \frac{(n-2)!}{k_1!\,k_2!\cdots k_n!}$$

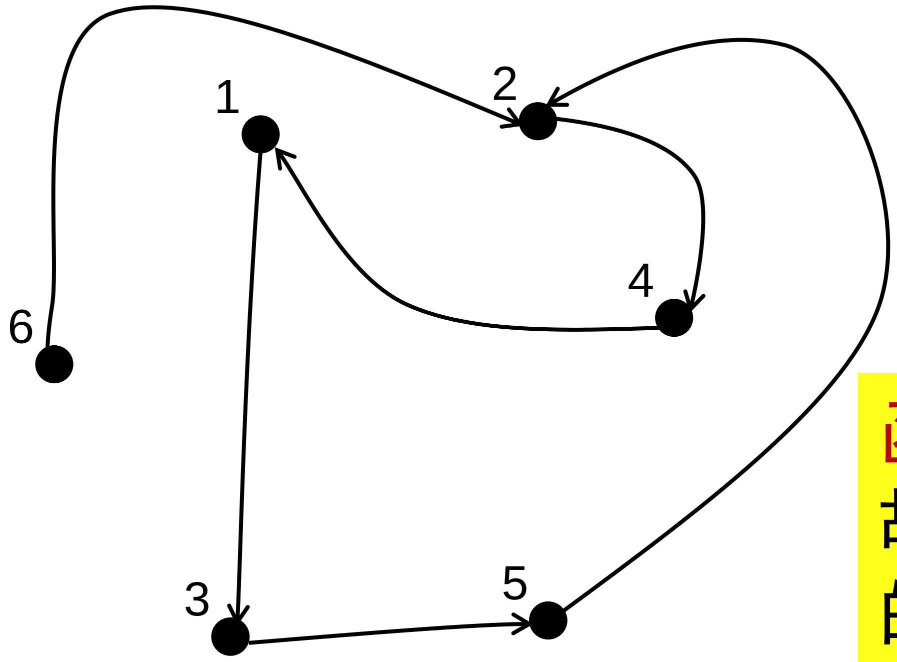
$$= \underbrace{(1+1+\cdots+1)^{n-2}} = n^{n-2}.$$

A proof with Vertebrates

有限域上的函数与函数图

- Function graph
  $f: V \to V$

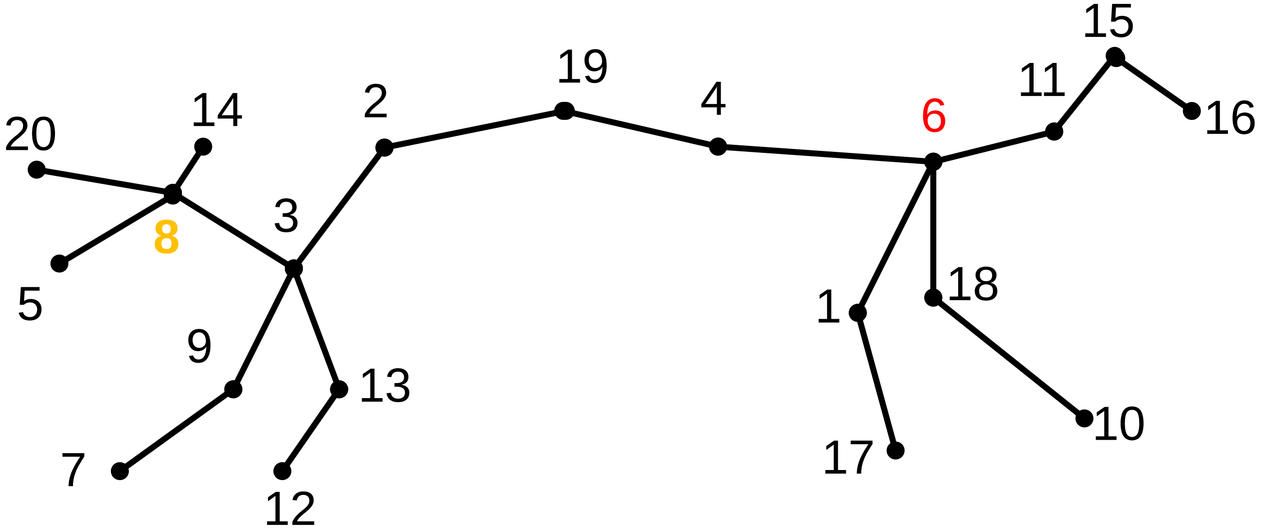
v	**1**	**2**	**3**	**4**	**5**	**6**
$f(v)$	3	4	5	1	2	2

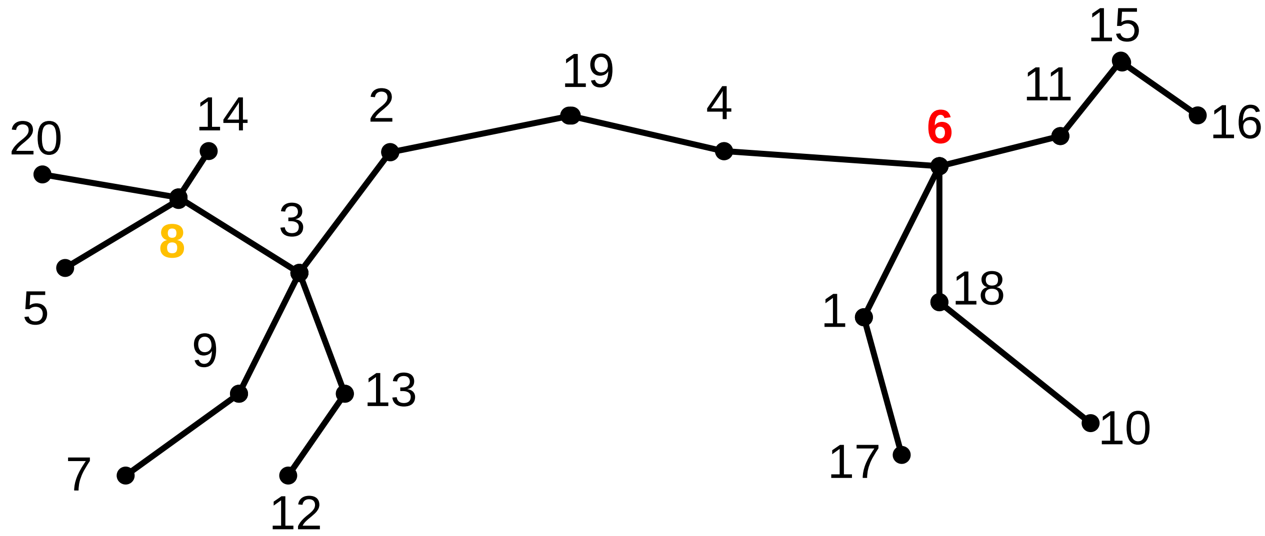


函数图与函数一一对应。故$|V| = n$时共有n^n种不同的函数图。

脊椎动物(Vertebrate)骨骼标本

- 骨骼标本：三元组(T, h, b)被称为骨骼标本若其中(1) T是一棵树；(2) $h, b \in V$。 h被称为颈椎骨， b被称为尾椎骨。
- 注意：h, b必须是树上的节点，除此外没有任何限制（可重合）。



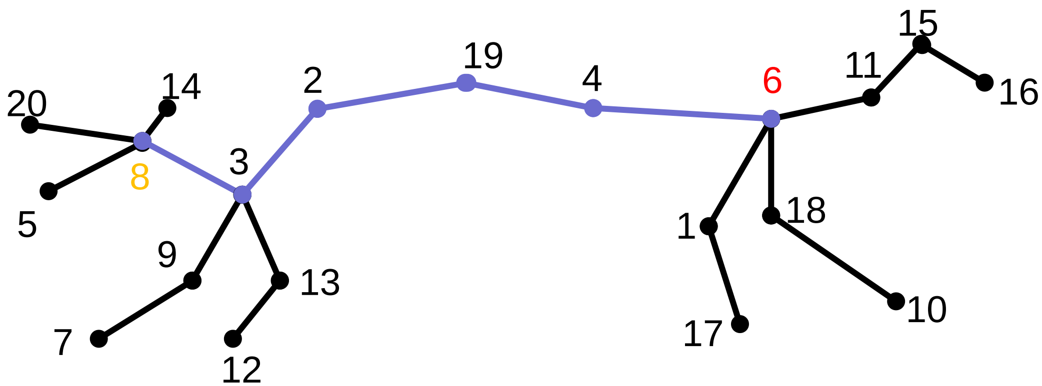


① 如果$|V| = n$，用T_n表示V上的树的所有可能棵数。

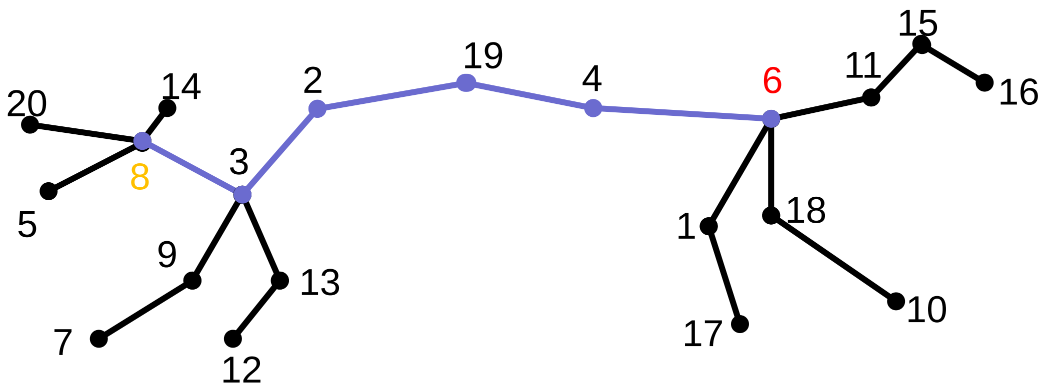
② 每一棵树T对应n^2种骨骼标本(T, h, b)。

③ 骨骼标本与V上的函数图一一对应。有n^n种。

④ 根据②③：　$T_n = \frac{n^n}{n^2} = n^{n-2}$



- 脊椎(Spine)：出现在从颈椎骨到尾椎骨的路径上的点被称为脊椎。



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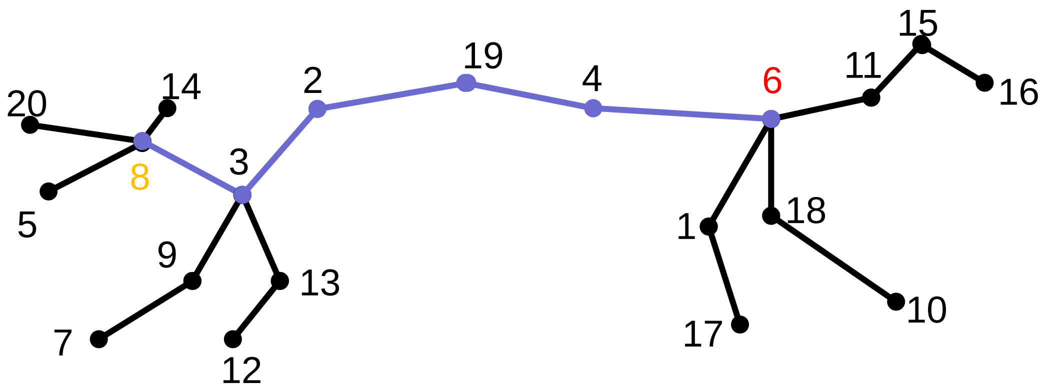
$\boldsymbol{v}$						
$f(v)$	8	3	2	19	4	6



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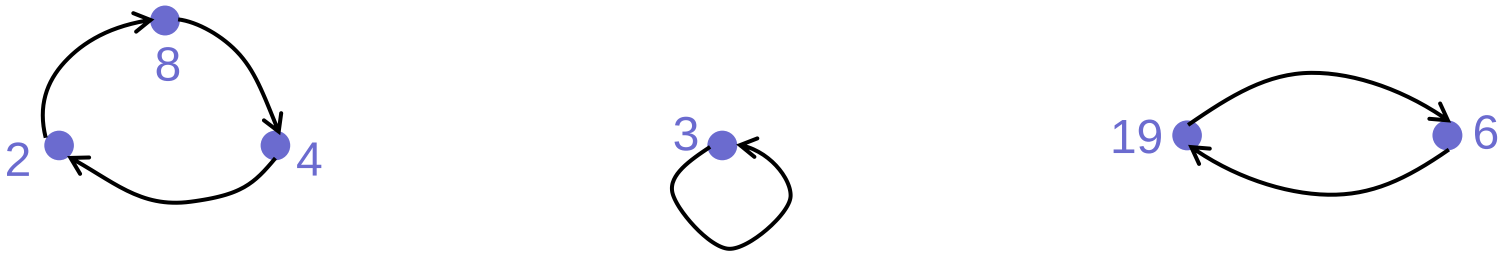

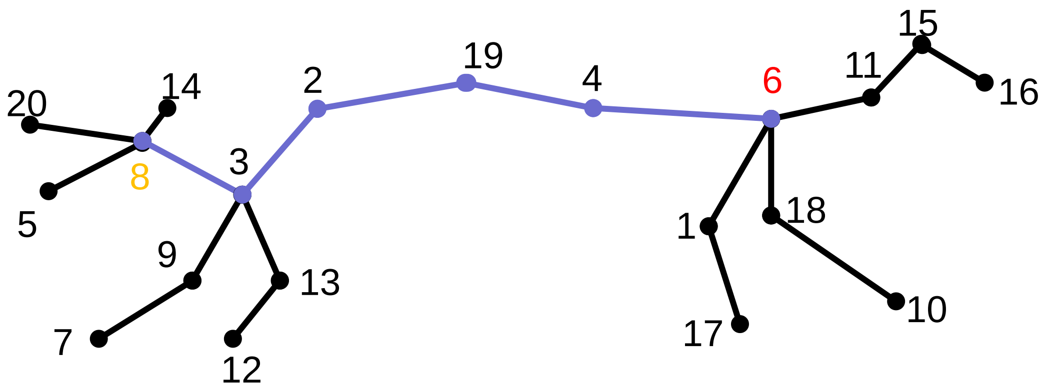

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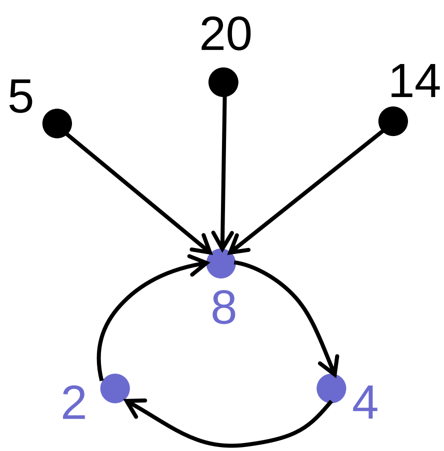
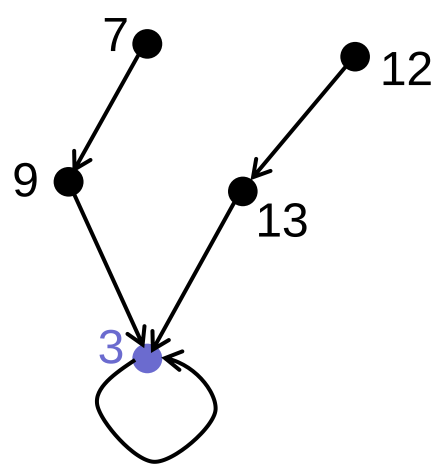
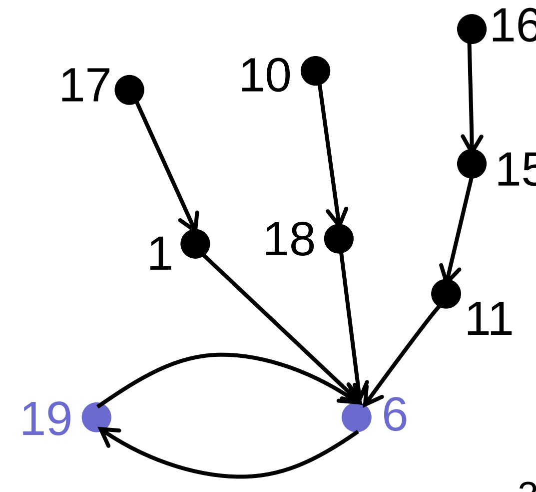


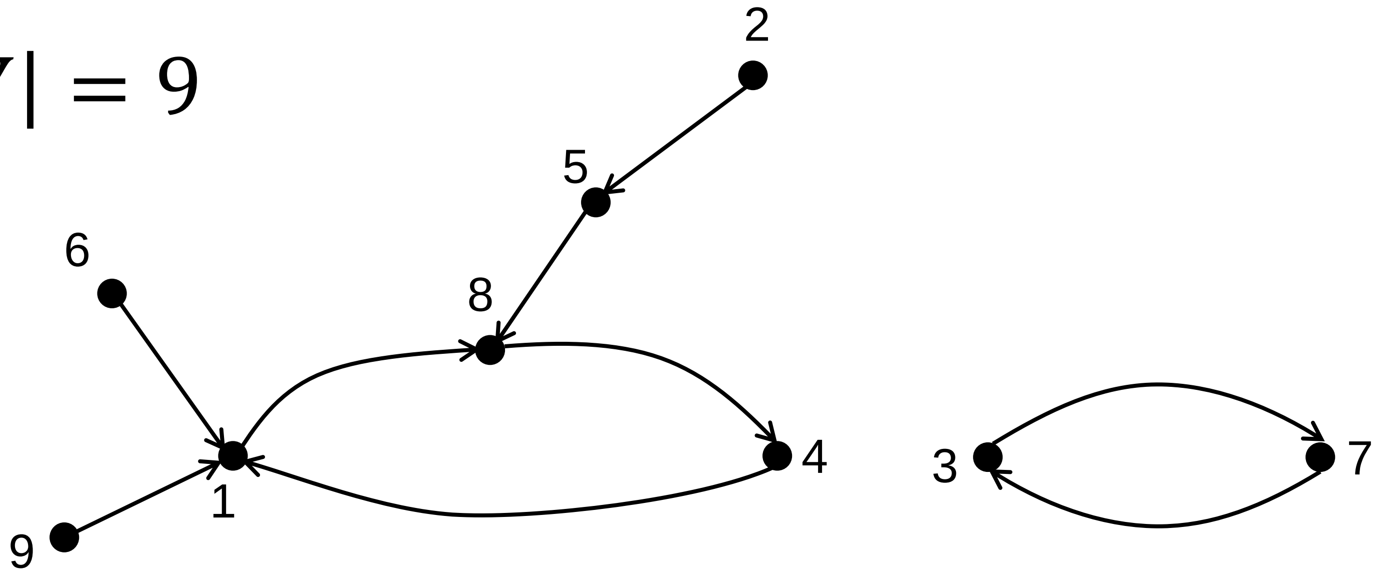




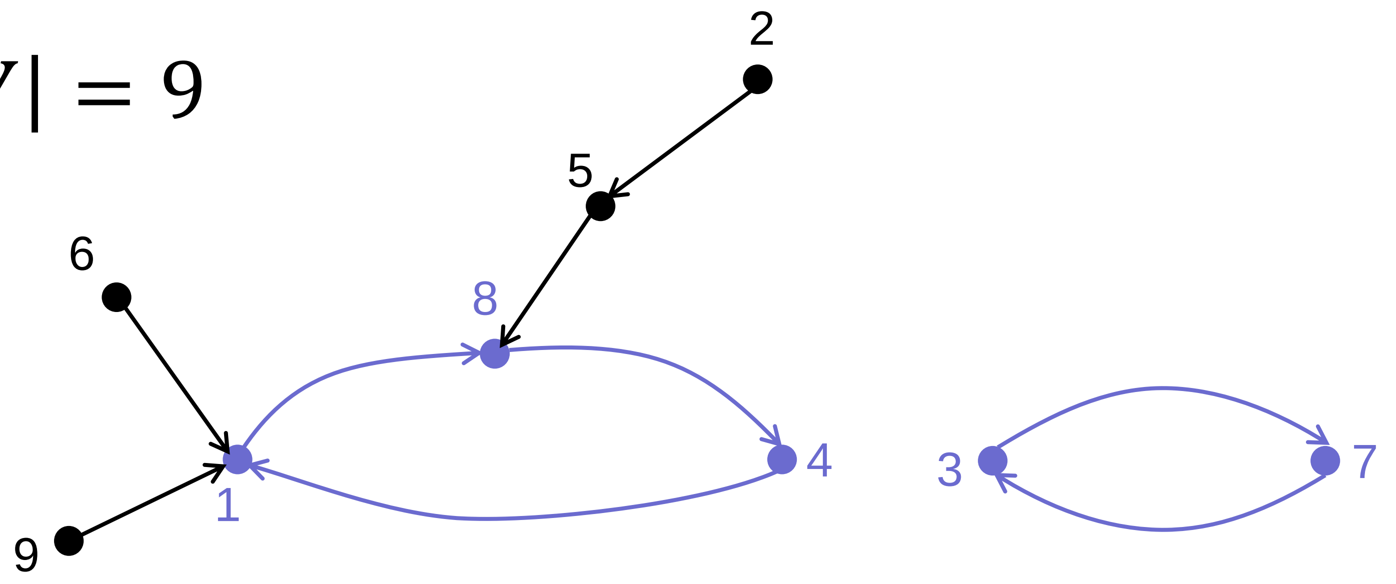
骨骼标本与V上的函数图一一对应

- V上的骨骼标本 ⟶ 函数 $f: V \to V$
- 函数 $f: V \to V$ ⟶ V上的骨骼标本

$$|V| = 9$$

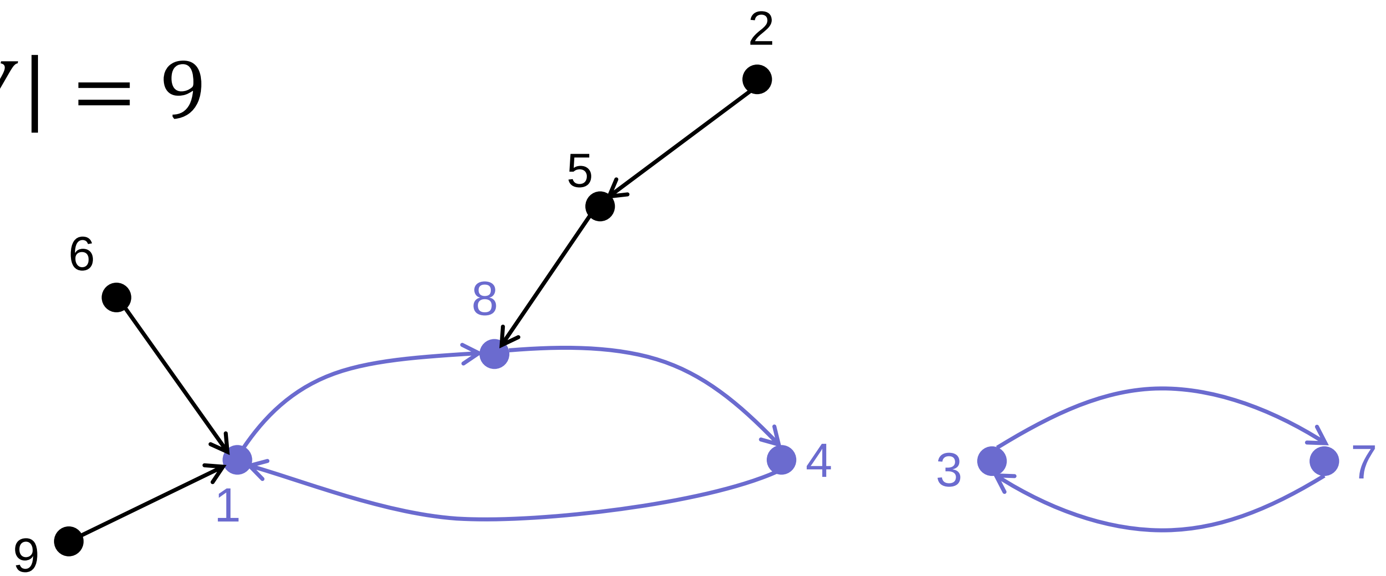


$|V| = 9$



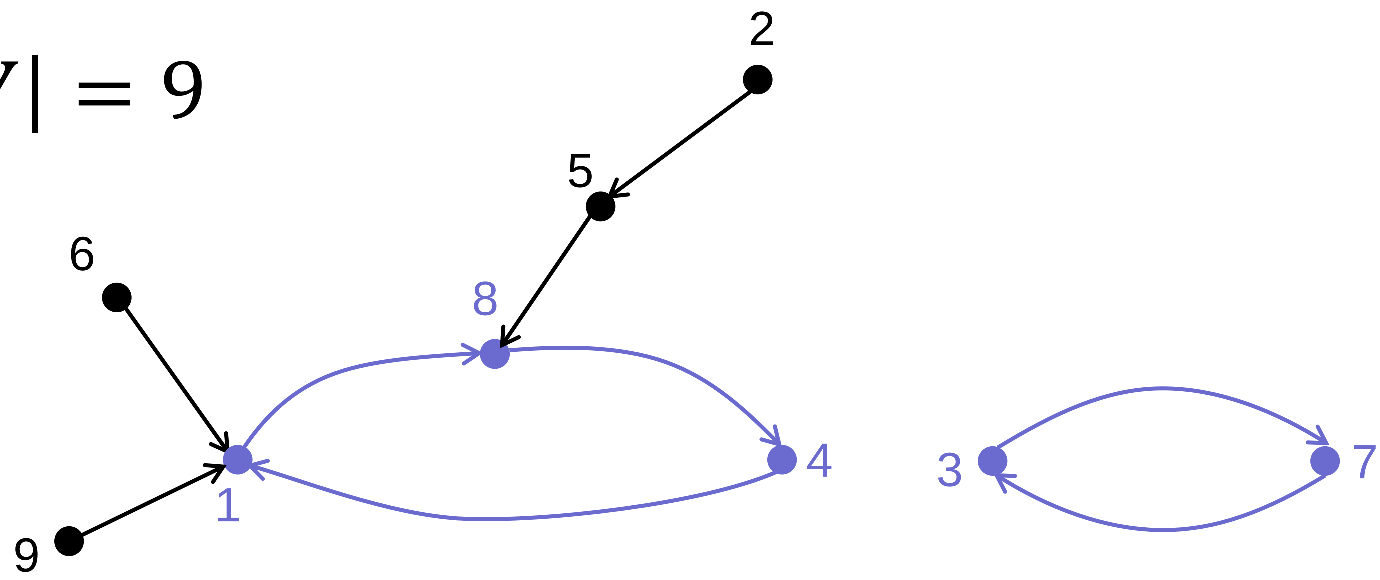
$\boldsymbol{v}$	1	3	4	7	8
$f(v)$					

$$|V| = 9$$



$\boldsymbol{v}$	1	3	4	7	8
$f(v)$	8	7	1	3	4

$$|V| = 9$$



v	1	3	4	7	8
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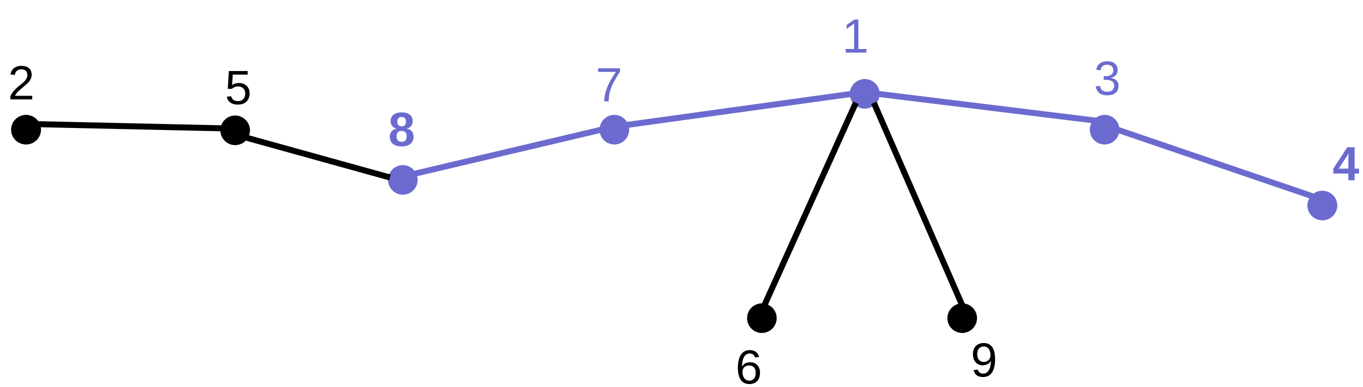
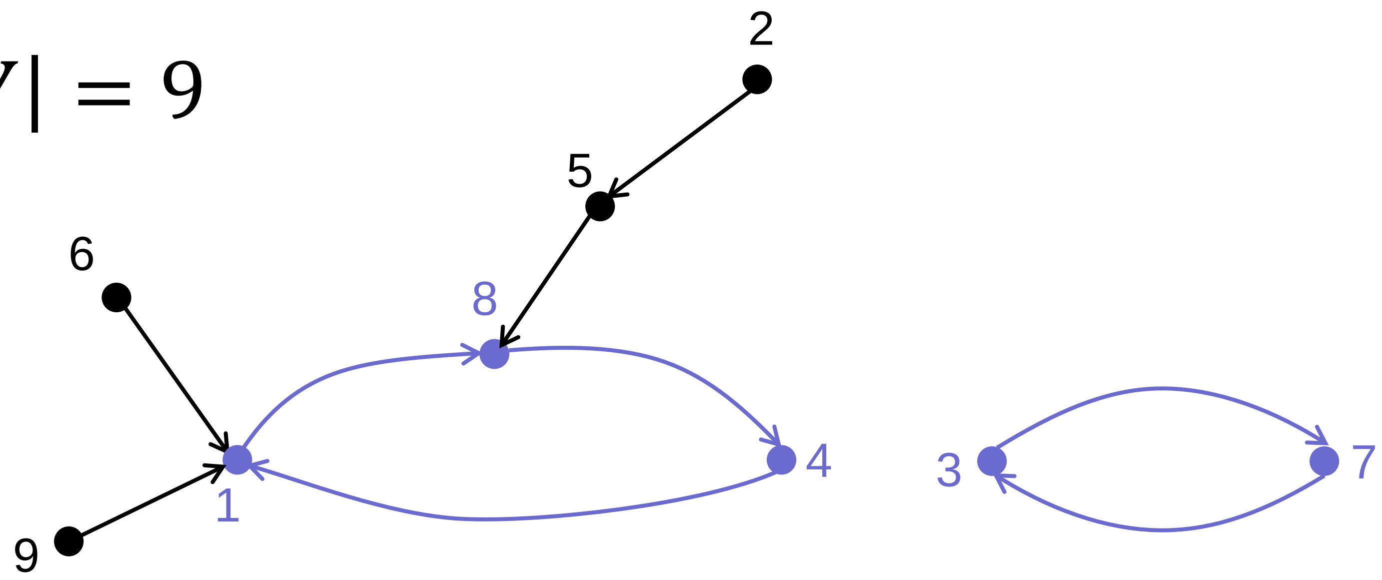


$$|V| = 9$$



v	1	3	4	7	8
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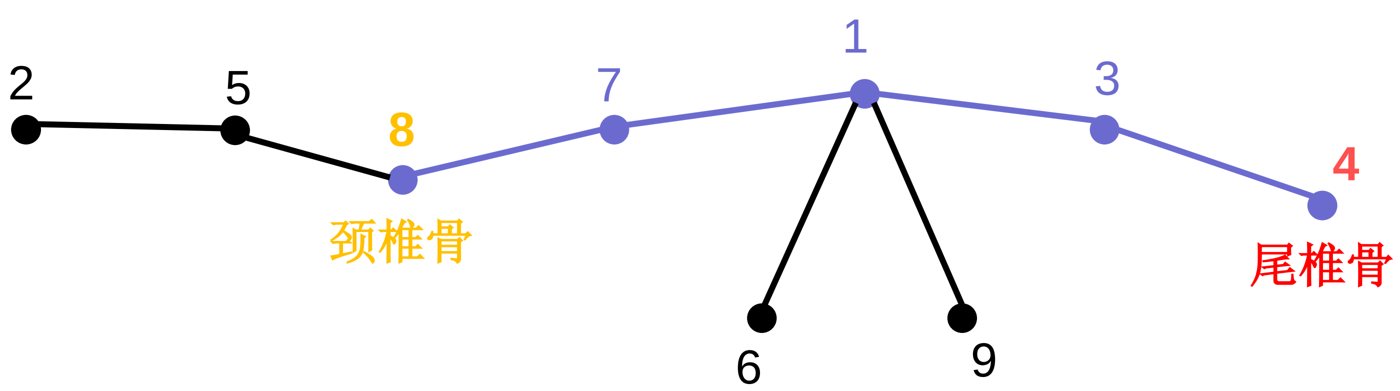

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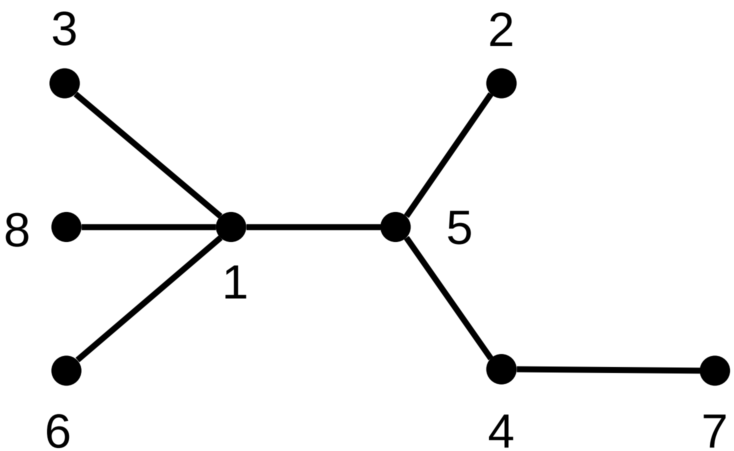
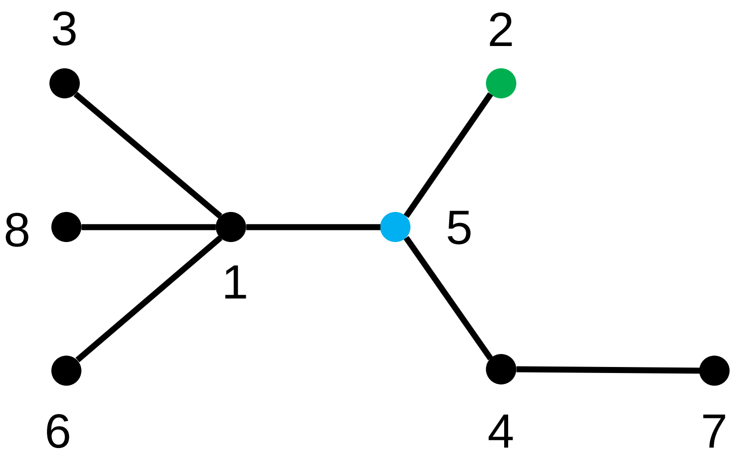
④ 根据②③： $$T_n = \frac{n^n}{n^2} = n^{n-2}$$

A Proof using the Prüfer code

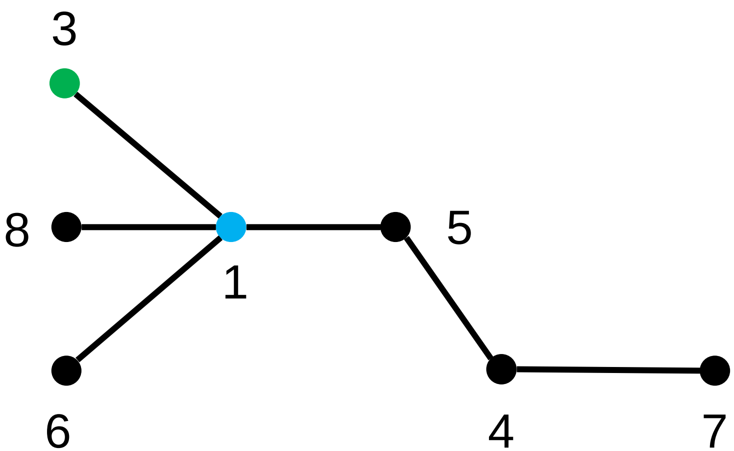


A Proof using the Prüfer code



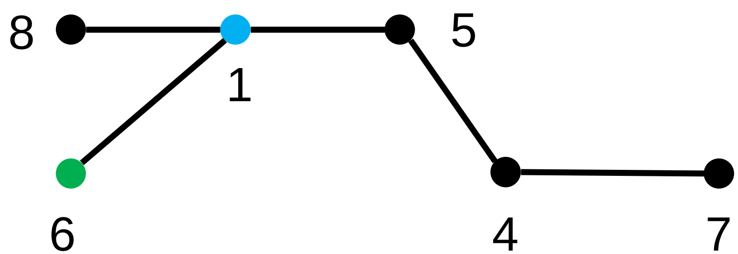
- (5)

A Proof using the Prüfer code



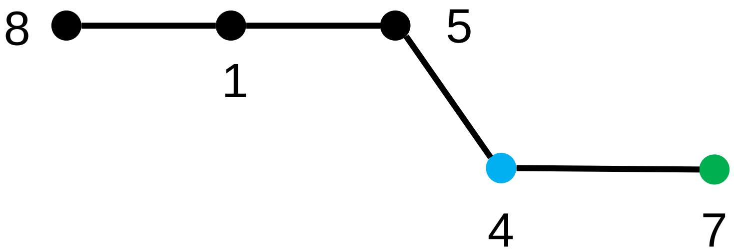
- (5)
- (5, 1)

A Proof using the Prüfer code



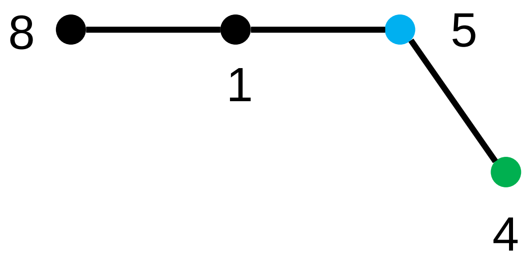
- （5）
- （5，1）
- （5，1，1）

A Proof using the Prüfer code



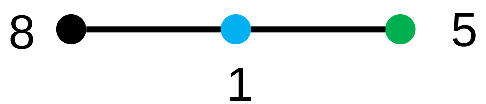
- （5）
- （5，1）
- （5，1，1）
- （5，1，1，4）

A Proof using the Prüfer code



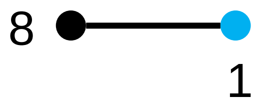
- （5）
- （5，1）
- （5，1，1）
- （5，1，1，4）
- （5，1，1，4，5）

A Proof using the Pr$\ddot{u}$fer code



- （5）
- （5，1）
- （5，1，1）
- （5，1，1，4）
- （5，1，1，4，5）
- （5，1，1，4，5，1）

A Proof using the Prüfer code

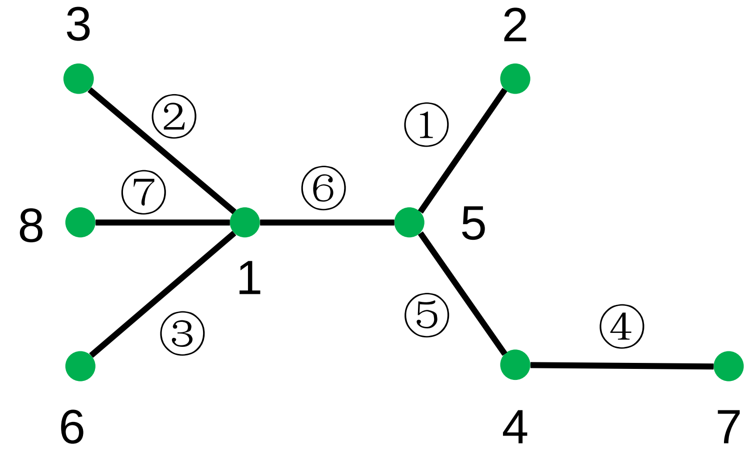


- （5）
- （5，1）
- （5，1，1）
- （5，1，1，4）
- （5，1，1，4，5）
- （5，1，1，4，5，1）

- Prüfer code for a spanning tree T of K_n：
$$p = P(T) = (p_1, p_2, \dots, p_{n-2}).$$

A Proof using the Prüfer code

（5， 1， 1， 4， 5， 1）



- Prüfer code for a spanning tree T of K_n：
$$p = P(T) = (p_1, p_2, \ldots, p_{n-2}).$$

A Proof using the Pr$\ddot{u}$fer code

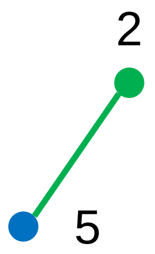
（5，1，1，4，5，1）

- Pr$\ddot{u}$fer code for a spanning tree T of K_n:
$$p = P(T) = (p_1, p_2, \ldots, p_{n-2}).$$
- Mapping between Pr$\ddot{u}$fer codes and spanning trees.

A Proof using the Pr$\ddot{u}$fer code

（5，1，1，4，5，1）

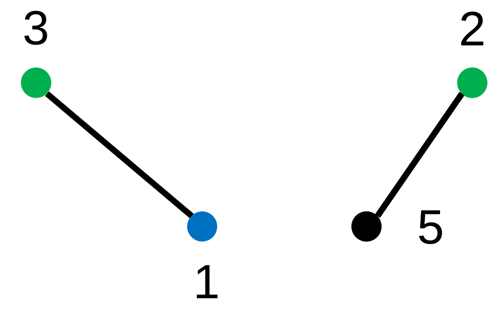
- $\min([n] \setminus \{5,1,1,4,5,1\}) = 2$



A Proof using the Pr$\ddot{u}$fer code

（5， 1， 1， 4， 5， 1）

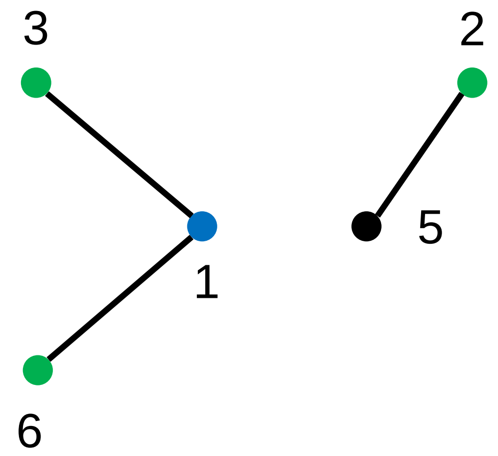
- $\min\left([n]\setminus\{5,1,1,4,5,1\}\right) = 2$
- $\min\left(\begin{matrix}[n]\setminus\{1,1,4,5,1\} \\ \setminus\{2\}\end{matrix}\right) = 3$



A Proof using the Pr$\ddot{u}$fer code

（5， 1， 1， 4， 5， 1）

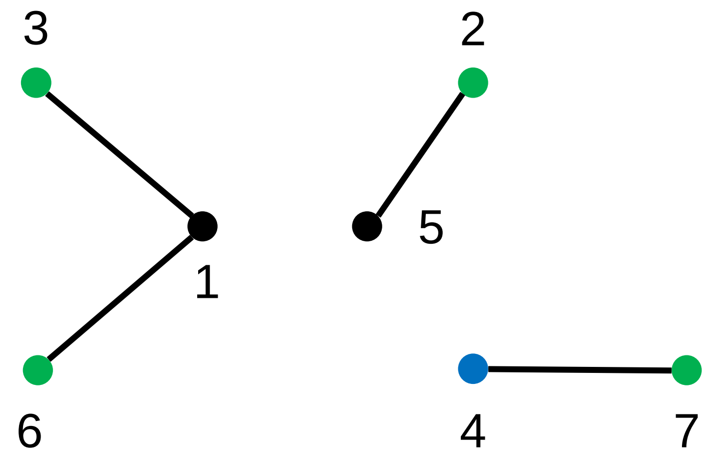
- $\min\left([n]\setminus\{5,1,1,4,5,1\}\right) = 2$
- $\min\left(\begin{matrix}[n]\setminus\{1,1,4,5,1\} \\ \setminus\{2\}\end{matrix}\right) = 3$
- $\min\left(\begin{matrix}[n]\setminus\{1,4,5,1\} \\ \setminus\{2,3\}\end{matrix}\right) = 6$



A Proof using the Prüfer code

（5， 1， 1， 4， 5， 1）

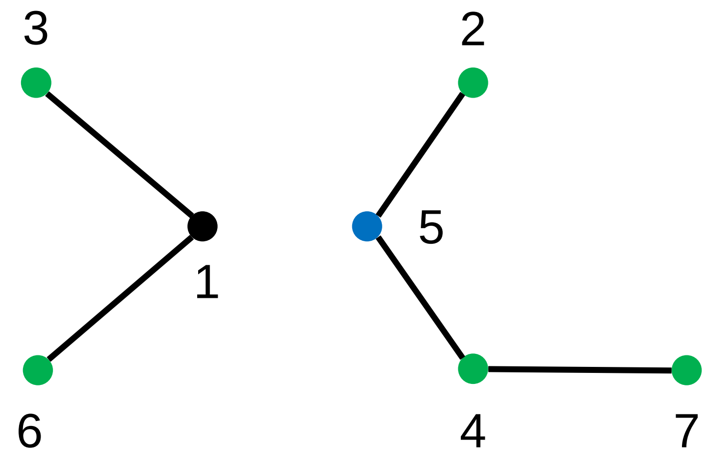
- $\min\left([n] \setminus \{5,1,1,4,5,1\}\right) = 2$
- $\min\begin{pmatrix}[n] \setminus \{1,1,4,5,1\} \\ \setminus\{2\}\end{pmatrix} = 3$
- $\min\begin{pmatrix}[n] \setminus \{1,4,5,1\} \\ \setminus\{2,3\}\end{pmatrix} = 6$
- $\min\begin{pmatrix}[n] \setminus \{4,5,1\} \\ \setminus\{2,3,6\}\end{pmatrix} = 7$



A Proof using the Pr$\ddot{u}$fer code

（5，1，1，4，5，1）

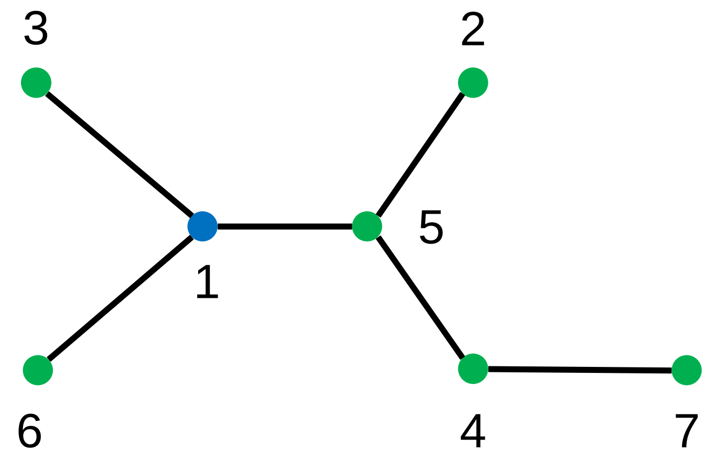
- $\min\left([n] \backslash \{5,1,1,4,5,1\}\right) = 2$
- $\min\begin{pmatrix} [n] \backslash \{1,1,4,5,1\} \\ \backslash \{2\} \end{pmatrix} = 3$
- $\min\begin{pmatrix} [n] \backslash \{1,4,5,1\} \\ \backslash \{2,3\} \end{pmatrix} = 6$
- $\min\begin{pmatrix} [n] \backslash \{4,5,1\} \\ \backslash \{2,3,6\} \end{pmatrix} = 7$
- $\min\begin{pmatrix} [n] \backslash \{5,1\} \\ \backslash \{2,3,6,7\} \end{pmatrix} = 4$



A Proof using the Pr$\ddot{u}$fer code

（5， 1， 1， 4， 5， 1）

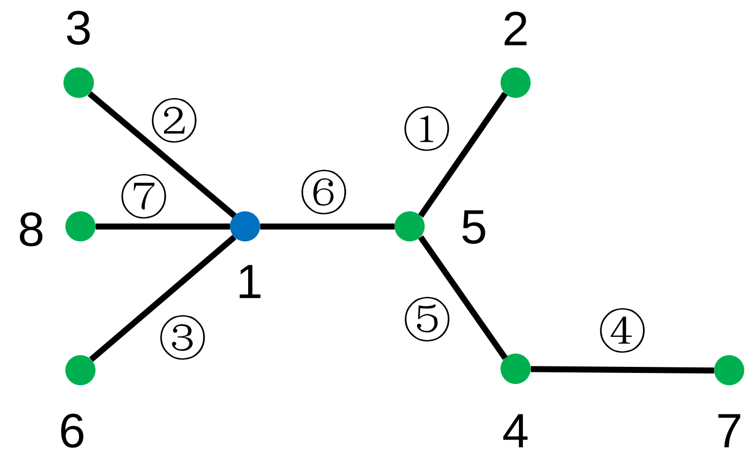
- $\min\left([n]\setminus\{5,1,1,4,5,1\}\right) = 2$
- $\min\begin{pmatrix}[n]\setminus\{1,1,4,5,1\}\\ \setminus\{2\}\end{pmatrix} = 3$
- $\min\begin{pmatrix}[n]\setminus\{1,4,5,1\}\\ \setminus\{2,3\}\end{pmatrix} = 6$
- $\min\begin{pmatrix}[n]\setminus\{4,5,1\}\\ \setminus\{2,3,6\}\end{pmatrix} = 7$
- $\min\begin{pmatrix}[n]\setminus\{5,1\}\\ \setminus\{2,3,6,7\}\end{pmatrix} = 4$
- $\min\begin{pmatrix}[n]\setminus\{1\}\\ \setminus\{2,3,6,7,4\}\end{pmatrix} = 5$



A Proof using the Prüfer code

（5，1，1，4，5，1）

- $\min\left([n]\setminus\{5,1,1,4,5,1\}\right)=2$
- $\min\begin{pmatrix}[n]\setminus\{1,1,4,5,1\}\\ \setminus\{2\}\end{pmatrix}=3$
- $\min\begin{pmatrix}[n]\setminus\{1,4,5,1\}\\ \setminus\{2,3\}\end{pmatrix}=6$
- $\min\begin{pmatrix}[n]\setminus\{4,5,1\}\\ \setminus\{2,3,6\}\end{pmatrix}=7$
- $\min\begin{pmatrix}[n]\setminus\{5,1\}\\ \setminus\{2,3,6,7\}\end{pmatrix}=4$
- $\min\begin{pmatrix}[n]\setminus\{1\}\\ \setminus\{2,3,6,7,4\}\end{pmatrix}=5$
- $[n]-\{2,3,6,7,4,5,1\}=8$



A Proof using the Prüfer code

- Prüfer code for a spanning tree T of K_n:
$$p = P(T) = (p_1, p_2, \dots, p_{n-2}).$$
- Bijection between Prüfer codes and spanning trees.

Proof working with determinants

$G = (V, E)$, where $V = \{1, 2, \dots, n\}$ $n \geq 2$,

$$E = \{e_1, e_2, \dots, e_m\}$$

Define $n \times n$ matrix Q -- the *Laplace matrix* for G:

$$q_{ii} = \deg_G(i) \qquad i = 1, 2, \dots, n$$

$$q_{ij} = \begin{cases} -1 & \{i, j\} \in E(G) \\ 0 & \text{other wise} \end{cases} \qquad i, j = 1, 2, \dots, n, i \neq j.$$

$$\begin{array}{cc} Q = \begin{bmatrix} \deg(1) & -1 & \cdots & 0 \\ -1 & \deg(2) & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \deg(n) \end{bmatrix}_{n\times n} & \begin{matrix} \\ \\ i \\ \\ \end{matrix} \\ \qquad\qquad\qquad\qquad j & \end{array}$$

Q_{ij} denote the $(n-1)\times(n-1)$ matrix arising from the matrix Q by deleting the ith row and jth column.

Theorem. For every graph G, $T(G) = \det Q_{11}$.

Application: $G = K_n$

$$Q = \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{bmatrix}_{n\times n}$$

$$Q_{11} = \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{bmatrix}_{(n-1)\times(n-1)}$$

$$\det(Q_{11}) = n^{n-2}$$

Theorem. For every graph G, $T(G) = \det Q_{11}$.

- Proof. (By induction) We show that the theorem holds for multigraphs (i.e., for graphs with multiple edges, no self-loops).
- For an edge

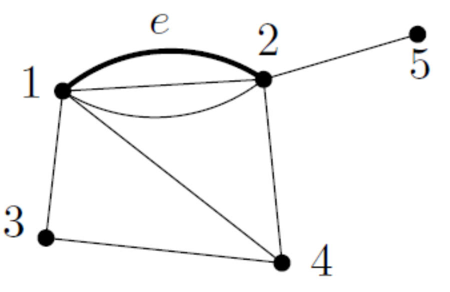
① $G - e$ Graph $(V, E\setminus\{e\})$

② $G{:}\,e$ contraction

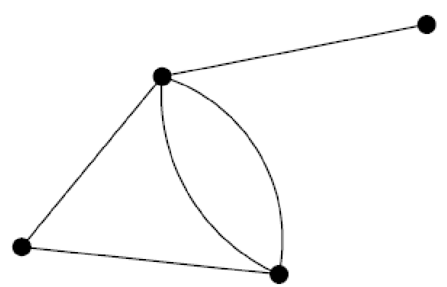
I. Remove the edge e

II. Merge the endpoints of e

III. Remove self-loops



G

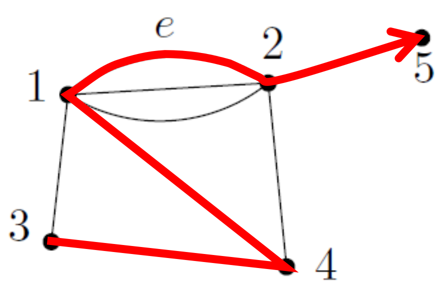


$G : e$

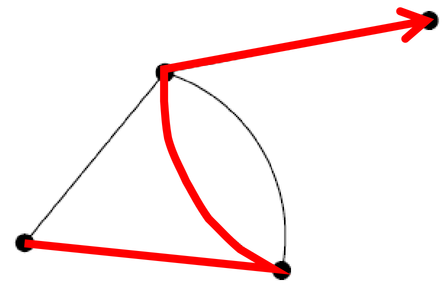
Theorem. For every graph G, $T(G) = \det Q_{11}$.

- Proof. For an edge

② $G: e$



G



$G : e$

$$T(G) = T(G-e) + T(G: e)$$

Theorem. For every graph G, $T(G) = \det Q_{11}$.

- Proof. For an edge $e \in G$

$$T(G) = T(G - e) + T(G{:}\, e) \qquad e = \{1,2\}$$

Q': the Laplacian of $G - e$

Q'': the Laplacian of $G{:}\, e$

${Q'}_{11} = Q_{11}$ except the element in the upper left corner minus 1.

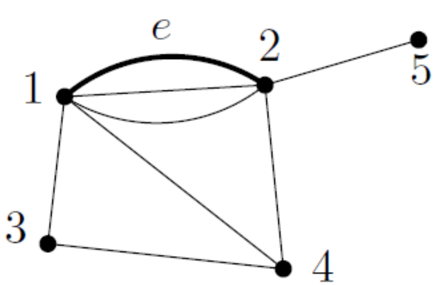
${Q''}_{11} = Q_{11,22}$.

Theorem. For every graph G, $T(G) = \det Q_{11}$.

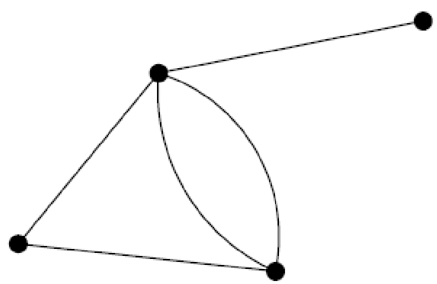
- Proof. For an edge $e \in G$

$$T(G) = T(G - e) + T(G\colon e) \qquad e = \{1,2\}$$

$$Q''_{11} = Q_{11,22}.$$



G



$G : e$

$$Q_{11} = \begin{pmatrix} 5 & 0 & -1 & -1 \\ 0 & 2 & -1 & 0 \\ -1 & -1 & 3 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \qquad Q''_{11} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Theorem. For every graph G, $T(G) = \det Q_{11}$.

- Proof. By induction on m that the results holds for every multigraph G with at most m edges.
- Base: $m = 0$ works.
- Vertex 1 is incident to at least one edge. Fix one of them and call it e. Numbering the other end of e to be 2. By induction

$$\begin{aligned} T(G) &= T(G - e) + T(G{:}\,e) \\ &= \det Q'_{11} + \det Q''_{11} \\ &= \det Q'_{11} + \det Q_{11,22} \\ &= \det Q_{11} \end{aligned}$$