Math. Struct. in Comp. Science (2011), *vol.* 21, *pp.* 943–996. © Cambridge University Press 2011
doi:10.1017/S0960129511000260 First published online 19 May 2011

The λ-calculus in the π-calculus[†]

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Received 16 December 2009; revised 19 January 2011

A general approach is proposed for transforming objects to methods on the fly in the framework of the π-calculus. The power of the approach is demonstrated by applying it to generate an encoding of the full lambda calculus in the π-calculus. The encoding is proved to preserve and reflect beta reduction, and is shown to be fully abstract with respect to Abramsky's applicative bisimilarity.

1. Introduction

The π-calculus of Milner *et al.* (1992) has been successful in both theory and practice. As a theoretical model, it has been shown that the name passing communication mechanism is both stable and powerful in that many variants of the π-calculus turn out to be just as powerful (Nestmann and Pierce 2000; Palamidessi 2003; Fu and Lu 2010). The investigations have also shown that the π-calculus captures much of the expressiveness of mobile computing. On the practical side, the π-calculus acts as a ubiquitous model for interpreting various phenomena in mobile and distributed computing. Studies have indicated that the π-calculus is extremely useful in programming objects, methods, specifications and protocols widely used in modern computing.

Two issues concerning the pragmatics of the π-calculus are particularly interesting. One is how the objects of a source model/calculus/language are interpreted and what kind of features are captured. The other is what particular variant of the π-calculus is used as a target model in a particular application. In the following we inspect these two aspects by taking a look at some previous work.

Walker's encoding of a parallel object-oriented language (Walker 1991; 1995) is carried out in the minimal π-calculus that has neither conditionals nor any forms of choice operation. This interpretation is natural since the naming policy of the π-calculus is a very good fit to the idea of the object-oriented paradigm. The reference of an object and the invoking of a method are interactions, and the passing of references is simply the passing of names. Amadio and Prasad's specification of the Mobile IP protocol (Amadio and Prasad 2000) makes use of the asynchronous π-calculus (Boudol 1992; Honda and Tokoro 1991a; Honda and Tokoro 1991b) with choice, match condition and parametric

[†] The authors are supported by the National Science Foundation of China (60873034, 61033002, 61003013).

definition. The choice and condition operators strictly enhance the expressive power of the π-calculus (Fu and Lu 2010). Parametric definition is normally strictly more powerful than the fixpoint or replication operators (Sangiorgi 1996). Because the π-calculus in Amadio and Prasad (2000) is mainly used as a description language for specifying protocols, the localisation operator is absent. The descriptive power of the π-calculus has also been explored with a view to specifying security protocols. Abadi and Gordon (1999) extended the π-calculus to Spi in an effort to deal with security issues. Baldamus *et al.* (2004) provides an interesting translation of Spi into the π-calculus with guarded choice and match. The basic idea of the translation is to understand the terms of Spi as objects. The ability of the π-calculus to create local names dynamically is essential for the interpretation of encryption and decryption.

A computation is typically composed of both data and control flows. The philosophy of the first-order π-calculus is that everything is seen as control flow management. To encode in the π-calculus is to think in terms of access (name) control. But how expressive is this new paradigm in comparison with traditional ones? We have already mentioned how object-oriented programming fits in the framework of the π-calculus. The functional paradigm and the higher order feature can also be realised by control flow management. Thomsen (1993; 1995) discussed the issue of how to translate higher order CCS into the π-calculus, and Sangiorgi (1993a; 1993b) proved that the higher order π-calculus is equivalent to the first-order π-calculus in a strong sense. Other work has confirmed that the π-calculus is more or less equivalent to many of its variants (Palamidessi 2003; Fu and Lu 2010). These results add considerable weight to the authority of the π-calculus.

All this work, and much other, has suggested that encoding in the π-calculus is very much like programming. As in all familiar programming paradigms, programming in the π-calculus has a certain methodology, the π-methodology. This methodology provides a general guideline on what entities are modelled, how the computations are simulated and which equalities are respected.

— In π-programming, there are two kinds of entities. The objects are simply π-processes, possibly with distinguished reference names. The methods are replicated forms of prefixed processes. A method can be invoked as often as necessary. The terms of a source language are normally interpreted as objects in the π-calculus, whereas types are typically interpreted as methods.

— In π-programs, computations are just reference management, which controls how names should and should not be passed around. The computational behaviour of a term is defined by the operators and the structure of the term. The structural aspect of the source language is captured in π-programs by suitable data structures, which can be explicit or implicit. Simulations of computations of source language are achieved by reference management and are organised through data structures.

— The intensional equalities of source models are interpreted by the observational equivalence of the π-calculus. The quality of an interpretation is judged by a full abstraction property. The existence of such a property is more likely if the intensional equality is observational. It is difficult for the observational theory of the π-calculus to capture the intensional equality if the latter is defined by some extra-logical theory.

Programming in the π-calculus can be greatly simplified if it has a richer set of operators. For example, guarded choice and the match operator are always convenient. However, in a particular application, the use of such operators may not be indispensable. A useful strategy is to encode the source language in some super model of the minimal π-calculus to start with, and then look for a way to transplant the encoding to a subcalculus. When dealing with a source calculus, it is interesting to investigate the minimal set of operators required to carry out a translation into the π-calculus. A result of this kind says a great deal about the nature of the source model.

Milner's interpretation (Milner 1992) of the lazy λ-calculus (Abramsky 1990) is an instructive example of using the general encoding strategy outlined above. An abstraction λ-term $\lambda x.M$ is coded up by an object with a special name, say 'λ'. The encoding of an application term MN has an underlying data structure of a binary tree, with two children being the encodings of M and N, respectively. A variable x is translated to a call, a leaf in the binary tree, that may invoke a method named x. Milner shows that his encoding preserves β-conversion. Sangiorgi (1993a; 1995) strengthened the result through a full abstraction theorem, stating that the observational equivalence induced by the encoding is precisely the open applicative bisimilarity, which is a generalisation of Abramsky's applicative bisimilarity (Abramsky 1990) to open λ-terms. By modifying the reference management of this encoding, Milner (1992) also provided an encoding of the call-by-value λ-calculus (Plotkin 1975). Milner's encodings have been studied in several different settings. For example, Sangiorgi (1993b) points out that the close relationship between the lazy λ-calculus (the call-by-name λ-calculus) and the higher order π-calculus can be explored to generate encodings in the first-order π-calculus, using the encoding from the higher order π-calculus to the first-order π-calculus. The encodings of the strong call-by-name λ-calculus have been studied in several variants of the π-calculus (Fu 1997; Parrow and Victor 1997; Fu 1999; Merro 2004).

Despite all the efforts, no encoding of the full λ-calculus has been proposed. One expects that such an encoding should satisfy at least two properties:

— It should preserve and reflect the operational semantics of the full λ-calculus.
— It should satisfy a full abstraction theorem with respect to some observational equivalence on the λ-terms.

The difficulty of encoding the full λ-calculus in the π-calculus is discussed in Milner (1992). Milner pointed out that, from the point of view of the π-calculus, the full λ-reduction strategy appears odd. In the reduction sequence

$$\begin{aligned}(\lambda x.M)N &\rightarrow^* (\lambda x.M')N'\\ &\rightarrow M'\{N'/x\}\\ &\rightarrow^* \ldots N_1' \ldots N_2' \ldots\end{aligned}$$

the β-reduction $(\lambda x.M')N' \rightarrow M'\{N'/x\}$ is a turning point. Before the β-reduction, the term N evolves by itself, but after it, two copies of the descendant N' of N may evolve independently in parallel. If N is modelled by a method, it is underneath a replication operator. But then one could hardly explain $N \rightarrow^* N'$. If N is interpreted as an object, then it seems difficult to model the cloning of N' into several copies. The idea of interpreting

$M'\{N'/x\}$ by syntactical substitution does not work either. To model the cloning by a higher order process, N has to appear in a higher order output prefix, which makes an interpretation of $N \to^* N'$ and $\lambda x.M \to^* \lambda x.M'$ very unlikely. These difficulties suggest that there is probably something about π-programming that we have not understood.

The design of an operationally sound and complete, and observationally fully abstract encoding of the full λ-calculus in the π-calculus has been an open issue for nearly twenty years. This problem is resolved in this paper.

The structure of the paper is as follows. Section 2 lays out the necessary background technicalities. Section 3 explains how to define data structures in the π-calculus. Section 4 defines the encoding of the full λ-calculus in the π-calculus. Section 5 discusses properties of the encoding. Section 6 establishes the correctness of the encoding. Section 7 provides conclusions and suggestions for further work. All the proofs of the lemmas stated in Section 5 and then used in Section 6 are given in Appendix A.

2. Technical background

This section covers the necessary background material on the λ-calculus, the π-calculus and Milner's encoding. In the rest of the paper we use $\{\diamond_0/\circ_0, \ldots, \diamond_n/\circ_n\}$ to denote a substitution that replaces $\circ_0, \ldots, \circ_n$, which is necessarily pairwise distinct, by $\diamond_0, \ldots, \diamond_n$, respectively.

2.1. *Lambda calculus*

Church's λ-calculus (Barendregt 1984) is an operational model of functional computation. The entities of the model are λ-terms and the computations are β-reductions. Let $\mathscr{V}$ be the set of the term variables, ranged over by $x, y, z, \ldots$. The set Λ of the λ-terms is defined by the grammar

$$M := x \mid \lambda x.M \mid MM'.$$

The variable x in $\lambda x.M$ is bound. A variable is free if it is not bound. A λ-term is closed if it does not contain any free variables. Let Λ^0 be the set of closed λ-terms. Let $\to^*$ be the reflexive and transitive closure of $\to$. The λ-contexts are defined by the grammar

$$C[\_] ::= \_ \mid \lambda x.C[\_] \mid (C[\_])M \mid M(C[\_]),$$

where M is a λ-term. We say that $C[\_]$ is a closing λ-context for M if $C[M] \in \Lambda^0$. The reduction strategy of the λ-terms is defined by the following rules:

$$\frac{}{(\lambda x.M)N \to M\{N/x\}}\ \beta \text{ reduction} \qquad \frac{M \to M'}{MN \to M'N}\ \text{structural rule}$$

$$\frac{N \to N'}{MN \to MN'}\ \text{eager evaluation} \qquad \frac{M \to M'}{\lambda x.M \to \lambda x.M'}\ \text{partial evaluation.}$$

Using the notion of λ-context, the four reduction rules can be replaced by the following single reduction rule:

$$\frac{}{C[(\lambda x.M)N] \to C[M\{N/x\}]}\ C[\_] \text{ is a } \lambda\text{-context.}$$

The intensional equality on the λ-terms is the well-known β-conversion $=_\beta$, which is the equality generated by $\rightarrow$. From the observational viewpoint, $=_\beta$ is too fine grained. Take, for instance, the closed term $\Omega \overset{\text{def}}{=} (\lambda x.xx)(\lambda x.xx)$. We have $\Omega \neq_\beta \Omega\Omega$, but $\Omega\Omega$ is clearly just as solitary as Ω. The importance of the observational theory of the λ-calculus has been emphasised by a number of people. Abramsky (1990), for example, introduced the applicative bisimilarity and developed a coherent observational theory for a particular reduction strategy, *viz.* the lazy λ-reduction strategy. The observational approach is, of course, applicable to other reduction strategies of the λ-calculus. The following definition of applicative bisimilarity is due to Abramsky.

Definition 1. A binary relation $\mathscr{R}$ on Λ^0 is an applicative bisimulation if the following properties hold whenever $M\mathscr{R}N$:

(i) If $M \rightarrow^* \lambda x.M'$, then $N \rightarrow^* \lambda x.N'$ for some N' such that $M'\{L/x\} \; \mathscr{R} \; N'\{L/x\}$ for every $L \in \Lambda^0$.
(ii) If $N \rightarrow^* \lambda x.N'$, then $M \rightarrow^* \lambda x.M'$ for some M' such that $M'\{L/x\} \; \mathscr{R} \; N'\{L/x\}$ for every $L \in \Lambda^0$.

The applicative bisimilarity $=_a$ is the largest applicative bisimulation.

The relation $=_a$ is obviously an equivalence. For λ-terms M, N containing, say, the free variable x, we could define $M =_a N$ if and only if $\lambda x.M =_a \lambda x.N$. Then $=_a$ becomes a congruence on the set of all λ-terms.

Lemma 1. If $M \rightarrow M'$, then $M =_a M'$.

Proof. If $M' \rightarrow^* \lambda x.M''$, then $M \rightarrow^* \lambda x.M''$. If $M \rightarrow^* \lambda x.M''$, then, by the Church–Rosser property, there exists some M''' such that $M' \rightarrow^* M'''$ and $\lambda x.M'' \rightarrow^* M'''$. Clearly, M''' must be of the form $\lambda x.M''''$. Therefore,

$$\{(M, M') \mid \Lambda^0 \ni M \rightarrow^* M'\}$$

is an applicative bisimulation. $\square$

It follows that beta conversion $=_\beta$ is included in $=_a$. The inclusion is strict since $\Omega \neq_\beta \Omega\Omega$ but $\Omega =_a \Omega\Omega$. Another concrete example of Lemma 1 is $(\lambda y.\mathbf{I})\Omega =_a \mathbf{I}$, where $\mathbf{I} \equiv \lambda x.x$. What this tells us is that the applicative bisimilarity and the β-conversion are not termination preserving. This is in contrast to the situation in the lazy λ-calculus. One implication of this fact is that, as long as we take the applicative bisimilarity as the equality on the λ-terms, the equivalence of the π-calculus should not be termination preserving either, or a full abstraction result would be impossible. An alternative would be to refine $=_a$ so that termination is taken into consideration. This would imply an overhaul of the theory of the λ-calculus since β-conversion would no longer be valid. In this paper, we shall stick with Definition 1.

An alternative characterisation of the applicative bisimilarity is given by the next lemma.

Lemma 2. The applicative bisimilarity $=_a$ is the largest among all $\mathscr{R}$ that satisfies the following properties:

(1) $\mathscr{R}$ is symmetric.

(2) If $M\mathscr{R}N \to N'$, then $M\mathscr{R}N'$.
(3) If $(\lambda x.M)\mathscr{R}N$, then $N \to^* \lambda x.N'$ for some N' such that $(\lambda x.M)\mathscr{R}(\lambda x.N')$.
(4) If $(\lambda x.M)\mathscr{R}(\lambda x.N)$, then $M\{L/x\}\mathscr{R}N\{L/x\}$ for every $L \in \Lambda^0$.

Proof. Since $M \to M'$ implies $M =_a M'$, the applicative bisimilarity satisfies the above properties. Conversely, a relation satisfying these properties is a subset of $=_a$. □

2.2. *π-calculus*

We assume that there is a set $\mathcal{N}$ of names and a set $\mathcal{N}_v$ of name variables. The following conventions will be enforced throughout the paper:

— The set $\mathcal{N}$ is ranged over by a, b, c, d.
— The union set $\mathcal{N} \cup \mathcal{N}_v$ is ranged over by $e, f, g, \ldots, x, y, z$.

So, for example, it is syntactically incorrect to write $a(c).T$. Moreover, the following additional assumption will be strictly respected:

If, for example, we write $(n)T$, then the explicit n in $(n)T$ should be understood as a name. Similarly, if we write $a(n).T$, then n in $a(n).T$ must be understood as a name variable.

It will become clear that our conventions are very convenient when defining the structural encoding of the λ-calculus in the π-calculus. We write $\widetilde{n}$ for a finite set of names/name variables $\{n_0, n_1, \cdots, n_i\}$, and we also abbreviate $(n_0 n_1 \cdots n_i)T$ to $(\widetilde{n})T$. When $\widetilde{n}$ is empty, $(\widetilde{n})T$ is just T.

The π-calculus (Milner *et al.* 1992) has a number of variants. The common part of all these variants is the minimal π-calculus, denoted by π^M, which is the variant used by Milner to encode the lazy λ-calculus. The grammar of the π^M-terms is given by the following BNF:

$$R, S, T, \ldots \;:=\; \pi.T \mid S \mid T \mid (a)T \mid !T,$$

where

$$\pi := \tau \mid n(x) \mid \overline{n}m.$$

The prefix π may be an internal action, an input $a(x)$ or an output $\overline{n}m$. In $a(x).T$, the name variable x is bound. A name variable is free if it is not bound. The name a in $(a)T$ is a local name. A name is global if it is not local. The bounded output prefix term $\overline{n}(c).T$ is defined by $(c)\overline{n}c.T$. We will write A, B, C, D, O, P, Q for the processes, which are terms without any free name variables. The operational semantics is defined by the following rules, in which μ ranges over the set $\mathscr{A} = \{ab, \overline{a}b, \overline{a}(b) \mid a, b \in \mathcal{N}\} \cup \{\tau\}$ of action labels:

Action

$$\frac{}{\tau.T \xrightarrow{\tau} T} \qquad \frac{}{\overline{a}b.T \xrightarrow{\overline{a}b} T} \qquad \frac{}{a(x).T \xrightarrow{ab} T\{b/x\}}\,.$$

Composition

$$\frac{S \xrightarrow{\mu} S'}{S \mid T \xrightarrow{\mu} S' \mid T} \qquad \frac{S \xrightarrow{ab} S' \quad T \xrightarrow{\overline{a}b} T'}{S \mid T \xrightarrow{\tau} S' \mid T'} \qquad \frac{S \xrightarrow{ab} S' \quad T \xrightarrow{\overline{a}(b)} T'}{S \mid T \xrightarrow{\tau} (b)(S' \mid T')}\,.$$

Localisation

$$\frac{T \xrightarrow{\overline{a}b} T'}{(b)T \xrightarrow{\overline{a}(b)} T'}\ a, b \text{ are distinct} \qquad \frac{T \xrightarrow{\mu} T'}{(b)T \xrightarrow{\mu} (b)T'}\ b \text{ is not in } \mu.$$

Recursion

$$\frac{T \mid !T \xrightarrow{\mu} T'}{!T \xrightarrow{\mu} T'}.$$

In the first composition rule, μ should not contain any global name in T.

The minimal π-calculus turns out to be surprisingly powerful. However, it is not very convenient when it comes to programming. Moreover, it is not yet clear if the full abstraction result (Theorem 2) is achievable if the target model is π^M. In this paper we shall work with π^{def}, the π-calculus with parametric definition and guarded choice. The set of π^{def}-terms is generated from the following grammar:

$$R, S, T, \ldots \ := \sum_{i \in I} \varphi_i \pi_i . T_i \mid S \mid T \mid (a)T \mid D(\widetilde{x}),$$

where I is a finite indexing set and φ_i is a condition. We write $\mathscr{P}$ for the set of π^{def}-processes. A condition is defined by the following grammar (Parrow and Sangiorgi 1995):

$$\varphi, \psi := \top \mid \bot \mid p = q \mid p \neq q \mid \varphi \wedge \psi,$$

where $p = q$ is a match and $p \neq q$ is a mismatch, $\top$ and $\bot$ are logical truth and falsity, respectively. The conjunction $\varphi \wedge \psi$ is often abbreviated to the concatenation $\varphi\psi$. The semantics of the *case term* $\sum_{i \in I} \varphi_i \pi_i . T_i$ is given by the following rules, in which $\Leftrightarrow$ stands for logical equivalence:

$$\frac{}{\sum_{i \in I} \varphi_i \pi_i . T_i \xrightarrow{\pi_i} T_i}\ \pi_i \text{ is an output or } \tau, \text{ and } \varphi_i \Leftrightarrow \top.$$

$$\frac{}{\sum_{i \in I} \varphi_i \pi_i . T_i \xrightarrow{ab} T_i\{b/x\}}\ \pi_i = a(x) \text{ and } \varphi_i \Leftrightarrow \top.$$

We will use extended choices of the form $\sum_{i=1..n} \varphi_i \mu_i . T_i$, where the prefixes could be bounded output prefixes. We often write $\varphi_1 \mu_1 . T_1 + \ldots + \varphi_n \mu_n . T_n$ for $\sum_{i=1..n} \varphi_i \mu_i . T_i$. Another alternative notation is

$$\begin{array}{l} \textbf{begin case} \\ \quad \varphi_1 \Rightarrow \mu_1 . T_1 \\ \quad\quad \vdots \\ \quad \varphi_n \Rightarrow \mu_n . T_n \\ \textbf{end case}. \end{array}$$

The well-known '**if_then_else**' can be defined in terms of the case terms. For example,

$$\textbf{if } p{=}q \wedge u{\neq}v \textbf{ then } \mu_1 . T_1 \textbf{ else } \mu_2 . T_2$$

is defined by the term

$$(p{=}q \wedge u{\neq}v)\mu_1 . T_1 + (p{\neq}q)\mu_2 . T_2 + (u{=}v)\mu_2 . T_2.$$

Occasionally, we shall mix several notations.

A parametric definition is given by a finite set of equations in the following form:

$$\begin{aligned} D_1(\widetilde{x_1}) &= T_1, \\ &\vdots \\ D_n(\widetilde{x_n}) &= T_n, \end{aligned}$$

where for each $i \in \{1, \dots, n\}$ the free names of T_i must appear in $\widetilde{x_i}$. An instantiation $D_i(\widetilde{n_i})$ of $D_i(\widetilde{x_i})$ is the term $T_i\{\widetilde{n_i}/\widetilde{x_i}\}$. In the above definition, $D_i(\widetilde{x_i})$ may have an instantiated occurrence in any of $T_1, \dots, T_n$. The rule defining the operational semantics of the parametric definition is

$$\frac{T_i\{\widetilde{n_i}/\widetilde{x_i}\} \xrightarrow{\mu} T'}{D_i(\widetilde{n_i}) \xrightarrow{\mu} T'}.$$

For clarity, we shall not always specify all the parameters when giving parametric definitions. Fu and Lu (2010) shows that the parametric definition is equivalent to the replication in terms of expressive power.

A binary relation $\mathscr{R}$ on the set of π^{def}-processes is:

— *equipollent* if whenever $P\mathscr{R}Q$ we have

$$\exists \mu \neq \tau. P \Longrightarrow \xrightarrow{\mu}$$

if and only if

$$\exists \mu' \neq \tau. Q \Longrightarrow \xrightarrow{\mu'},$$

where $\Longrightarrow$ is the reflexive and transitive closure of $\xrightarrow{\tau}$;

— *extensional* if it is closed under composition and localisation;

— a *weak bisimulation* if it satisfies the following *weak bisimulation property*:

- if $Q\mathscr{R}^{-1}P \xrightarrow{\tau} P'$, then $Q \Longrightarrow Q'\mathscr{R}^{-1}P'$ for some Q';
- if $P\mathscr{R}Q \xrightarrow{\tau} Q'$, then $P \Longrightarrow P'\mathscr{R}Q'$ for some P'.

The *weak bisimilarity* $\approx$ is the largest extensional, equipollent, weak bisimulation on the set of π^{def}-processes. For the present calculus, $\approx$ is precisely the barbed bisimilarity of Milner and Sangiorgi (1992).

There is a variant of the π-calculus, called the polyadic π-calculus (Milner 1997), in which more than one name can be passed around in a single communication. To simulate polyadic communication in the monadic π-calculus, we introduce the following abbreviations:

$$\begin{aligned} a(x_1, .., x_i).T &\stackrel{\mathrm{def}}{=} a(v).v(x_1)...v(x_i).T \\ \overline{a}\langle n_1, .., n_i\rangle.T &\stackrel{\mathrm{def}}{=} \overline{a}(c).\overline{c}n_1...\overline{c}n_i.T \\ \overline{a}(b_1, .., b_i).T &\stackrel{\mathrm{def}}{=} \overline{a}(c).\overline{c}(b_1)...\overline{c}(b_i).T. \end{aligned}$$

Another abbreviation we will use is $\prod_{i\in\{1,..,n\}} T_i$, which stands for $T_1 \mid \dots \mid T_n$.

We conclude this section by introducing a structural congruence over the π-processes.

Definition 2. The structural congruence $\equiv$ is the smallest congruence over the π-terms satisfying the following equalities:

(1) $S \equiv T$ whenever S is α-convertible to T.
(2) $T \,|\, \mathbf{0} \equiv T$, $S \,|\, T \equiv T \,|\, S$ and $R \,|\, (S \,|\, T) \equiv (R \,|\, S) \,|\, T$.
(3) $!T \equiv T \,|\, !T$.
(4) $(a)\mathbf{0} \equiv \mathbf{0}$ and $(a)(b)T \equiv (b)(a)T$.
(5) $(a)(S \,|\, T) \equiv S \,|\, (a)T$ if a does not appear global in S.
(6) $(a)a(x).T \equiv \mathbf{0}$ and $(a)\overline{a}m.T \equiv \mathbf{0}$.
(7) $(a)!a(x).T \equiv \mathbf{0}$ and $(a)!\overline{a}n.T \equiv \mathbf{0}$.

In this paper, the structural congruence is introduced to help define and reason about relations on processes. It is not part of the language definition, and is not quite the same as the standard congruence. The following useful fact will be used without any reference.

Lemma 3. If $S' \equiv S \xrightarrow{\mu} T$, then $S' \xrightarrow{\mu} T' \equiv T$ for some T'.

We will write $P \nrightarrow$ to mean that P cannot do any τ-action.

2.3. *Milner's translation*

There have been a number of encodings of different variants of the λ-calculus in the π-calculus. In this subsection we use Milner's encoding of the lazy λ-calculus as an example to explain the difficulties of interpreting all four reduction rules of the full λ-calculus. Milner's basic idea was to explore the fact that an abstraction term $\lambda x.M$ can be seen as a process ready to interact at the name λ. Since $\lambda x.M$ is a function, it is naturally encoded as a π-term in input prefix form. Now consider the application term $(\lambda x.M)N$. It is clear that the encoding of $(\lambda x.M)N$ should not be able to interact at λ at this point. But structurally the encoding of $\lambda x.M$ is a π-term in input prefix form, and the interface name must be different from λ. This name has to be a local name since the only interface of the λ-terms with the outside world is λ. We conclude that the encodings of the λ-terms are parameterised on names. Milner's encoding is as follows:

$$\begin{aligned}
[\![x]\!](f) &\stackrel{\text{def}}{=} \overline{x}f \\
[\![\lambda x.M]\!](f) &\stackrel{\text{def}}{=} f(x).f(u).[\![M]\!](u) \\
[\![MN]\!](f) &\stackrel{\text{def}}{=} (cd)([\![M]\!](c) \,|\, \overline{c}d.\overline{c}f \,|\, !d(v).[\![N]\!](v)).
\end{aligned}$$

A closed λ-term M is encoded by $[\![M]\!](f)$. The encoding of the abstraction $\lambda x.M$ is the object $f(x).f(u).[\![M]\!](u)$. The underlying structure of $[\![MN]\!](f)$ is a binary tree: the left child $[\![M]\!](c)$ is an object accessible at the local name c; and the right child $!d(v).[\![N]\!](v)$ is a method with name d. Notice that an object is in prefix form whereas a method is in replication form. The part $\overline{c}d.\overline{c}f$ is the control management attached to the root. Its role is to pass the method name d to $[\![M]\!](c)$ so that the method can be called as many times as necessary. The control management also assigns the name f to the final object. The encoding $[\![x]\!](f)$ is simply a call that may invoke the method named x.

The encoding as it stands is sound for two of the four rules: the β-reduction rule and the structural rule. To accommodate the partial evaluation rule, the process $[\![M]\!](u)$ in $[\![\lambda x.M]\!](f)$ must not be blocked by the input prefixes. One possibility for getting rid of the blocking is to use outputs instead of inputs. Similarly, the object $[\![N]\!](v)$ in $[\![MN]\!](f)$ is blocked by the prefix and the replication operator. Both must go if the eager evaluation rule is to be respected. However, if the replication disappears in front of $[\![N]\!](v)$, it must reappear somewhere! One way to overcome the problem could be to make the control management freeze the execution of (the descendant of) $[\![N]\!](v)$ immediately after the contraction of the β-redex MN has been simulated. In summary, to account for the operational semantics of the full λ-calculus, Milner's encoding must be modified to something like

$$\begin{aligned}
[\![x]\!]\langle f\rangle &\stackrel{\text{def}}{=} \overline{x}f\\
[\![\lambda x.M]\!]\langle f\rangle &\stackrel{\text{def}}{=} (c_x d)(\overline{f}c_x.\overline{f}d \mid ([\![M]\!]\langle d\rangle)\{c_x/x\})\\
[\![MN]\!]\langle f\rangle &\stackrel{\text{def}}{=} (cd)([\![M]\!]\langle c\rangle \mid App \mid [\![N]\!]\langle d\rangle).
\end{aligned}$$

If we try to design *App*, we soon realise that we have to identify a local name with another (local or global) name since there are too many output prefixes, and some of the names carried by the output prefixes must be identified to simulate β-reduction. But this is completely ruled out by the semantics of the π-calculus, so it might be easier to start afresh.

3. Data structures defined in the π-calculus

A data structure is an organised set of data that admits a set of predefined operations. The elements of an instance of a data structure are linked in a way prescribed by the operations on the data structure. In the π-calculus, a data structure is modelled by a set of processes of a certain shape. Both the elements of the data structure and the operations on the data structure are processes. The link relationship among the elements is indicated, often implicitly, by the reference relationship. In this section, we explain how to define data structures in the π-calculus and introduce the structural trees that will be used in our encoding of the λ-terms.

We shall look first at how to encode lists as π-processes. A list of paired names $(\langle u_1, v_1\rangle, \ldots, \langle u_n, v_n\rangle)$ is programmed as a list of processes linked by pointers. The element $\langle u_i, v_i\rangle$ is interpreted by the process $\overline{a_i}\langle u_i, v_i, a_{i+1}\rangle$. It is accessed through a_i. The pointer a_{i+1} is the reference of the next pair. In order to indicate the end of the list, we need a special name, say $\top$ or $\bot$[†]. The list is defined in Figure 1, together with the following three operations on the list:

— The process $Add(x, y, u)$ adds the pair $\langle x, y\rangle$ to the head of the list u.
— The process $Remove(x, y, u)$ deletes the pair $\langle x, y\rangle$ from u.
— The process $Find(x, u, v)$ checks whether x is the first parameter of a pair in u: if it is, it outputs the second parameter at v; otherwise it outputs $\bot$ at v.

The above example explains the general picture. Most of the data structures are special instances of the tree structure. Instead of looking at more examples of data structures coded up in the π-calculus, we shall focus in the following on the general tree structure.

[†] In this paper, we use $\bot$ for the boolean constant *false* as well as the name that plays the role of *false*. This overloading is harmless, and the symbol $\top$ is used similarly.

$$\begin{array}{rcl}
[\![()]\!]_a & \stackrel{\text{def}}{=} & \overline{a}\langle\bot,\bot,\bot\rangle \\
[\![(\langle p_0,q_0\rangle,\ldots,\langle p_n,q_n\rangle)]\!]_a & \stackrel{\text{def}}{=} & (a')(\overline{a}\langle p_0,q_0,a'\rangle \mid [\![(\langle p_1,q_1\rangle,\ldots,\langle p_n,q_n\rangle)]\!]_{a'})
\end{array}$$

$$\begin{array}{rcl}
\mathit{Add}(x,y,u) & = & u(zz'u').(u'')(\overline{u}\langle x,y,u''\rangle \mid \overline{u''}\langle z,z',u'\rangle) \\
\mathit{Remove}(x,y,u) & = & u(zz'u').\textbf{if } u'=\bot \textbf{ then } \overline{u}\langle z,z',u'\rangle \\
& & \qquad\qquad \textbf{else if } x=z\wedge y=z' \textbf{ then } u'(zz'u'').\overline{u}\langle z,z',u''\rangle \\
& & \qquad\qquad\quad \textbf{else } (\overline{u}\langle z,z',u'\rangle \mid \mathit{Remove}(x,y,u')) \\
\mathit{Find}(x,u,v) & = & u(zz'u').(\overline{u}\langle z,z',u'\rangle \mid \textbf{if } u'=\bot \textbf{ then } \overline{v}\bot \\
& & \qquad\qquad\quad \textbf{else if } x=z \textbf{ then } \overline{v}z' \textbf{ else } \mathit{Find}(x,u',v))
\end{array}$$

Fig. 1. Encoding of list

3.1. *Tree structure*

The point of introducing the tree structure into the π-calculus is that an expression or a program is naturally constructed in a tree-like structure. When translating a source language into the π-calculus, the tree structure is a useful tool, which can be modified, improved and refined to obtain a correct interpretation.

A binary tree defined in the π-calculus is a process of the following form:

$$\prod_{i\in\{0,1,\ldots,k\}} \overline{n_i}\langle p_i,l_i,r_i,t_i\rangle$$

where each concurrent component $\overline{n_i}\langle p_i,l_i,r_i,t_i\rangle$ represents a node.

The names appearing in $\overline{n_i}\langle p_i,l_i,r_i,t_i\rangle$ suggest the following interpretation:

— n_i is the name of the present node. In other words, it is the reference of the node $\overline{n_i}\langle p_i,l_i,r_i,t_i\rangle$.

— p_i, l_i, r_i are the names of the parent, and left and right children, respectively.

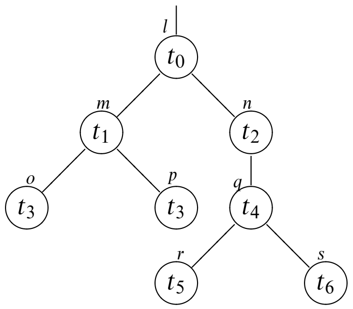
— t_i is the tag, which carries additional information attached to the node.

In a particular application, there can be more than one tag to a node. Some of these tags indicate a piece of global information, while others carry local information. A node of a tree is a term that reveals everything about itself. The revealed information includes its position in the tree and the name of an object or method attached to the node.

We use the special name ⊥ to indicate that a node has no parent/left child/right child. Consider the process T defined by

$$\begin{array}{rl}
T = & \overline{l}\langle\bot,m,n,t_0\rangle \\
& \mid \overline{m}\langle l,o,p,t_1\rangle \mid \overline{n}\langle l,q,\bot,t_2\rangle \\
& \quad \mid \overline{o}\langle m,\bot,\bot,t_3\rangle \mid \overline{p}\langle m,\bot,\bot,t_3\rangle \mid \overline{q}\langle n,r,s,t_4\rangle \\
& \qquad \mid \overline{r}\langle q,\bot,\bot,t_5\rangle \mid \overline{s}\langle q,\bot,\bot,t_6\rangle,
\end{array}$$

which represents the tree



In this tree, all the node names are exposed. In practice, we are only interested in those trees where only a restricted number of nodes are accessible. For instance, $T(l)$ defined in (1) is only accessible at l.

$$T(l) = (mnopqrs)T. \tag{1}$$

It is always a good strategy if a tree is only accessible at the root.

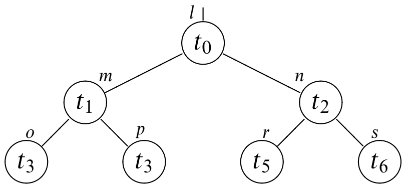
It is straightforward to implement a tree-traversal algorithm in the π-calculus using a depth first or breadth first strategy. Such a traversal is required, for instance, to check if a certain property is valid for some/all nodes of the trees. In a concurrent computation scenario, properties are dynamically established. It follows that a traversal will need to be executed again and again. At the programming level, this means that a while loop has to be introduced. The implementation of this idea in the π-calculus would inevitably introduce an infinite τ-loop. This is an undesirable feature in most circumstances since it may preempt useful actions such as the simulation of a β-reduction.

3.2. *Operations on trees*

The flat structure of the trees defined above makes it easy to define operations on trees. The process

$$\begin{aligned} n(p,l,r,t).\textbf{if } l \neq \perp \textbf{ then } l(p',l',r',t').(&\overline{n}\langle p,l',r',t\rangle \\ &\mid l'(p_1,l_1,r_1,t_1).\overline{l'}\langle n,l_1,r_1,t_1\rangle \\ &\mid r'(p_2,l_2,r_2,t_2).\overline{r'}\langle n,l_2,r_2,t_2\rangle) \end{aligned}$$

interacts with T and transforms it into the tree



This example shows that as long as we have access to an individual node of a tree, we can manoeuvre our way through the tree to find the necessary information and use the information to manipulate the tree.

When programming with trees, the trees have objects or methods attached to their nodes, and their attachment to the nodes may be static or dynamic.

Static associations, which are implemented by the static composition operator, have the following shape:

$$\overline{n_i}\langle p_i, l_i, r_i, t_i\rangle \mid L(p_i, l_i, r_i, t_i).$$

Here the node $\overline{n_i}\langle p_i, l_i, r_i, t_i\rangle$ has a standby operation $L(p_i, l_i, r_i, t_i)$, which does not go away when the node is being visited. The static association is good enough if the methods attached to the nodes of a tree are fixed. The operation $L(p_i, l_i, r_i, t_i)$ can be designed in such a way that anybody who wants to invoke it must first get the associated node information. In this case, $L(p_i, l_i, r_i, t_i)$ can only be invoked locally, which is the preferred way of using static associations.

If the operations attached to the nodes of a tree are allowed to be updated, it is more convenient to use dynamic associations, which are implemented by the dynamic choice operator, and take the form

$$\overline{n_i}\langle p_i, l_i, r_i, t_i\rangle + \psi\tau.L(p_i, l_i, r_i, t_i).$$

The advantage of dynamic associations is that after the node has been visited, it could be reinstated to something like

$$\overline{n_i}\langle p_i, l_i, r_i, t_i\rangle + \psi'\tau.L'(p_i, l_i, r_i, t_i)$$

where the dynamically associated operation has been updated. Note that even if the node information is discharged once $\psi\tau.L(p_i, l_i, r_i, t_i)$ is put into operation, it is often restored later. The dynamic association cannot be implemented in π^M since the choice operator is not definable in π^M (Fu and Lu 2010).

3.3. *Turning objects into methods*

In the π-calculus, it is impossible to construct a general process, say Q, that produces the replicated form $!O$ of a process O by interacting with O. The reason is very simple. The process Q may contain only a finite number of names. There is no way for Q to reproduce $!a_0(x_0).\cdots.a_n(x_n).\overline{a_i}x_i$ from $a_0(x_0).\cdots.a_n(x_n).\overline{a_i}x_i$ if the set $\{a_0, \cdots, a_n\}$ is large enough. However, since a data structure is a well-organised set of data, it is possible to have a π-process L that transforms every instance O of that data structure into some method from which a copy of O can be made whenever necessary, thus achieving the effect of $!O$. Two copies of O may only differ by being rooted at different places. As an example,

```
$Freeze(z,v) \;=\; z(p,l,r,t).(v_0v_1)$
        begin case
          $l{\neq}\bot\wedge r{\neq}\bot \Rightarrow \tau.(Freeze(l,v_0) \mid Freeze(r,v_1)$
                  $\mid\, !v(p',n').(l'r')(\overline{n'}\langle p',l',r',t\rangle \mid \overline{v_0}\langle n',l'\rangle \mid \overline{v_1}\langle n',r'\rangle))$
          $l{\neq}\bot\wedge r{=}\bot \Rightarrow \tau.(Freeze(l,v_0) \mid\, !v(p',n').(l')(\overline{n'}\langle p',l',\bot,t\rangle \mid \overline{v_0}\langle n',l'\rangle))$
          $l{=}\bot\wedge r{\neq}\bot \Rightarrow \tau.(Freeze(r,v_1) \mid\, !v(p',n').(r')(\overline{n'}\langle p',\bot,r',t\rangle \mid \overline{v_1}\langle n',r'\rangle))$
          $l{=}\bot\wedge r{=}\bot \Rightarrow \tau.!v(p',n').\overline{n'}\langle p',\bot,\bot,t\rangle$
        end case
```

Fig. 2. Freezing a tree

suppose D is a tree defined by

$$\begin{aligned} D \stackrel{\text{def}}{=}\; & (\bar{l}\langle\bot,m,n,t_0\rangle + t_0.Op(l,\bot,m,n,t_0)) \\ & \mid (\overline{m}\langle l,o,p,t_1\rangle + t_1.Op(m,l,o,p,t_1)) \\ & \mid (\overline{n}\langle l,q,\bot,t_2\rangle + t_2.Op(n,l,q,\bot,t_2)) \\ & \mid (\overline{o}\langle m,\bot,\bot,t_3\rangle + t_3.Op(o,m,\bot,\bot,t_3)) \\ & \mid (\overline{p}\langle m,\bot,\bot,t_3\rangle + t_3.Op(p,m,\bot,\bot,t_3)) \\ & \mid (\overline{q}\langle n,r,s,t_4\rangle + t_4.Op(q,n,r,s,t_4)) \\ & \mid (\overline{r}\langle q,\bot,\bot,t_5\rangle + t_5.Op(r,q,\bot,\bot,t_5)) \\ & \mid (\overline{s}\langle q,\bot,\bot,t_6\rangle + t_6.Op(s,q,\bot,\bot,t_6)), \end{aligned}$$

which has dynamically associated operations. Let $D(l)$ be defined by

$$D(l) \stackrel{\text{def}}{=} (mnopqrs)D.$$

Consider the parametric definition of $Freeze(z,v)$ given in Figure 2. The process

$$D(l) \mid Freeze(l,v)$$

may engage in a series of interactions. By the end of the engagement, the tree $D(l)$ has been turned into a blueprint of $D(l)$, which looks something like

$$\begin{aligned} (v_0v_1)(&!v(p',n').(l'r')(\overline{n'}\langle p',l',r',t_0\rangle \mid \overline{v_0}\langle n',l'\rangle \mid \overline{v_1}\langle n',r'\rangle) \\ & \mid !v_0(p'',n'').(l''r'')\ldots \\ & \mid !v_1(p''',n''').(l'''r''')\ldots \\ & \mid \ldots). \end{aligned}$$

In the blueprint, the operations attached to the nodes are gone. However, it contains enough information for the tree $D(l)$ to be restored as many times as necessary. If the blueprint interacts with $\overline{v}\langle\bot,l\rangle$, a replica of the tree $T(l)$, defined in (1), is produced. We could then use the process

$$\begin{aligned} Restore(l) \;=\; & l(p',l',r',t').(Restore(l') \mid Restore(r') \\ & \mid (\bar{l}\langle p',l',r',t'\rangle + t'.Op(l,p',l',r',t'))) \end{aligned}$$

to transfer $T(l)$ back to $D(l)$.

If we think of a tree structure as a piece of running program, the *Freeze* and *Restore* processes defined above are especially interesting. What *Freeze* does is to terminate the execution of part of the program and place the fingerprint of its structure in store. A copy of the terminated program can be revived by *Restore*, using the fact that the operations attached to the nodes are uniform in the sense that they are parameterised over the node information. In fact, the copy can be produced in such a way that it immediately replaces the subtree rooted at any specific node. The recursive instantiations of the parameters z, v are crucial for $\mathit{Freeze}(z, v)$ to produce the blueprint properly. It is worth remarking that the dynamic association greatly facilitates the definition of both *Freeze* and *Restore*.

Technically, the definition of a process like *Freeze* is tricky. Caution should be exercised to make sure that the correct instantiations of name parameters are carried out top down. This idea is crucial to the encoding studied in this paper.

3.4. *Operations in trees*

Suppose an expression $op(e_1, \ldots, e_n)$ is modelled by a tree structure in the π-calculus. The root of the tree has $n+1$ children interpreting $op, e_1, \ldots, e_n$, respectively. In the general tree representation, the expression is evaluated as follows to admit as much parallel execution as possible:

— The node modelling op moves first, and finds its parent information using the parent link.
— Using the parent information, it goes down the tree and checks if the evaluations of the expressions $e_1, \ldots, e_n$ are all finished.
— If the values of $e_1, \ldots, e_n$ are all ready, it evaluates $op(e_1, \ldots, e_n)$, otherwise, it either undoes everything that has been done or keeps on checking.

Notice that parallel executions are required for modelling the full β-reduction. However, the problem with the above evaluation policy is that, from the viewpoint of the interleaving semantics, the evaluation could be engaged in an infinite sequence of internal actions. This is definitely not desirable. The infinite internal chattering is caused by a strategy that goes up and down the tree to see if some statement is valid or not.

One way to prevent the unnecessary internal chattering is to introduce additional tags on the nodes. These tags not only indicate the state and the position of the nodes, but may also act as interfaces for synchronising actions.

To simulate the operation, a tree structure should contain enough tags that enable direct interactions between the nodes. The key point is that traversal is only required for tree manipulation, and is not needed for property checking or for potential interactions, since all the useful properties are indicated by the tags on the nodes and all the potential interactions can happen immediately. The evaluations of $e_1, \ldots, e_n$ could coordinate each other such that when all are done, an interaction with the node representing op initiates the evaluation of $op(e_1, \ldots, e_n)$.

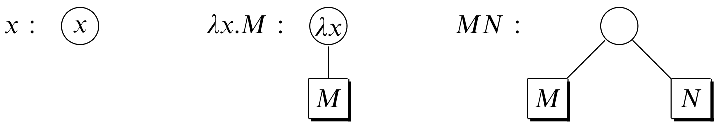
The application operation of the full λ-calculus is an example that fits into the scenario described above. In our π-interpretation of the full λ-calculus, an additional tag is used to indicate the existence of a redex.

4. Encoding in the π-calculus

A λ-term is interpreted as a binary tree definable in the π-calculus. The nodes of the tree are associated with operations that admit a proper simulation of β-reduction. The encoding is based on the syntactical assumption that an element of $\mathscr{V}$ is an element of $\mathscr{N} \cup \mathscr{N}_v$.

4.1. *Structural trees for λ-terms*

The interpretation tree of a λ-term has three kinds of nodes. An application term MN is interpreted by a tree whose root has two children, the interpretations of M, N. An abstraction term $\lambda x.M$ is interpreted by a node with one child, the interpretation of M. A variable is modelled by a leaf. The following diagrams illustrate this:



Formally, the node information is described by the following vector:

name	*parent*	*lchild*	*rchild*	*var*	*fun*

The parameters of the vector have the following readings:

— *name* is the name of the node;
— *parent* is the name of the parent of the node – if the node does not have a parent, then $parent = \bot$;
— *lchild* (*rchild*) is the name of the left (right) child of the node – if the node does not have any child, then $lchild = rchild = \bot$;
— if the node corresponds to an abstraction term $\lambda x.M$ or a variable term x, then $var = x$, otherwise $var = \bot$;
— if the node is the left child of its parent, then $fun = \top$, otherwise $fun = \bot$.

According to the above interpretation, the three kinds of node can be characterised as follows:

— $\bot \neq lchild \neq rchild \neq \bot$. This is an application node, which has two children.
— $lchild = rchild \neq \bot$. This is an abstraction node, which has one child.
— $lchild = rchild = \bot$. This is a variable node, which is a leaf.

So the information about the children can tell you the type of a node. But it does not say if the node is in a functional position, like the M in MN, or in a non-functional position, like the M in $\lambda x.M$ or in NM. This additional information is provided by the parameter fun.

4.2. *Encoding of λ-terms*

The encoding of the λ-calculus is given in Figure 3. The interpretation $[\![L]\!]_\lambda$ of λ-term L contains a name λ, which is used to access the structural tree, and is accessible only if L is an abstraction term. The process $[\![L]\!]_\lambda$ also makes use of the global names $\bot$ and

$$
\begin{array}{rcl}
[\![L]\!]_\lambda & \stackrel{\text{def}}{=} & (n,s)([\![L]\!]^{\perp}_{n,\lambda} \mid Sem) \\
\hline
[\![x]\!]^{f}_{n,p} & = & L(n,p,\perp,\perp,x,f) \\
[\![\lambda x.M]\!]^{f}_{n,p} & = & (m,x)(L(n,p,m,m,x,f) \mid [\![M]\!]^{\perp}_{m,n}) \\
[\![MN]\!]^{f}_{n,p} & = & (l,r)(L(n,p,l,r,\perp,f) \mid [\![M]\!]^{\top}_{l,n} \mid [\![N]\!]^{\perp}_{r,n})
\end{array}
$$

Fig. 3. Encoding of the λ-calculus

$\top$, which play the role of the logical values. However, neither $\perp$ nor $\top$ appears as an interface name in $[\![L]\!]_\lambda$.

In the standard semantics of the λ-calculus, β-reductions are carried out one at a time, which fits in quite well with the interleaving semantics. In the translation, the semaphore *Sem* is used to prevent the simulations of two β-reductions from interfering with each other. It is given by the following recursive definition:

$$
\begin{aligned}
Sem &= \overline{s}.Sem^- \\
Sem^- &= s.Sem.
\end{aligned}
$$

The simulation of a β-reduction begins by turning the positive state *Sem* into the negative state Sem^-. When the simulation ends, it turns Sem^- back to *Sem*. The encoding $[\![M]\!]^{f}_{n,p}$ defines a node of the structural tree. The subscripts n,p are the names of the present node and its parent, respectively. The superscript f indicates whether the node is the left child of its parent or not. The encoding also makes use of the global name s. The translation $[\![x]\!]^{f}_{n,p}$ is a leaf of the structural tree. The interpretation $[\![\lambda x.M]\!]^{f}_{n,p}$ is a tree, the root of which corresponds to the abstraction operator λx and has only one child; and the child is the root of the subtree $[\![M]\!]^{\perp}_{m,n}$. The structure of $[\![MN]\!]^{f}_{n,p}$ should now be clear. It is a tree with two immediate subtrees $[\![M]\!]^{\top}_{l,n}$ and $[\![N]\!]^{\perp}_{r,n}$. The superscript $\top$ of $[\![M]\!]^{\top}_{l,n}$ indicates that M is in the function position of MN, whereas the superscript $\perp$ in $[\![N]\!]^{\perp}_{r,n}$ means that N is in the value position of MN.

The process $L(n,p,l,r,v,f)$, the node named n, is defined in Figure 4. The intended meanings of the parameters of $L(n,p,l,r,v,f)$ were explained in Section 4.1. It consists of two parts: one identifying the position of the node and the other defining the associated operation that can be enacted at the node. The process $\overline{n}\langle p,l,r,v,f\rangle$ declares itself and proclaims the control information related to the node. Different kinds of nodes have different influences on the tree structure.

— $l = r = \perp$.
  In this case $L(n,p,l,r,x,f)$ must be a leaf of the tree, standing for the variable x. The process $(a)\overline{v}\langle n,p,f,a\rangle.\overline{a}\langle\perp,\perp,\perp\rangle$ simply makes a copy of the frozen tree rooted at x. It will become clear that the frozen tree is much more complicated than the one in the previous section.

— $l = r \neq \perp \wedge f{=}\top$.
  This is an abstraction node that is the left child of its parent, corresponding to the left-hand side of an application term. The operation of the node $Abs(n,p,l,x,f)$ can immediately kick off a β-reduction.

$$
\begin{aligned}
L(n,p,l,r,v,f) \;=\; & \overline{n}\langle p,l,r,v,f\rangle + \textbf{begin case}\\
& \quad l=r=\bot \Rightarrow (a)\overline{v}\langle n,p,f,a\rangle.\overline{a}\langle \bot,\bot,\bot\rangle\\
& \quad l=r\neq\bot\wedge f=\top \Rightarrow \mathit{Abs}(n,p,l,v,f)\\
& \quad l=r\neq\bot\wedge p=\lambda \Rightarrow \mathit{RAbs}(n,p,l,v,f)\\
& \textbf{end case}
\end{aligned}
$$

$$
\begin{aligned}
\mathit{Abs}(n,p,m,x,f) \;=\; & s.p(p^1,l^1,r^1,v^1,f^1).m(p_1,l_1,r_1,v_1,f_1).(L(p,p^1,l_1,r_1,v_1,f^1)\mid\\
& \textbf{begin case}\\
& \quad l_1=r_1\neq\bot \Rightarrow l_1(p_2,l_2,r_2,v_2,f_2).(L(l_1,p,l_2,r_2,v_2,f_2)\\
& \quad \mid (b)(\mathit{Backup}(r^1,x,b,\bot)\mid b.\overline{s}))\\
& \quad l_1\neq r_1 \Rightarrow l_1(p_2,l_2,r_2,v_2,f_2).r_1(p_2',l_2',r_2',v_2',f_2').(L(l_1,p,l_2,r_2,v_2,f_2)\\
& \quad \mid L(r_1,p,l_2',r_2',v_2',f_2')\mid (b)(\mathit{Backup}(r^1,x,b,\bot)\mid b.\overline{s}))\\
& \textbf{end case})
\end{aligned}
$$

$$
\begin{aligned}
\mathit{RAbs}(n,\lambda,m,x,f) \;=\; & \lambda(z).s.m(p_1,l_1,r_1,v_1,f_1).\\
& (b)(\mathit{Backup}(z,x,b,\top)\mid b.\overline{s}.L(m,\lambda,l_1,r_1,v_1,\bot))
\end{aligned}
$$

$$
\begin{aligned}
\mathit{Backup}(n,x,b,e) \;=\; & n(p,l,r,v,f).\\
& \textbf{begin case}\\
& \quad l=\bot\vee r=\bot \Rightarrow \overline{b}.!x(n',p',f',a).(o)(\mathit{Find}(v,a,o)\mid o(v').\\
& \qquad \textbf{begin case}\\
& \qquad\quad v'=\bot\wedge e=\top \Rightarrow \tau.[\![\Omega]\!]^{f'}_{n'p'}\\
& \qquad\quad v'=\bot\wedge e=\bot \Rightarrow \tau.L(n',p',\bot,\bot,v,f')\\
& \qquad\quad v'\neq\bot \Rightarrow \tau.L(n',p',\bot,\bot,v',f')\\
& \qquad \textbf{end case})\\
& \quad l=r\neq\bot \Rightarrow \tau.(x'b')(\mathit{Backup}(l,x',b',e)\mid b'.\overline{b}.!x(n',p',f',a).\\
& \qquad (m'a'v')(L(n',p',m',m',v',f')\mid \overline{x'}\langle m',n',\bot,a'\rangle\mid \overline{a'}\langle v,v',a\rangle))\\
& \quad \bot\neq l\neq r\neq\bot \Rightarrow \tau.(x_0x_1b_0b_1)(\mathit{Backup}(l,x_0,b_0,e)\mid \mathit{Backup}(r,x_1,b_1,e)\\
& \qquad \mid b_0.b_1.\overline{b}.!x(n',p',f',a).(l'r')(L(n',p',l',r',\bot,f')\\
& \qquad \mid \overline{x_0}\langle l',n',\top,a\rangle\mid \overline{x_1}\langle r',n',\bot,a\rangle))\\
& \textbf{end case}
\end{aligned}
$$

Fig. 4. Definition of the node operation

— $l = r \neq \bot \wedge p = \lambda$.
This is an abstraction node, which happens to be the root. In this case, we have that $\mathit{RAbs}(n, p, l, x, f)$ can start a β-reduction if the environment provides a term at the special interface λ. The associated operation is not the same as the one in the previous case. This should not be a problem since we are only interpreting closed λ-terms.

— $l = r \neq \bot \wedge p \neq \lambda \wedge f = \bot$.
This is an abstraction node that is the right child or the only child of its parent. It cannot invoke any β-reductions, so it does not change the structure of the tree.

— $l \neq r$.
This is an application node that does not invoke any action. Its role is to organise the control flow of the reduction.

Note that the process $L(n,p,l,r,x,f)$ can be viewed as an enriched solution for *App* mentioned in Section 2.

The *Backup*(n,x,b,e) process makes a replica, or a blueprint, of the tree that encodes a λ-term. The parameters should be understood as follows:

— n is the name of the root of the tree to be replicated;
— x is the name that may be used to access the replica;
— b is the control name that releases the replica after it has been completely produced;
— e indicates if the tree rooted at n is from the environment or not.

The operational behaviour of *Backup*(n,x,b,e) will be analysed later in the paper.

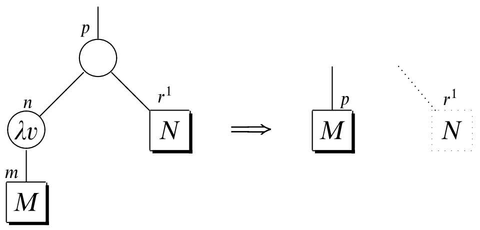
4.3. *The simulation of β-reduction*

When $L(n,p,l,r,v,f)$ is an abstraction node with $f = \top$, a β-redex is present. In this case, $L(n,p,l,r,v,f)$ can set the semaphore to Sem^{-} and begin to simulate the β-reduction. The simulation consists of two phases:

(1) In the first phase:

— The child m of the node n is moved to the position of the parent p of node n.
— The node n is deleted.
— The node name m is dropped.
— The right child r^1 of p is temporarily detached.

The change in the tree structure can be depicted as follows:



(2) In the second phase, the subtree rooted at r^1 is backed up. This is done by the process *Backup*$(r^1,v,b,\perp)$, which replicates the subtree from the root to the leaves in such a way that a copy of the subtree can be readily generated by providing a new name of the root and its parent's name. The parameter b is to make sure that the simulation of the β-reduction is not complete until every node of the subtree is backed up. When the backup is over, Sem^{-} will be set back to Sem.

If $L(n,p,l,r,v,f)$ is an abstraction node situated at the root, it can backup a λ-term from the environment. This situation is indicated by instantiating e by $\top$, where e is the fourth parameter of *Backup*. In this case the backup process needs to do some more work and make sure that the result of the backup is the interpretation of a well-defined *closed* λ-term. It achieves this by treating all the interpretations of free variables as if they were the interpretations of the closed λ-term Ω. In order to do this properly, a tree structure is introduced to record the interpretations of the closed variables of the λ-term, reflecting naturally the nested structure of the closed variables. When the backup comes down to a leaf that encodes the variable v, it checks the status of v in the term being backed up using the process *Find* defined in Figure 1. There are three possibilities:

— If *Find* returns with $v' = \bot$ and $e = \top$, the term being backed up comes from the environment and the variable v is not the interpretation of a bound variable. In this case, we substitute Ω for v in the backup.

— If *Find* returns with $v' = \bot$ and $e = \bot$, the term being backed up comes from within and v is not bound in that subterm. In this case, the variable v should be backed up since every restored copy shares the same v.

— If the name v' returned by *Find* is not $\bot$, then v corresponds to a closed variable in the λ-term being backed up. In this case, a new local name must be produced every time a copy of the backup is made.

The condition $l = \bot \vee r = \bot$ in the definition of *Backup* works for the terms from within and the terms from without. The term from the environment might be corrupted, but as long as one of l or r is $\bot$, the backup procedure will regard it as representing a variable.

One might wonder why the tree a of the paired names must be revisited every time a copy of the replica is restored, and even why the tree is required at restore time. The point is that bound variables in different copies must not get confused. The nodes of the tree named a are dispersed underneath different replication operators so that the second parameters of the paired names are all localised underneath the replication operators.

After the simulation of $(\lambda x.M)N \longrightarrow M\{N/x\}$, the process $Backup(n, x, b, e)$ will have transformed the interpretation of $(\lambda x.M)N$ into a process of the form

$$\begin{aligned}(xx_0x_1\ldots x_k)(\ldots \mid\ & !x(n', p', f', a).(\ldots \mid \overline{x_0}\langle\ldots\rangle \mid \ldots)\\ \mid\ & !x_0(n'_0, p'_0, f'_0, a_0).(\ldots \mid \overline{x_1}\langle\ldots\rangle \mid \ldots)\\ \mid\ & \ldots\\ \mid\ & !x_k(n'_k, p'_k, f'_k, a_k).(\ldots)).\end{aligned}$$

An occurrence of the variable x in M is interpreted as $(a)\overline{x}\langle n, p, f, a\rangle.\overline{a}\langle\bot\bot\bot\rangle$. It can make a copy of the interpretation of N by interacting with $!x(n', p', f', a).\_$, starting a chain of replication. Notice that since *Find* works from leaf to root, the *Backup* process can deal properly with terms like $\lambda y.(\lambda y.y)y$.

5. Properties of the encoding

In this section, we study the properties of the encoding, including determining what operations the translations can perform and what forms they may evolve into. This section forms a preamble to the next section, which shows the correctness of the encoding.

By the definition of the encoding, for every closed λ-term L the only public interface of $[\![L]\!]_\lambda$ is λ. So the possible actions of $[\![L]\!]_\lambda$ are either $\xrightarrow{\tau}$ or $\xrightarrow{\lambda z}$ for some name z:

— If the next action of $[\![L]\!]_\lambda$ is a τ-action, the simulation of $(\lambda x.M)N \rightarrow M\{N/x\}$ is initiated by this action and will be followed by a number of τ-actions carried out in two stages:

  - In the *reduction stage*, the tree of $(\lambda x.M)N$ is adjusted to the tree of M, while the tree of N is frozen.
  - In the *replication stage*, every occurrence of x in M is replaced by a replica of N.

  The reduction stage is not immediately followed by the replication stage. This is because a replication of N is fired by each occurrence of x. These replications may not only interleave among themselves, but also interleave with other reduction stages.

— If the next action of $[\![L]\!]_\lambda$ is a λz-action, then $L \equiv \lambda x.M$ for some M, and the translation is ready to input a λ-term from the environment. Suppose $[\![L]\!]_\lambda \xrightarrow{\lambda q} P$ for some P. By synchronising with the semaphore, the process P can start to simulate the input of a λ-term N from the environment. This simulation may also be divided into two stages:

  - In the *input stage*, the tree of $\lambda x.M$ is adjusted to the tree of M, while the tree structure of the 'λ-term N' provided by the environment is frozen. The term N might contain a few grammatical errors, which are corrected in the input stage.
  - In the *replication stage*, every occurrence of x in M is replaced by a replica of the corrected version of N.

We shall analyse these two cases in detail in Sections 5.1 and 5.2, respectively. To do the analysis, we find it helpful to introduce some additional notation and definitions:

— To simplify the account, we shall pretend that we are working with the polyadic π^{def}. For example, we shall write $P \xrightarrow{n(p,l,r,v,f)} P'$ to mean that the names p, l, r, v, f are received in that order at n in an atomic action.

— The translation $[\![L]\!]^f_{n,p}$ of the λ-term L has an underlying tree structure. This tree takes the form

$$(\widetilde{v})(\widetilde{n}) \prod_{i \in I} L(n_i, p_i, l_i, r_i, v_i, f_i)$$

for some finite set I, where $\widetilde{v}$ is the set of the bound variables of L, and $\widetilde{n}$ is the set of the node names of the tree excluding the root name n and the name of its parent p. For each $i \in I$, the component $L(n_i, p_i, l_i, r_i, v_i, f_i)$ is a node of the tree and n_i is the name of the node. In the following we shall write $T^{n,p,f}_L$ for the tree

$$\prod_{i \in I} L(n_i, p_i, l_i, r_i, v_i, f_i). \tag{2}$$

We also write $L(n', p', l', r', v', f') \in T^{n,p,f}_L$ to indicate that $L(n', p', l', r', v', f')$ is a component of $T^{n,p,f}_L$.

— We write $RT^{n,p,f}_M$ for the composition of the nodes in $T^{n,p,f}_M$ apart from the root located at n.

— If N is a sub-term of L, that is $L \equiv C[N]$ for some closing λ-context, we write

$$T^{n,p,f}_L \equiv T^{n,p,f}_{C[\_]} \mid T^{n',p',f'}_N$$

to mean that $T^{n,p,f}_{C[\_]}$ is what is left after removing the nodes of $T^{n',p',f'}_N$ from $T^{n,p,f}_L$. Here n' is the name of the sub-term N and p' is its parent's name.

— Suppose $T_L^{z,\perp,\perp}$ is of the form (2). The notation $T(z, L)$ stands for the process

$$(\widetilde{v})(\widetilde{n}) \prod_{L(n,p,l,r,v,f)\in T_L^{z,\perp\perp}} \overline{n}\langle p, l, r, v, f\rangle$$

where $\widetilde{v}$ is the set of the bound variables of L, and $\widetilde{n}$ is the set of the node names excluding z.

— A *rooted binary tree* T_r is a directed graph with the following properties:

- There is a node, called the root, that does not have any incoming edges. Every other node has precisely one incoming edge.
- Every node has zero, one or two outgoing edges. If a node has two children, the children are ordered and are called the left and right child, respectively.

A *labelled tree* LT is a rooted binary tree with each of its node labelled by a pair $\langle n, \vartheta\rangle$, where n is a name and ϑ is either a name or $\circ$. We write $\mathbf{1}_z$ for the single node labelled tree with label $\langle z, \circ\rangle$, and **LT** for the set of all labelled trees.

— Suppose LT_1 and LT_2 are two labelled trees. We write $LT_1 \Subset LT_2$ if LT_1 can be obtained from LT_2 by pruning all the subtrees from some nodes.

— Suppose n is the label of a specific leaf of the labelled tree LT. We write $LT \cdot (n, l, r, v)$ for the labelled tree that extends LT as follows:

- If $l = \perp \vee r = \perp$, then $LT \cdot (n, l, r, v)$ is obtained from LT by changing the second parameter of the label of the specific leaf to v.
- If $l = r \neq \perp$, then $LT \cdot (n, l, r, v)$ is obtained from LT by changing the second parameter of the label of the specific leaf to v and attaching to the leaf a single child labelled $\langle l, \circ\rangle$.
- If $\perp \neq l \neq r \neq \perp$, then $LT \cdot (n, l, r, v)$ is obtained from LT by attaching to the specific leaf a left child labelled $\langle l, \circ\rangle$ and a right child labelled $\langle r, \circ\rangle$.

— Let LT be a labelled tree. The notation $I(LT)$ denotes the multi-set of all the first parameters of the labels of the leaves of LT whose second parameter is $\circ$.

5.1. *The reduction action*

In this subsection we consider the τ-actions of the translations. Suppose O contains the β-redex $(\lambda x.M)N$. We may indicate this fact by writing $O \equiv C[(\lambda x.M)N]$ where $C[\_]$ is a closing context for $(\lambda x.M)N$. By the definition of the encoding and structural congruence, we have

$$[\![O]\!]_\lambda \equiv (s)(\widetilde{x})(\widetilde{n})\left(Sem \mid T_O^{n,\lambda,\perp}\right)$$

where $\widetilde{x}$ is the set of the bound variables in O and $\widetilde{n}$ is the set of the internal node names of the tree $T_O^{n,\lambda,\perp}$. We can rearrange the bound names in $T_O^{n,\lambda,\perp}$ by applying structural congruence rules:

$$\begin{aligned} T_O^{n,\lambda,\perp} &\equiv T_{C[\_]}^{n,\lambda,\perp} \mid T_{(\lambda x.M)N}^{n',p',f'}, \\ T_{(\lambda x.M)N}^{n',p',f'} &\equiv L(n', p', l, r, \perp, f') \mid L(l, n', m, m, x, \top) \mid T_M^{m,l,\perp} \mid T_N^{r,n',\perp}. \end{aligned}$$

The β-reduction $C[(\lambda x.M)N] \rightarrow C[M\{N/x\}]$ is simulated in two stages, which are described in the following subsections.

5.1.1. *The reduction stage.* A β-redex is resolved in a reduction stage. By the end of a reduction stage all the occurrences of a bound variable are tied to the replicated form of the encoding of a λ-term. This is achieved mainly by the *Backup* process. By the definition of $L(n,p,l,r,v,f)$ given in Figure 4,

$$L(l,n',m,m,x,\top) \xrightarrow{s} L_1 \equiv n'(p^1,l^1,r^1,v^1,f^1).\cdots$$
$$Sem \xrightarrow{\overline{s}} Sem^-.$$

So the reduction stage begins by setting Sem to the state Sem^-:

$$[\![O]\!]_\lambda \xrightarrow{\tau} P_1 \equiv (s)(\widetilde{x})(\widetilde{n})(Sem^- \mid T^{n,\lambda,\perp}_{C[\_]} \mid L(n',p',l,r,\perp,f') \mid L_1 \mid T^{m,l,\perp}_{M} \mid T^{r,n',\top}_{N}).$$

The process L_1 may interact with $L(n',p',l,r,\perp,f')$ and $T^{m,l,\perp}_{M}$. According to the structures of M, P_1 may perform:

— two τ-actions in the case of $M \equiv w$ for some w;
— three τ-actions in the case of $M \equiv \lambda w.M_1$ for some w, M_1; or
— four τ-actions in the case of $M \equiv M_1M_2$ for some M_1, M_2.

After these τ-actions, P_1 will have evolved into P_2, that is, we have

$$\begin{aligned} P_1 \overset{\tau}{\Longrightarrow} P_2 &\equiv (s)(\widetilde{x})(\widetilde{n})(Sem^- \mid T^{n,\lambda,\perp}_{C[\_]} \mid T^{n',p',f'}_{M} \mid (b)(Backup(r,x,b,\perp) \mid b.\overline{s}) \mid T^{r,n',\top}_{N}) \\ &\equiv (s)(\widetilde{x})(\widetilde{n})(Sem^- \mid T^{n,\lambda,\perp}_{C[M]} \mid (b)(Backup(r,x,b,\perp) \mid b.\overline{s}) \mid T^{r,n',\top}_{N}). \end{aligned}$$

Now $Backup(r,x,b,\perp)$ is able to input at r. It has three kinds of transition described as follows, where $Copy(x,x_0,x_1,w,e)$ abbreviates $!x(n',p',f',a).\cdots$:

— If $x_0 = x_1 = \perp$, the first case of $Backup(n,x,b,e)$ applies and

$$Backup(r,x,b,\perp) \xrightarrow{r(p,\perp,\perp,w,f)} \xrightarrow{\overline{b}} Copy(x,\perp,\perp,w,\perp).$$

— If $x_0 = x_1 \neq \perp$, the second case of $Backup(n,x,b,e)$ applies and

$$\begin{aligned} Backup(r,x,b,\perp) \xrightarrow{r(p,m,m,w,f)} \xrightarrow{\tau} &(x'b')(b'.\overline{b}.Copy(x,x',x',w,\perp) \\ &\mid Backup(m,x',b',\perp)). \end{aligned}$$

— If $x_0 \neq x_1$, the third case of $Backup(n,x,b,e)$ applies and

$$\begin{aligned} Backup(r,x,b,\perp) \xrightarrow{r(p,l',r',\perp,f)} \xrightarrow{\tau} &(x_0x_1b_0b_1)(b_0.b_1.\overline{b}.Copy(x,x_0,x_1,w,\perp) \\ &\mid Backup(l',x_0,b_0,\perp) \mid Backup(r',x_1,b_0,\perp)). \end{aligned}$$

By the above analysis, $Backup(r,x,b,\perp)$ must interact with $T^{r,n',\top}_{N}$, reading the nodes of the tree recursively from root to leaf. When it reaches a leaf, it backs up and unlocks all the b_i's in reverse order. Process P_2 evolves into the following process P_3 when the prefix b before $\overline{s}$ is demolished:

$$P_2 \overset{\tau}{\Longrightarrow} P_3 \equiv (s)(\widetilde{x})(\widetilde{n})(Sem^- \mid T^{n,\lambda,\perp}_{C[M]} \mid \overline{s} \mid [\![x := N]\!]_\perp)$$

where $[\![x{:=}N]\!]_e$ is the process

$$(x_0x_1\dots x_k)[\![x{:=}N]\!]_e^{x_0x_1\dots x_k}$$

and $[\![x{:=}N]\!]_e^{x_0x_1\dots x_k}$ is the process

$$!x(n',p',f',a).\_\,|\,!x_0(n'_0,p'_0,f'_0,a_0).\_\,|\,\dots\,|\,!x_k(n'_k,p'_k,f'_k,a_k).\_.$$

The parameter e, which is either $\perp$ or $\top$, is the parameter e in $Backup(n,x,b,e)$.

As the final step in the reduction stage, the process P_3 resets Sem^- to Sem in a single step:

$$P_3 \xrightarrow{\tau} P_4 \equiv (s)(\widetilde{x})(\widetilde{n'})(Sem\,|\,T^{n,\lambda,\perp}_{C[M]}\,|\,[\![x{:=}N]\!]_\perp),$$

where $\widetilde{n'}$ is obtained from $\widetilde{n}$ by removing several names, bearing in mind that in the simulation of the λ-reduction two nodes are removed and all the nodes related to N are removed.

The process $[\![x{:=}N]\!]_\perp$ is similar to the blueprint discussed in Section 3. If it interacts with $\overline{x}\langle n,p,f,a\rangle$, a replica of the tree of N is produced and is rooted at n with p being the parent node. It should be obvious that $(x)[\![x{:=}N]\!]_e \equiv \mathbf{0}$.

5.1.2. *The replication stage.* In a replication stage, a copy of a replica is made in the position of an occurrence of the variable that is tied to the replica. This is achieved by providing the replica with the location name of the variable. The instantiation of an x in $C[M]$ can start as soon as $[\![x{:=}N]\!]_\perp$ is ready since a leaf node $L(n_i,p_i,\perp,\perp,x,f_i)$ in the tree of M can carry out the action $\overline{x}\langle n_i,p_i,f_i,a\rangle$ and turn into $\overline{a}\langle\perp,\perp,\perp\rangle$, where a is a newly generated local name. The process $[\![x{:=}N]\!]_\perp$ has three kinds of transition:

— If $N\equiv N_1N_2$ for some N_1 and N_2, then

$$\begin{aligned}[\![x{:=}N]\!]_\perp \xrightarrow{x(n',p',f',a)} (\widetilde{x})([\![x{:=}N]\!]_\perp^{\widetilde{x}}\,|\,(l'r')(L(n',p',l',r',\perp,f')\\ |\,\overline{x_0}\langle l',n',\top,a\rangle\,|\,\overline{x_1}\langle r',n',\perp,a\rangle)).\end{aligned}$$

The application node has been copied to n'. Now the two processes $\overline{x_0}\langle l',n',\top,a\rangle$ and $\overline{x_1}\langle r',n',\perp,a\rangle$ can interact with $[\![x{:=}N]\!]_\perp$ and get the copy of N_1 and N_2, respectively.

— If $N\equiv\lambda v.N_1$ for some N_1 and v, then

$$\begin{aligned}[\![x{:=}N]\!]_\perp \xrightarrow{x(n',p',f',a)} (\widetilde{x})([\![x{:=}N]\!]_\perp^{\widetilde{x}}\,|\,(m'a'v')(L(n',p',m',m',v',f')\\ |\,\overline{x'}\langle m',n',\perp,a'\rangle\,|\,\overline{a'}\langle v,v',a\rangle)).\end{aligned}$$

After the abstraction node has been copied to n', the process $\overline{x'}\langle m',n',\perp,a'\rangle$ interacts with $[\![x{:=}N]\!]_\perp$ and gets the copy of N_1. The name a' is used to record the new name v' that is correlated with v. The free variable v in N_1 will be replaced by a new local name v' every time N_1 is replicated.

— If $N\equiv w$ for some w, then

$$\begin{aligned}[\![x{:=}N]\!]_\perp &\xrightarrow{x(n',p',f',a)} (\widetilde{x})([\![x{:=}N]\!]_\perp^{\widetilde{x}}\,|\,(o)(Find(w,a,o)\,|\,o(v').\cdots))\\ &\overset{\tau}{\Longrightarrow} (\widetilde{x})([\![x{:=}N]\!]_\perp^{\widetilde{x}}\,|\,L(n',p',\perp,\perp,w,f')).\end{aligned}$$

Since we have the τ-action

$$(a)(Find(w,a,o)\,|\,\overline{a}\langle\bot,\bot,\bot\rangle) \xrightarrow{\tau} \overline{a}\langle\bot,\bot,\bot\rangle\,|\,\overline{o}\langle\bot\rangle,$$

a leaf with variable w is copied to n'.

To avoid confusing the bound names in two copies of N, the parameter a is introduced. It is easy to see that all the pointers $\{a_i\}_{i\in I}$ form a tree with root a. For an application node, the pointer a_i is transmitted to both the left and right subtrees. For an abstraction node with variable v, a new name v' is generated and the pair (v,v') is inserted with a new pointer a_j whose parent is a_i. Pointer a_j will be transmitted to the single subtree. For a variable v, $Find(v,a_i,o)$ searches the tree from a_i to the root and returns the result on channel o. If $v'=\bot$, a leaf with variable v is copied at n'. But if $v'\neq\bot$, instead of a leaf with variable v, a leaf with variable v' is copied.

When two occurrences of x are copying N, the bound names will not be confused because we assign a fresh root a for every x. So every x in M is replaced by a λ-term N', which is α-convertible to N. After all the occurrences of x in M have been replaced by N, the replication stage ends. Formally,

$$\begin{aligned} P_4 \Longrightarrow P_5 &\equiv (s)(\widetilde{x'})(\widetilde{n'})(Sem\,|\,(x)[\![x{:=}N]\!]_\bot\,|\,T^{n,\lambda,\bot}_{C[M\{N/x\}]})\\ &\equiv (s)(\widetilde{x'})(\widetilde{n'})(Sem\,|\,T^{n,\lambda,\bot}_{C[M\{N/x\}]})\\ &\equiv [\![C[M\{N/x\}]]\!]_\lambda, \end{aligned}$$

where $\widetilde{x'}=\widetilde{x}\cup\widetilde{x''}\setminus\{x\}$ and $\widetilde{x''}$ is the set of the bound variables introduced by the copies of N for all the occurrences of x in M.

Ideally, the simulation of β-reductions would proceed as follows: when a reduction stage has completed, the replication stage begins, then, after all the replications are complete, a new reduction stage starts, and so on. In other words the reduction $O\to O'\to O''\to\ldots$ is simulated by

$$[\![O]\!]_\lambda \underbrace{\overset{\tau}{\Longrightarrow}}_{reduction} \underbrace{\Longrightarrow}_{replication} [\![O']\!]_\lambda \underbrace{\overset{\tau}{\Longrightarrow}}_{reduction} \underbrace{\Longrightarrow}_{replication} \cdots.$$

In practice, the encoding can start a reduction stage provided Sem is ready; and it can begin a replication stage of some x provided $[\![x{:=}N]\!]_e$ is available. If we use $\xrightarrow{\tau}_1$ to denote the τ-action performed in a reduction stage, and $\xrightarrow{\tau}_2$ in a replication stage, a transition sequence looks like

$$[\![O]\!]_\lambda \overset{\tau}{\Longrightarrow}_1 \overset{\tau}{\Longrightarrow}_2 \overset{\tau}{\Longrightarrow}_1 \overset{\tau}{\Longrightarrow}_2 \cdots.$$

The interleaving between the replication and reduction stages is harmless, though it does make the analysis of the simulations a little nasty. The semaphore and the fact that $L(n,p,l,r,v,f) \xrightarrow{\overline{n}\langle p,l,r,v,f\rangle} \mathbf{0}$ for every node are the basic mechanisms ensuring that the interleaving does not have an adverse effect on the simulation.

5.2. *Input actions*

When the λ-term being encoded is an abstraction $\lambda x.M$, the translation may also perform the input action λz. According to the definition of the encoding and the structural congruence,

$$[\![\lambda x.M]\!]_\lambda \equiv (s)(Sem \mid (\widetilde{x})(\widetilde{n})(L(n,\lambda,m,m,x,\bot) \mid T_M^{m,n,\bot})),$$

where $\widetilde{x}$ is the set of all the bound variables in M and $\widetilde{n}$ is the set of all the names of the tree node. By the definition of the node,

$$L(n,\lambda,m,m,x,\bot) \xrightarrow{\lambda z} L_1 \equiv s.m(p_1,l_1,r_1,v_1,f_1).(b)(Backup(z,x,b,\top) \\ \mid b.\overline{s}.L(m,\lambda,l_1,r_1,v_1,\bot)).$$

Consequently,

$$[\![\lambda x.M]\!]_\lambda \xrightarrow{\lambda z} P_1 \equiv (s)(Sem \mid (\widetilde{x})(\widetilde{n})(L_1 \mid T_M^{m,n,\bot})).$$

At this stage, the translation can still engage in a β-reduction. But if it sets the semaphore to Sem^-, the translation moves to an input stage, followed by a replication stage.

5.2.1. *Input stages.* In an input stage, a replica of the encoding of a term imported from the environment is made. This is again achieved mainly by the *Backup* process. To kick off the input stage, P_1 first consumes the prefix s in L_1, and then reads the root of T_M:

$$P_1 \xrightarrow{\tau} P_2 \xrightarrow{\tau} Q_0 \equiv (s)(Sem^- \mid (\widetilde{x})(\widetilde{n})(b)(Backup(z,x,b,\top) \\ \mid b.\overline{s}.L(m,\lambda,l_1,r_1,v_1,\bot) \mid RT_M^{m,n,\bot})).$$

The root m will not be reinstated before $Backup(z,x,b,\top)$ ends. This is different from the situation in the reduction stage, where the tree structure is adjusted at the same time as the process $Backup(r,x,b,\bot)$ begins. The reason is that the root of $T_M^{m,n,\bot}$ might also be an abstraction node. If we change its parent n into λ, then a new action $\xrightarrow{\lambda z'}$ for some z' is possible. The action $\xrightarrow{\lambda z'}$ may be harmless to the correctness of the encoding, but we prefer the present encoding for its robustness.

The process $Backup(z,x,b,\top)$ works in a similar way to $Backup(z,x,b,\bot)$. It backs up the tree rooted at z from the environment. For each closed λ-term N, we have

$$(z)(Q_0 \mid T(z,N)) \overset{\tau}{\Longrightarrow} (s)(\widetilde{x})(\widetilde{n})(Sem \mid T_M^{n,\lambda,\bot} \mid [\![x{:=}N]\!]_\top).$$

While we are completely assured that the grammar of $T(z,N)$ is in its correct form, we are not at all certain if the term being imported from the environment is good. A term may be ill-defined because:

— It contains free variables; the name used to denote these free variables might even be $\bot$, $\top$ or λ.
— The tree nodes do not have the desired format.
— The node names are name variables.
— Two nodes share the same name.
— A node has no children, but neither its left or right child parameter is the special name $\bot$.

However, the process $Backup(z,x,b,\top)$ works correctly even if a term has these problems by essentially *not using* any names imported from the environment in any *prefix*. We now explain how the process $Backup(z,x,b,\top)$ corrects these syntactical errors by considering all possibilities for the input action $\xrightarrow{n(p,l,r,v,f)}$:

— Consider the process $Backup(n,x,b,e)$ defined in Figure 3. The names received to instantiate the parameters p,f of the prefix $n(p,l,r,v,f)$ do not appear anywhere in the rest of the process. In fact, the parent and function information of a node is ignored by $Backup(n,x,b,e)$. The information conveyed in the four parameters n,l,r,v is enough for the backup process.

— If $l=\perp \wedge r \neq \perp$ or $l \neq \perp \wedge r=\perp$, an incorrect form of a node is encountered, and the process $Backup(z,x,b,\top)$ handles it as if it were a leaf.

— If $v=\perp$ or $v=\top$ or $v=\lambda$, these two names are very special in our encoding, but it does not matter if they are imported as v in an abstraction node or a leaf because in the replacement stage, the variable in an abstraction node will be renamed and the leaf with free variables will be replaced by Ω.

Consequently, $Backup(z,x,b,\top)$ acts correctly for all inputs from the environment.

Since environments do not always provide sufficient information, the backup procedure may either succeed or get stuck. In the case of a successful backup, we have the following transition:

$$\begin{array}{rcl}
Q_0 & \xrightarrow{n_0(p_0,l_0,r_0,v_0,f_0)} & Q_1 \\
 & \overset{\tau}{\Longrightarrow} & Q_1' \\
 & \xrightarrow{n_1(p_1,l_1,r_1,v_1,f_1)} & Q_2 \\
 & \overset{\tau}{\Longrightarrow} & Q_2' \\
 & & \vdots \\
 & \xrightarrow{n_k(p_k,l_k,r_k,v_k,f_k)} & Q_k \\
 & \overset{\tau}{\Longrightarrow} & Q_k' \\
 & \equiv & (s)(Sem^- \mid (\widetilde{x})(\widetilde{n})(\overline{s}.L(m,\lambda,l_1,r_1,v_1,\perp) \mid RT_M^{m,n,\perp}) \mid [\![x{:=}N]\!]_\top),
\end{array}$$

where $n_0=z$, $k \geqslant 1$ and N is a closed λ-term. After it has dealt with all the leaves of a tree, the process Q_k broadcasts this fact upward through the tree, with the help of b_i's, to release $[\![x{:=}N]\!]_\top$. Finally, the process Q_k' resets Sem^- to Sem through the action

$$Q_k' \xrightarrow{\tau} P_3 \equiv (s)(Sem \mid (\widetilde{x})(\widetilde{n})(T_M^{m,\lambda,\perp} \mid [\![x{:=}N]\!]_\top)),$$

which completes the input stage. So a successful backup always imports, as it were, a closed λ-term from the environment, no matter how corrupted the information provided by the environment is.

The backup procedure may fail when it is ready to read node information but the environment refuses to provide any. In this case nothing can ever happen. One might wonder why it is not a good idea to let the input stage get stuck once corruption is detected. The truth is that a more complex encoding would be necessary to implement that strategy since there are many ways an input term may be corrupted.

5.2.2. *Replication stages.* A replication stage in the current situation is almost the same as the replication stage in Section 5.1.2. The only difference is that the parameter e in $[\![x{:=}N]\!]_e$ takes the value $\top$ in the present case. As in the previous situation, we have

$$\begin{aligned} P_3 \Longrightarrow P_4 &\equiv (s)(\widetilde{x'})(\widetilde{n'})(Sem \,|\, (x)[\![x{:=}N]\!]_\top \,|\, T^{n,\lambda,\perp}_{M\{N/x\}}) \\ &\equiv (s)(\widetilde{x'})(\widetilde{n'})(Sem \,|\, T^{n,\lambda,\perp}_{M\{N/x\}}) \\ &\equiv [\![M\{N/x\}]\!]_\lambda, \end{aligned}$$

where $\widetilde{x'} = \widetilde{x} \cup \widetilde{y} \setminus \{x\}$, $\widetilde{y}$ contains the newly generated bound variables in every occurrence of N, and $\widetilde{n'}$ is the set of the names of the nodes of the original tree apart from the root name plus the names of the nodes in the translation of N.

As in the previous subsection, the replication stage interleaves with other stages, and this kind of interleaving is again harmless.

5.3. *Correspondence property*

It is not difficult to see that the key step of the simulation of a β-reduction is indicated by the change of the state of the semaphore. By changing the semaphore from its positive state Sem to its negative state Sem^-, the interpretation initiates the simulation. After the semaphore is set back to Sem, the translation signals the completion of the simulation. This suggests we classify all the descendants of the translations according to the state of the semaphore. If M is an abstraction term $\lambda x.M'$, then the interpretation of M can perform a labelled action λz for each z. After the labelled action, the descendants of the interpretation can also be classified by the two states of the semaphore. These observations are formalised in the following definition.

Definition 3. Let $M, \lambda x.M'$ be closed λ-terms, z be a name and LT be a labelled tree. The sets $\mathscr{T}_M$, $\mathscr{B}_M$, $\mathscr{LT}^z_{\lambda x.M'}$, $\mathscr{LB}^z_{\lambda x.M'}$ and $\mathscr{L}^{LT}_{\lambda x.M'}$ are defined by the following inductions:

(1) $[\![M]\!]_\lambda \in \mathscr{T}_M$.

(2) Suppose $P \in \mathscr{T}_M$. Then:

(a) If $P \xrightarrow{\tau} P' \equiv (\widetilde{n})(Sem \,|\, Q)$ for some Q and $\widetilde{n}$ with $s \in \widetilde{n}$, then $P' \in \mathscr{T}_M$.

(b) If $P \xrightarrow{\tau} P' \equiv (\widetilde{n})(Sem^- \,|\, Q)$ for some Q and $\widetilde{n}$ with $s \in \widetilde{n}$, and if there exists some Q' such that $Q \Longrightarrow Q' \not\rightarrow$ and

$$(\widetilde{n})(Sem^- \,|\, Q') \xrightarrow{\tau} (\widetilde{n})(Sem \,|\, Q'') \equiv [\![M']\!]_\lambda$$

for some M', then $P' \in \mathscr{B}_{M'}$.

(c) If $M \equiv \lambda x.M'$ and $P \xrightarrow{\lambda z} P'$, then $P' \in \mathscr{LT}^z_M$.

(3) Suppose $P \in \mathscr{B}_M$. Then:

(a) If $P \xrightarrow{\tau} P' \equiv (\widetilde{n})(Sem^- \,|\, Q)$ for some Q and $\widetilde{n}$ with $s \in \widetilde{n}$, then $P' \in \mathscr{B}_M$.

(b) If $P \xrightarrow{\tau} P' \equiv (\widetilde{n})(Sem \,|\, Q)$ for some Q and $\widetilde{n}$ with $s \in \widetilde{n}$, then $P' \in \mathscr{T}_M$.

(c) If $M \equiv \lambda x.M'$ and $P \xrightarrow{\lambda z} P'$, then $P' \in \mathscr{LB}^z_M$.

(4) Suppose $P \in \mathscr{LT}^z_{\lambda x.M'}$. Then:

(a) If $P \xrightarrow{\tau} P' \equiv (\widetilde{n})(Sem \,|\, Q)$ for some Q and $\widetilde{n}$ with $s \in \widetilde{n}$, then $P' \in \mathscr{LT}^z_{\lambda x.M'}$.

(b) If $P \xrightarrow{\tau} P' \equiv (\widetilde{n})(Sem^- \,|\, Q)$ and if there exist some P'', P''' such that $P' \xrightarrow{\tau} P'' \xrightarrow{z(p,l,r,v,f)} P'''$, then $P' \in \mathscr{L}^{\mathbf{1}_z}_{\lambda x.M'}$.

(5) Suppose $P \in \mathscr{LB}^z_{\lambda x.M'}$. Then:

If $P \xrightarrow{\tau} P' \equiv (\widetilde{n})(Sem^- \,|\, Q)$ for some Q and $\widetilde{n}$ with $s \in \widetilde{n}$, then $P' \in \mathscr{LB}^z_{\lambda x.M'}$.

(6) Suppose $P \in \mathscr{L}^{LT}_{\lambda x.M'}$. Then:

(a) If $P \xrightarrow{\tau} P' \equiv (\widetilde{n})(Sem^- \,|\, Q)$ for some Q and $\widetilde{n}$ with $s \in \widetilde{n}$, then $P' \in \mathscr{L}^{LT}_{\lambda x.M'}$.

(b) If $P \xrightarrow{n(p,l,r,v,f)} P'$, then $P' \in \mathscr{L}^{LT\cdot(n,l,r,v)}_{\lambda x.M'}$.

(c) If $P \xrightarrow{\tau} P' \equiv (\widetilde{n})(Sem \,|\, Q)$ for some Q and $\widetilde{n}$ with $s \in \widetilde{n}$, and if there exist some Q' and some closed λ-term N such that $Q \Longrightarrow Q' \not\rightarrow$ and $(\widetilde{n})(Sem \,|\, Q') \equiv [\![N]\!]_\lambda$, then $P' \in \mathscr{T}_N$.

The π-processes in $\mathscr{T}_M$ are the translations of the λ-term M. So each closed λ-term is not just interpreted by a single π-process, but rather by a set of π-processes. The set $\mathscr{B}_M$ consists of the translations of the λ-term M with an unfinished backup task for the last reduction. A process in $\mathscr{T}_M$ has no unfinished 'β-reduction'. This is indicated by the fact that the semaphore is in the positive state Sem. On the other hand, a process in $\mathscr{B}_M$ does have an unfinished 'β-reduction', indicated by the negative state Sem^- of the semaphore. If the λ-term is an abstraction, then the π-processes in $\mathscr{T}_M$ and $\mathscr{B}_M$ can do the observable action λz, which takes the π-processes to $\mathscr{LT}^z_M$ and $\mathscr{LB}^z_M$, respectively. A π-process in $\mathscr{L}^{LT}_{\lambda x.M'}$ can repeatedly read the node information from the environment until all the leaves of the labelled tree are labelled with $\bot$. While it is doing this, it expands the labelled tree LT and stays in $\mathscr{L}^{LT}_{\lambda x.M'}$. By the end of the procedure, the shape of the labelled tree LT is precisely the shape of the interpretation tree of some closed λ-term N.

Definition 3 summarises the correspondence between the β-reductions of the closed λ-terms and the τ-actions of the interpretations of the λ-terms. The next lemma explains the correspondence in one direction.

Lemma 4. Suppose $M \to M'$ for closed λ-terms M, M'. The following statements are valid:

(1) If $P \in \mathscr{T}_M$, then $\exists P'.P \overset{\tau}{\Longrightarrow} P' \in \mathscr{T}_{M'}$.
(2) If $P \in \mathscr{B}_M$, then $\exists P'.P \overset{\tau}{\Longrightarrow} P' \in \mathscr{T}_{M'}$.
(3) If $P \in \mathscr{LT}^z_M$, then $\exists P'.P \overset{\tau}{\Longrightarrow} P' \in \mathscr{LT}^z_{M'}$.
(4) If $P \in \mathscr{LB}^z_M$, then $\exists P'.P \overset{\tau}{\Longrightarrow} P' \in \mathscr{LT}^z_{M'}$.

Conversely, the actions of the interpretations of a closed λ-term reflect the β-reductions of the λ-term. This is described in the following lemma.

Lemma 5. Suppose $M, \lambda x.M'$ are closed λ-terms. Then:

(1) There is no infinite τ-action sequence

$$P_0 \xrightarrow{\tau} P_1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} P_i \xrightarrow{\tau} \cdots$$

in any of $\mathscr{T}_M$, $\mathscr{B}_M$, $\mathscr{LT}^z_{\lambda x.M'}$, $\mathscr{LB}^z_{\lambda x.M'}$ and $\mathscr{L}^{LT}_{\lambda x.M'}$.

(2) Suppose $P \in \mathscr{T}_M$. Then:

(a) If $P \xrightarrow{\tau} P'$, then either $P' \in \mathscr{T}_M$ or M_1 exists such that $M \rightarrow M_1$ and $P' \in \mathscr{B}_{M_1}$.

(b) If $P \xrightarrow{\lambda z} P'$, then M must be an abstraction term and $P' \in \mathscr{L}\mathscr{T}^z_M$. If M is an abstraction term, then $\exists P'.P \Longrightarrow\xrightarrow{\lambda z} P' \in \mathscr{L}\mathscr{T}^z_M$.

(c) If $P \xrightarrow{\mu} P'$, then either $\mu = \tau$ or $\mu = \lambda z$ for some z.

(3) Suppose $P \in \mathscr{B}_M$. Then:

(a) If $P \xrightarrow{\tau} P'$, then $P' \in \mathscr{B}_M \cup \mathscr{T}_M$.

(b) If $P \xrightarrow{\lambda z} P'$, then M must be an abstraction term and $P' \in \mathscr{L}\mathscr{B}^z_M$. If M is an abstraction term, then $\exists P'.P \Longrightarrow\xrightarrow{\lambda z} P' \in \mathscr{L}\mathscr{B}^z_M$.

(c) If $P \xrightarrow{\mu} P'$, then either $\mu = \tau$ or $\mu = \lambda z$ for some z.

(4) Suppose $P \in \mathscr{L}\mathscr{T}^z_{\lambda x.M'}$. Then:

(a) If $P \xrightarrow{\tau} P'$, then either $P' \in \mathscr{L}\mathscr{T}^z_{\lambda x.M'} \cup \mathscr{L}^{\mathbf{1}_z}_{\lambda x.M'}$ or M_1 exists such that $M' \rightarrow M_1$ and $P' \in \mathscr{L}\mathscr{B}^z_{\lambda x.M_1}$.

(b) There exists P' such that $P \xrightarrow{\tau} P' \in \mathscr{L}^{\mathbf{1}_z}_{\lambda x.M'}$.

(c) P can and can only perform τ-actions.

(5) Suppose $P \in \mathscr{L}\mathscr{B}^z_{\lambda x.M'}$. Then:

(a) If $P \xrightarrow{\tau} P'$, then $P' \in \mathscr{L}\mathscr{B}^z_{\lambda x.M'} \cup \mathscr{L}\mathscr{T}^z_{\lambda x.M'}$.

(b) P can and can only perform τ-actions.

(6) Suppose $P \in \mathscr{L}^{LT}_{\lambda x.M'}$. Then:

(a) If $P \xrightarrow{\tau} P'$, then either $P' \in \mathscr{L}^{LT}_{\lambda x.M'}$ or $P' \in \mathscr{T}_{M'\{N/x\}}$ for some closed λ-term N.

(b) If $P \xrightarrow{\tau} P' \in \mathscr{T}_{M'\{N/x\}}$ for some closed λ-term N, then for every closed λ-term $\lambda x.M''$ and every $P_1 \in \mathscr{L}^{LT}_{\lambda x.M''}$ some P'_1 exists such that $P_1 \overset{\tau}{\Longrightarrow} P'_1 \in \mathscr{T}_{M''\{N/x\}}$.

(c) If $P \xrightarrow{n(p,l,r,v,f)} P'$, then $P' \in \mathscr{L}^{LT\cdot(n,l,r,v)}_{\lambda x.M'}$. Moreover, for every closed λ-term $\lambda x.M''$ and every $P_1 \in \mathscr{L}^{LT}_{\lambda x.M''}$ some P'_1 exists such that $P_1 \overset{n(p,l,r,v,f)}{\Longrightarrow} P'_1 \in \mathscr{L}^{LT\cdot(n,l,r,v)}_{\lambda x.M''}$.

(d) P can and can only do either τ-actions or input actions at names in $I(TL)$.

If we want to construct a relation $\mathscr{R}$ from Λ^0 to $\mathscr{P}$ that demonstrates the correspondence between the closed λ-terms and their translations, then, inevitably, $\mathscr{R}$ should contain (M, P) for every process $P \in \mathscr{T}_M \cup \mathscr{B}_M$. However, we need to throw more elements into $\mathscr{R}$ in order to close up the argument. Hence, we have the next definition.

Definition 4. For closed λ-term $\lambda x.M$ and N, let $\mathscr{C}\mathscr{T}^N_{\lambda x.M}$, $\mathscr{C}\mathscr{B}^N_{\lambda x.M}$ and $\mathscr{C}^N_{\lambda x.M}$ be the smallest sets satisfying the following properties:

(1) If $P \in \mathscr{L}\mathscr{T}^z_{\lambda x.M}$, then $(z)(P \mid T(z, N)) \in \mathscr{C}\mathscr{T}^N_{\lambda x.M}$.

(2) If $P \in \mathscr{L}\mathscr{B}^z_{\lambda x.M}$, then $(z)(P \mid T(z, N)) \in \mathscr{C}\mathscr{B}^N_{\lambda x.M}$.

(3) If $P \in \mathscr{L}^{\mathbf{1}_z}_{\lambda x.M}$, then $(z)(P \mid T(z, N)) \in \mathscr{C}^N_{\lambda x.M}$.

(4) If $P \in \mathscr{C}^N_{\lambda x.M}$ and $P \xrightarrow{\tau} P' \equiv (\widetilde{n})(Sem^- \mid Q)$ for some Q and $\widetilde{n}$ with $s \in \widetilde{n}$, then $P' \in \mathscr{C}^N_{\lambda x.M}$.

By definition, the π-process $T(z, N)$ provides the static information of the interpretation structure of N. The only action it can ever do is to interact with $Backup(z, x, b, \top)$. Intuitively, an element of $\mathscr{C}\mathscr{T}^N_{\lambda x.M}$ is an interpretation of $\lambda x.M$ that is destined to be involved in the import of N from an environment. The difference between $\mathscr{C}\mathscr{B}^N_{\lambda x.M}$ and $\mathscr{C}\mathscr{T}^N_{\lambda x.M}$ is the same as that between $\mathscr{B}_M$ and $\mathscr{T}_M$. A process in $\mathscr{C}^N_{\lambda x.M}$ is an interpretation of $\lambda x.M$ that is already engaged in the import.

The classification described in Definition 4 also enjoys a kind of correspondence property.

Lemma 6. Suppose $\lambda x.M$ and N are closed λ-terms. Then:

(1) There is no infinite τ-action sequence

$$P_0 \xrightarrow{\tau} P_1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} P_i \xrightarrow{\tau} \cdots$$

in any of $\mathscr{C}\mathscr{T}^N_{\lambda x.M}$, $\mathscr{C}\mathscr{B}^N_{\lambda x.M}$ and $\mathscr{C}^N_{\lambda x.M}$.

(2) If $P \in \mathscr{C}\mathscr{T}^N_{\lambda x.M} \cup \mathscr{C}\mathscr{B}^N_{\lambda x.M} \cup \mathscr{C}^N_{\lambda x.M}$ and $M\{N/x\} \to M'$, then $P \overset{\tau}{\Longrightarrow} P' \in \mathscr{T}_{M'}$ for some P'.

(3) Suppose $P \in \mathscr{C}\mathscr{T}^N_{\lambda x.M}$. Then:

(a) If $P \xrightarrow{\tau} P'$, then either $P' \in \mathscr{C}\mathscr{T}^N_{\lambda x.M} \cup \mathscr{C}^N_{\lambda x.M}$ or M' exists such that $M \to M'$ and $P' \in \mathscr{C}\mathscr{B}^N_{\lambda x.M'}$.

(b) There exists P' such that $P \xrightarrow{\tau} P' \in \mathscr{C}\mathscr{T}^N_{\lambda x.M} \cup \mathscr{C}^N_{\lambda x.M}$.

(c) P can only do τ-actions.

(4) Suppose $P \in \mathscr{C}\mathscr{B}^N_{\lambda x.M}$. Then:

(a) If $P \xrightarrow{\tau} P'$, then $P' \in \mathscr{C}\mathscr{B}^N_{\lambda x.M} \cup \mathscr{C}\mathscr{T}^N_{\lambda x.M}$.

(b) There exists P' such that $P \xrightarrow{\tau} P'$.

(c) P can only do τ-actions.

(5) Suppose $P \in \mathscr{C}^N_{\lambda x.M}$. Then:

(a) If $P \xrightarrow{\tau} P'$, then $P' \in \mathscr{C}^N_{\lambda x.M} \cup \mathscr{T}_{M\{N/x\}}$.

(b) There exists P' such that $P \xrightarrow{\tau} P'$.

(c) P can only do τ-actions.

We will need the following technical lemma in the proof of the full abstraction theorem given in the next section.

Lemma 7. For every closed λ-term M and all P, Q in

$$\mathscr{T}_M \cup \mathscr{B}_M \cup \Big(\bigcup_{L\{N/x\} \equiv M} \mathscr{C}\mathscr{T}^N_{\lambda x.L} \cup \mathscr{C}\mathscr{B}^N_{\lambda x.L} \cup \mathscr{C}^N_{\lambda x.L}\Big),$$

we have $P \approx Q$.

Every element of $\mathscr{T}_M \cup \mathscr{B}_M \cup (\bigcup_{L\{N/x\} \equiv M} \mathscr{C}\mathscr{T}^N_{\lambda x.L} \cup \mathscr{C}\mathscr{B}^N_{\lambda x.L} \cup \mathscr{C}^N_{\lambda x.L})$ can be regarded as an interpretation of the λ-term M according to our encoding. The above lemma essentially states that all the interpretations of a closed λ-term are equivalent.

The proofs of Lemmas 4, 5, 6 and 7 are given in Appendix A.

6. Correctness of the encoding

In this section we justify the encoding defined in Section 4. It has become traditional for such a justification to consist of two parts:

(1) The interpretation $[\![M]\!]_\lambda$ of a closed λ-term M should simulate the β-reduction of M; and it should not introduce any additional internal actions other than those simulations. This property is often referred to as operational soundness and completeness.

(2) The encoding should relate the applicative bisimilarity on the closed λ-terms to the observational equivalence on the interpretations. This correspondence is the so-called full abstraction property.

We shall discuss these two properties in the following subsections.

6.1. *Operational soundness and completeness*

What is required operationally for the π-process P to simulate a λ-term M? Obviously, the β-reduction of M should be simulated non-trivially by the τ-actions of P. Conversely, a sequence of τ-actions of P must, essentially, reflect the β-reduction of M. It may well be the case that P needs to perform some internal adjustments before the real simulation, but these internal adjustments should be completed in a finite number of steps. In addition to this bisimulation property, we should have $M \rightarrow^* \lambda x.M'$ for some M' if and only if $P \Longrightarrow\xrightarrow{\lambda z} P'$ for some fresh z and some P'. Moreover, P' must also be able to simulate M' in such a way that, for each closed λ-term N, we have $(z)(P' \mid T(z,N))$ simulates $M\{N/x\}$. These remarks lead to the following definition, which is meant to capture the operational soundness and completeness.

Definition 5. Let $\mathscr{R}$ be a relation from Λ^0 to $\mathscr{P}$. It is said to be a *subbisimilarity* if the following properties hold:

(1) $\forall M \in \Lambda^0.\exists P.M\mathscr{R}P$.

(2) If $P\mathscr{R}^{-1}M \rightarrow M'$, then $\exists P'.P \overset{\tau}{\Longrightarrow} P'\mathscr{R}^{-1}M'$.

(3) If $M\mathscr{R}P \xrightarrow{\tau} P'$, then either $\exists M'.M \rightarrow M'\mathscr{R}P'$ or $M\mathscr{R}P'$.

(4) If $M\mathscr{R}P_0$ and $P_0 \xrightarrow{\tau} P_1 \cdots \xrightarrow{\tau} P_i \xrightarrow{\tau} \cdots$ is an infinite sequence of τ-actions, then there must be some $k \geqslant 1$ and M' such that $M \rightarrow M'\mathscr{R}P_k$.

(5) If $\lambda x.M\mathscr{R}P$, then $P \Longrightarrow P' \xrightarrow{\lambda z} P''$ for some fresh z and some P', P'' such that $\lambda x.M\mathscr{R}P'$ and $M\{N/x\}\ \mathscr{R}\ (z)(P'' \mid T(z,N))$ for every $N \in \Lambda^0$.

(6) If $M\mathscr{R}P$ and $P \xrightarrow{\lambda z} P'$ for some fresh z, then $M \equiv \lambda x.M'$ for some x, M' such that $M'\{N/x\}\ \mathscr{R}\ (z)(P' \mid T(z,N))$ for every $N \in \Lambda^0$.

Requirements (2) and (4) imply that the encoding is termination preserving. One might wonder why process $T(z,N)$ is used in clauses (5) and (6) instead of $T_L^{z,\perp,\perp}$. The reason is that we really do not want N to reduce at this point, just as we do not want N to reduce at the point the substitution $\{N/x\}$ is applied to M.

We are now ready to show that our encoding of the λ-calculus preserves and reflects the operational semantics.

Theorem 1. There is a subbisimilarity from Λ^0 to $\mathscr{P}$.

Proof. Let $\mathscr{R}$ be the union of the relations

$$\{(M,P) \mid M \in \Lambda^0 \wedge P \in \mathscr{T}_M \cup \mathscr{B}_M\}$$

and

$$\left\{ (M\{N/x\},P) \,\middle|\, \begin{array}{l} \lambda x.M, N \in \Lambda^0 \text{ and} \\ P \in \mathscr{CT}^N_{\lambda x.M} \cup \mathscr{CB}^N_{\lambda x.M} \cup \mathscr{C}^N_{\lambda x.M} \end{array} \right\}.$$

We argue that $\mathscr{R}$ is a subbisimilarity. This amounts to verifying the six properties of Definition 5:

(1) $\forall M \in \Lambda^0 . M \mathscr{R} [\![M]\!]_\lambda \in \mathscr{T}_M$.

(2) Suppose $M\mathscr{R}P$ and $M \to M'$. There are two cases:

— $P \in \mathscr{T}_M \cup \mathscr{B}_M$. Then by Lemma 4 (1, 2) some P' exists such that $P \overset{\tau}{\Longrightarrow} P' \in \mathscr{T}_{M'}$.

— $P \in \mathscr{CT}^N_{\lambda x.M_1} \cup \mathscr{CB}^N_{\lambda x.M_1} \cup \mathscr{C}^N_{\lambda x.M_1}$ and $M \equiv M_1\{N/x\}$. By Lemma 6 (2), some P' exists such that $P \overset{\tau}{\Longrightarrow} P' \in \mathscr{T}_{M'}$.

(3) Suppose $M\mathscr{R}P \xrightarrow{\tau} P'$. From Lemma 5 (2a, 3a) and Lemma 6 (3a, 4a, 5a), it is easy to see that either $M\mathscr{R}P'$ or $M \to M'\mathscr{R}P'$ for some M'.

(4) Suppose $M\mathscr{R}P_0$ and $P_0 \xrightarrow{\tau} P_1 \cdots \xrightarrow{\tau} P_i \xrightarrow{\tau} \cdots$ is an infinite sequence of τ-actions. It follows from Lemma 5 (1, 2a, 3a) and Lemma 6 (1, 3a, 4a, 5a) that there must be some $k > 0$ and some M' such that $M \to M'\mathscr{R}P_k$.

(5) Suppose $\lambda x.M\mathscr{R}P$. Then:

— If $P \in \mathscr{T}_{\lambda x.M} \cup \mathscr{B}_{\lambda x.M}$, it follows from Lemma 5 (2b, 3b) and Definition 3 (2c, 3c) that some P' exists such that $P \xrightarrow{\lambda z} P' \in \mathscr{LT}^z_{\lambda x.M} \cup \mathscr{LB}^z_{\lambda x.M}$. Therefore

$$(z)(P' \mid T(z,N)) \in \mathscr{CT}^N_{\lambda x.M} \cup \mathscr{CB}^N_{\lambda x.M}$$

by definition. Hence

$$M\{N/x\}\mathscr{R}(z)(P' \mid T(z,N)).$$

— If $\lambda x.M \equiv M'\{N/y\}$ and $P \in \mathscr{CT}^N_{\lambda y.M'} \cup \mathscr{CB}^N_{\lambda y.M'} \cup \mathscr{C}^N_{\lambda y.M'}$, then it follows from Lemma 6 (3a, 3b, 4a, 4b, 5a, 5b) that some P' exists such that $P \overset{\tau}{\Longrightarrow} P' \in \mathscr{T}_{M'\{N/y\}}$. So this case is reduced to the previous case.

(6) Suppose $M\mathscr{R}P$ and $P \xrightarrow{\lambda z} P'$. Then, from Lemma 6 (3c, 4c, 5c), P must be in $\mathscr{T}_M$ or $\mathscr{B}_M$. It follows from Lemma 5 (2b, 3b) that $M \equiv \lambda x.M'$ for some M' and, consequently,

$$P' \in \mathscr{LT}^z_{\lambda x.M'} \cup \mathscr{LB}^z_{\lambda x.M'}.$$

So

$$(z)(P' \mid T(z,N)) \in \mathscr{CT}^N_{\lambda x.M'} \cup \mathscr{CB}^N_{\lambda x.M'}$$

by definition. Hence $M'\{N/x\}\mathscr{R}(z)(P' \mid T(z,N))$. □

6.2. *Full abstraction*

Milner (1992) pointed out that the bisimulation equivalence on his encodings of the lazy λ-calculus induces a relation weaker than β-conversion. Sangiorgi (1994; 1995) pointed out that the induced relation is precisely the open applicative bisimilarity on the open λ-terms. Sangiorgi's open applicative bisimilarity extends Abramsky's applicative bisimilarity from the set of the closed λ-terms to the set of the open λ-terms. We hope to prove a similar result for the present encoding. Such a result, if attainable, would only hold for the closed λ-terms. Take, for instance, the open terms Ωxx and Ωx. Clearly, Ωxx and Ωx are open applicative bisimilar. But $[\![\Omega xx]\!]_u$ is not bisimilar to $[\![\Omega x]\!]_u$. So we should focus on the closed λ-terms.

The difficulty in obtaining a full abstraction result is to make sure that the soundness property is valid. The most significant contribution of this paper is the soundness of our encoding with respect to the applicative bisimilarity. All the complications of our encoding are required to achieve soundness.

Proposition 1. For all closed λ-terms M, N, we have $M =_a N$ implies $[\![M]\!]_\lambda \approx [\![N]\!]_\lambda$.

Proof. The relations $\mathscr{R}_1$, $\mathscr{R}_2$ and $\mathscr{R}_3$ are defined as follows:

$$\mathscr{R}_1 \stackrel{\text{def}}{=} \left\{ (P,Q) \,\middle|\, \begin{array}{l} \exists M,N \in \Lambda^0.(M =_a N \wedge \\ P \in \mathscr{T}_M \cup \mathscr{B}_M \wedge Q \in \mathscr{T}_N \cup \mathscr{B}_N) \end{array} \right\}$$

$$\mathscr{R}_2 \stackrel{\text{def}}{=} \left\{ (P,Q) \,\middle|\, \begin{array}{l} \exists \lambda x.M, \lambda x.N \in \Lambda^0.\exists z.(\lambda x.M =_a \lambda x.N \wedge \\ P \in \mathscr{L}\mathscr{T}^z_{\lambda x.M} \cup \mathscr{L}\mathscr{B}^z_{\lambda x.M} \wedge Q \in \mathscr{L}\mathscr{T}^z_{\lambda x.N} \cup \mathscr{L}\mathscr{B}^z_{\lambda x.N}) \end{array} \right\}$$

$$\mathscr{R}_3 \stackrel{\text{def}}{=} \left\{ (P,Q) \,\middle|\, \begin{array}{l} \exists \lambda x.M, \lambda x.N \in \Lambda^0.(\lambda x.M =_a \lambda x.N \wedge \\ LT \text{ is a labelled tree } \wedge P \in \mathscr{L}^{LT}_{\lambda x.M} \wedge Q \in \mathscr{L}^{LT}_{\lambda x.N}) \end{array} \right\}.$$

These relation are clearly symmetric. Let $\mathscr{R}$ be defined by the relation

$$\mathscr{R} \stackrel{\text{def}}{=} \mathscr{R}_1 \cup \mathscr{R}_2 \cup \mathscr{R}_3.$$

We shall prove that $\mathscr{R}$ is a weak bisimulation. By definition, there are three cases:
Case I: $(P,Q) \in \mathscr{R}_1$.

— $P \in \mathscr{T}_M$. By Lemma 5 (2c), there are two cases:
  - If $P \xrightarrow{\tau} P'$, then by Lemma 5 (2a), either $P' \in \mathscr{T}_M$ or there exists some M' such that $M \to M'$ and $P' \in \mathscr{B}_{M'}$. It follows from $M' =_a M =_a N$ that $P' \mathscr{R}_1 Q$.
  - If $P \xrightarrow{\lambda z} P'$, then, by Lemma 5 (2b) and Definition 3 (2c), $M \equiv \lambda x.M'$ for some M' such that $P' \in \mathscr{L}\mathscr{T}^z_M$. Since $M =_a N$, it must be the case that $N \to^* \lambda x.N'$ for some N' such that $M =_a \lambda x.N'$. Then by Lemma 4 (1, 2) some Q'' exists such that $Q \stackrel{\tau}{\Longrightarrow} Q'' \in \mathscr{T}_{\lambda x.N'}$. By Lemma 5 (2b) and Definition 3 (2c), we have $Q'' \xrightarrow{\lambda z} Q' \in \mathscr{L}\mathscr{T}^z_{\lambda x.N'}$. Therefore $P' \mathscr{R}_2 Q'$.

— $P \in \mathscr{B}_M$. By Lemma 5 (3c), there are two cases:
  - If $P \xrightarrow{\tau} P'$, then $P' \in \mathscr{B}_M \cup \mathscr{T}_M$ by Lemma 5 (3a). Hence, $P' \mathscr{R}_1 Q$.
  - If $P \xrightarrow{\lambda z} P'$, then $M \equiv \lambda x.M'$ for some x, M' and $P' \in \mathscr{L}\mathscr{B}^z_M$ by Lemma 5 (3b) and Definition 3 (3c). Since $M =_a N$, it must be the case that $N \to^* \lambda x.N'$ for some N' such that $M =_a \lambda x.N'$. Then by Lemma 4 (1, 2), some Q'' exists such that $Q \stackrel{\tau}{\Longrightarrow} Q'' \in \mathscr{T}_{\lambda x.N'}$. By Lemma 5 (2b) and Definition 3 (2c), we have $Q'' \xrightarrow{\lambda z} Q' \in \mathscr{L}\mathscr{T}^z_{\lambda x.N'}$. So $P' \mathscr{R}_2 Q'$.

Case II: $(P, Q) \in \mathscr{R}_2$.

— $P \in \mathscr{L}\mathscr{T}^z_{\lambda x.M}$. By Lemma 5 (4c), there is one case:

– If $P \xrightarrow{\tau} P'$, then by Lemma 5 (4a), either $P' \in \mathscr{L}\mathscr{T}^z_{\lambda x.M} \cup \mathscr{L}^{\mathbf{1}_z}_{\lambda x.M}$ or there exists some M' such that $M \rightarrow M'$ and $P' \in \mathscr{L}\mathscr{B}^z_{\lambda x.M'}$. If $P' \in \mathscr{L}\mathscr{T}^z_M$, then $P'\mathscr{R}_2 Q$. If $P' \in \mathscr{L}^{\mathbf{1}_z}_{\lambda x.M}$, then by Lemma 5 (1, 4a, 4b, 5a), there exist some Q'', Q' such that $Q \Longrightarrow Q'' \in \mathscr{L}\mathscr{T}^z_{\lambda x.N}$ and $Q'' \xrightarrow{\tau} Q' \in \mathscr{L}^{\mathbf{1}_z}_{\lambda x.N}$. So $P'\mathscr{R}_3 Q'$. If $P' \in \mathscr{L}\mathscr{B}^z_{M'}$, then $P'\mathscr{R}_2 Q$ because $M' =_a M =_a N$.

— $P \in \mathscr{L}\mathscr{B}^z_{\lambda x.M}$. By Lemma 5 (5b), there is one case:

– If $P \xrightarrow{\tau} P'$, then $P' \in \mathscr{L}\mathscr{T}^z_{\lambda x.M} \cup \mathscr{L}\mathscr{B}^z_{\lambda x.M}$ by Lemma 5 (5a). Hence $P'\mathscr{R}_2 Q$.

Case III: $(P, Q) \in \mathscr{R}_3$.

— $P \in \mathscr{L}^{LT}_{\lambda x.M}$. By Lemma 5 (6d), there are two cases:

– According to Lemma 5 (6a), if $P \xrightarrow{\tau} P'$, then either $P' \in \mathscr{L}^{LT}_{\lambda x.M}$ or $P' \in \mathscr{T}_{M\{L/x\}}$ for some closed λ-term L. In the former case, $P'\mathscr{R}_3 Q$. In the latter case, $Q \overset{\tau}{\Longrightarrow} Q' \in \mathscr{T}_{N\{L/x\}}$ for some Q' due to Lemma 5 (6b).

– If $P \xrightarrow{n(p,l,r,v,f)} P'$, then $P' \in \mathscr{L}^{LT\cdot(n,l,r,v)}_{\lambda x.M}$ by Lemma 5 (6c). For the same reason, $Q \overset{n(p,l,r,v,f)}{\Longrightarrow} Q' \in \mathscr{L}^{LT\cdot(n,l,r,v)}_{\lambda x.N}$ for some Q'. Clearly, $P'\mathscr{R}_3 Q'$.

We may now conclude that $\mathscr{R}$ is a weak bisimulation. Hence $\mathscr{R} \subseteq \approx$. It then follows from the definition of $\mathscr{R}$ that $M =_a N$ implies $[\![M]\!]_\lambda \approx [\![N]\!]_\lambda$. $\square$

The proof of the above proposition is actually given for the polyadic π^{def}, but, in fact, the result is also valid for the monadic π^{def}. The translation of the polyadic π^{def} to the monadic π^{def} given in Section 2.2 is not sound with respect to $\approx$. For instance, $a(x, y) \mid b(u, v)$ is equivalent to $a(x, y).b(u, v) + b(u, v).a(x, y)$ in the polyadic π^{def}. But their translations in the monadic π^{def} are not equivalent. However, if we focus on the set of the interpretations of the closed λ-terms, the encoding of Section 2.2 is fully abstract, and we will now explain why. If we take the set of the translations of the polyadic π^{def}-processes as the set of the observers for the monadic π^{def}-processes, then we get an equivalence relation $\approx_m$ on the set of the monadic π^{def}-processes. This equivalence is strictly weaker than $\approx$, that is $\approx \subsetneq \approx_m$. But if two bisimilar polyadic π^{def}-processes are interpretations of closed λ-terms, then their translations into the monadic π^{def}-calculus are equivalent with respect to $\approx_m$. A formal proof is tedious, though the idea is very simple. The main points are as follows. Suppose $P\mathscr{R}Q$, where $\mathscr{R}$ is the relation defined in the above proof. The set of the actions P, Q can perform may be classified into three groups.

— The first group consists of the actions that, essentially, simulate the β-reductions. These interactions are all carried out at the local names. The set of the observers can never interfere with these interactions. Formally, we have, for example,

$$(a)C[a(x, y).T \mid \overline{a}\langle b, c\rangle.O] \approx (a)C[a(z).z(x).z(y).T \mid \overline{a}(e).\overline{e}(b).\overline{e}(c).O],$$

where $C[\_]$ represents some environment.

— The second group has the λ-action like λz. The name λ carries only one parameter. It is automatically a monadic prefix. An observation at λ in the polyadic π^{def} is the same as in the monadic π^{def}.

— The third group contains the input actions, which import the node information from the environment. Now suppose $P \in \mathscr{L}^{LT}_{\lambda x.M}$, $Q \in \mathscr{L}^{LT}_{\lambda x.N}$ and $\lambda x.M =_a \lambda x.N$. After it has been translated to the monadic π^{def}, P may perform an unintended corrupt action due to the decomposition of polyadic actions to monadic actions. But the point is that Q can do precisely the same unintended corrupt actions. This is because the input actions are all contributed by the *Backup* subroutine. According to the definition of *Backup*, the effect of inputting bad information is local. Moreover, *Backup* is designed in such a way that no matter what information it gets from the environment, it always discards the information it receives and produces good replicas.

In fact, we do not need to refer to the encoding of the polyadic π^{def} into the monadic π^{def}. If we understand the polyadic notation as an abbreviation in the monadic calculus, the above proof of Proposition 1 can be rephrased to produce a longer proof for the encoding in the monadic π^{def}. The present proof can be seen as a shorthand version of that longer proof.

In the other direction, the encoding also reflects the applicative bisimilarity in the sense of the next proposition.

Proposition 2. If $M, N \in \Lambda^0$, then $[\![M]\!]_\lambda \approx [\![N]\!]_\lambda$ implies $M =_a N$.

Proof. We will prove that the symmetric relation

$$\mathscr{R} \stackrel{\mathrm{def}}{=} \{(M,N) \mid [\![M]\!]_\lambda \approx [\![N]\!]_\lambda\}$$

is an applicative bisimulation. If $M \to M'$, then $M' =_a M$ by Lemma 1. It follows from Proposition 1 that $[\![M']\!]_\lambda \approx [\![M]\!]_\lambda \approx [\![N]\!]_\lambda$ and, consequently, $M'\mathscr{R}N$. Therefore we only need to consider the case in which at least one of M, N is an abstraction term.

Now suppose $[\![M]\!]_\lambda \approx [\![N]\!]_\lambda$ and $M \equiv \lambda x.M'$. Then $[\![M]\!]_\lambda \xrightarrow{\lambda z} P' \in \mathscr{L}\mathscr{T}^{z}_{\lambda x.M'}$ for some P' and some fresh name z. This action must be simulated by

$$[\![N]\!]_\lambda \Longrightarrow Q''' \xrightarrow{\lambda z} Q'' \Longrightarrow Q' \approx P'$$

for some Q''', Q'', Q'. According to Lemma 5 (2a, 2b, 3a, 3b), we have $Q''' \in \mathscr{T}_{\lambda x.N_1} \cup \mathscr{B}_{\lambda x.N_1}$ for some N_1 such that $N \to^* \lambda x.N_1$. Hence, $Q'' \in \mathscr{L}\mathscr{T}^{z}_{\lambda x.N_1} \cup \mathscr{L}\mathscr{B}^{z}_{\lambda x.N_1}$. Using Lemma 5 (4a, 5a), we get that

$$Q' \in \mathscr{L}\mathscr{T}^{z}_{\lambda x.N_2} \cup \mathscr{L}\mathscr{B}^{z}_{\lambda x.N_2} \cup \mathscr{L}^{\mathbf{1}_z}_{\lambda x.N_2}$$

for some N_2 such that $N_1 \to^* N_2$. By the congruence property of $\approx$, we have

$$(z)(P' \mid T(z,L)) \approx (z)(Q' \mid T(z,L))$$

for each closed λ-term L. By definition,

$$(z)(P' \mid T(z,L)) \in \mathscr{C}\mathscr{T}^{L}_{\lambda x.M'}$$

and

$$(z)(Q' \mid T(z,L)) \in \mathscr{CT}^{L}_{\lambda x.N_2} \cup \mathscr{CB}^{L}_{\lambda x.N_2} \cup \mathscr{C}^{L}_{\lambda x.N_2}.$$

Now

$$(z)(P' \mid T(z,L)) \approx [\![M'\{L/x\}]\!]_\lambda \qquad (3)$$

and

$$(z)(Q' \mid T(z,L)) \approx [\![N_2\{L/x\}]\!]_\lambda. \qquad (4)$$

The equivalence (3) can be proved by induction on the structures of P' and $T(z,L)$. Notice that the structure of P' is derived from that of M'. The equivalence (4) can be proved similarly. It then follows from (3), (4) and Lemma 7 that

$$[\![M'\{L/x\}]\!]_\lambda \approx [\![N_2\{L/x\}]\!]_\lambda.$$

In summary, $N \rightarrow^* \lambda x.N_2$ and $N_2\{L/x\}\mathscr{R}M'\{L/x\}$ for all $L \in \Lambda^0$.

We can now conclude that $\mathscr{R}$ is an applicative bisimulation. □

The full abstraction now follows from Propositions 1 and 2.

Theorem 2. Suppose $M, N \in \Lambda^0$. Then $[\![M]\!]_\lambda \approx [\![N]\!]_\lambda$ if and only if $M =_a N$.

7. Conclusion

In this paper we have advocated a more disciplined methodology for programming with the π-calculus. The idea that the π-calculus naturally supports an object-oriented paradigm has been popularised by the work of Walker (Walker 1991; 1995). An object is a general π-process that may or may not terminate, whereas a method is a replicated form of process that can be invoked a potentially infinite number of times. One contribution of this paper is to formalise the procedure of turning an object into a method in an on-the-fly manner, where the method is supposed to be a replicated form of the object. From a technical viewpoint, the paper exploits the use of data structures to facilitate the object-to-method transfer. An object must be designed with an underlying data structure so that it can be suspended, blueprinted and restored in a replicated form. The use of data structures is crucial to all three phases. To demonstrate the power of the methodology, and associated technicalities, we have applied it to resolving the issue of interpreting the full operational semantics of the λ-calculus.

Our encoding is given at a lower level than Milner's encoding. Two questions arise:

— Is this level of detail necessary?
— What benefit do we get from this low-level interpretation?

We are not in a position to give a definite answer to the first question, but as Milner (1992) pointed out, the structural semantics of the full λ-calculus is incompatible with that of the π-calculus, so a simple structural interpretation of the former in the latter is highly unlikely, and low-level programming appears to be inevitable. The answer to the second question is related. The encoding is more an implementation than a structural translation. One may think of the π-calculus as providing a machine language upon which the 'higher order programming language', the full λ-calculus, is implemented. The significance of the present work, in the light of language implementation, is that it points out the problems and some possible solutions when implementing something in the π-calculus. The extra technicality used in the encoding to achieve the full abstraction is precisely what is necessary when implementing a typed higher order language in the untyped machine language of the π-calculus. At a more fundamental level, the technical novelty exhibited in the encoding is extremely useful in establishing expressiveness results – see Fu and Zhu (2010) and Fu (2010b) for interesting examples, and Fu (2010a) for a systematic exposition.

There are many possible variations of the encoding. At one extreme, is a more deterministic encoding than the one proposed in this paper. In this deterministic variant, the backup procedure is immediately followed by an instantiation procedure whose purpose is to replace all the occurrences of the variable by the imported term. The instantiation procedure is based on a tree-traversal algorithm that inspects all the nodes of the structural tree. The deterministic encoding enjoys an easier correctness proof since it is easier to construct the subbisimilarity. The proof of Theorem 1 is simplified because a lot of parallelism is removed in the deterministic encoding. But it just does not look nice. At the opposite extreme, one could imagine an encoding that does not use the semaphore at all. Such an encoding can probably be regarded as beautiful, but it would make the correctness proof a headache due to the presence of a multitude of interleaving activities. The encoding given in this paper is a trade-off between the simplicity of the encoding and the tractability of the correctness proof.

An important question remains unanswered in this paper. Can the encoding be defined in π^M? The π^{def}-calculus has two constructs that are not present in π^M, *viz.* parametric definition and the case construction. The latter can be decomposed into mismatch, match and guarded choice constructs. Let us consider the possibility of doing away with these constructs:

— It is a kind of folklore that parametric definition is equivalent to replication in the variants of the π-calculus with guarded choice only. This issue is discussed in Milner (1997) and is systematically examined in Fu and Lu (2010). Parametric definition offers a more concise way of expressing a programming idea than replication. But the encoding of this paper can be equivalently given in the variant $\pi^!$ of π^{def}, which uses replication instead of parametric definition.

— The case processes are not required for modelling internal β-reductions. For instance, we could add another four more parameters to the definition of a node. Then, to read from a node named n, we can apply a process of the form

$$n(p,l,r,v,f,c_0,c_1,c_2,c).(c_0.C_0 \,|\, c_1.C_1, \,|\, c_2.C_2 \,|\, \overline{c}).$$

The names received for the parameters c_0, c_1, c_2, c are local names. If $\perp \neq l \neq r \neq \perp$, the received names for c, c_2 are equal. In this case, only C_2 will be fired. The other two cases can be treated similarly. The definition of $L(p,l,r,v,f)$ makes use of static rather than dynamic associations, so this part of the encoding can be done in π^M.

— The real difficulty arises with the part of the encoding that deals with importing terms from the environments since the names read from the environment could be free, which renders the above strategy useless. One solution could be to change from

> the information retrieval viewpoint to an information verification viewpoint. So an interpretation of the abstraction term $\lambda x.M$ does not import any term tree from an environment, but, instead, builds up a replicated form of a term tree by making an enquiry about the shape of the term tree. For instance the interpretation of $\lambda x.M$ may contain a component
>
> $$\overline{\lambda}(u).\overline{u}(c_0).\overline{u}(c_1).\overline{u}(c_2).u(c).(c_0.\_ \mid c_1.\_ \mid c_2.\_ \mid \overline{c}.\_).$$
>
> The process places the query (c_0, c_1, c_2) on the environment. The environment then answers by sending back one of the local names c_0, c_1, c_2 as a verification code. The process then generates part of the tree according to the code and then makes further enquiries if necessary. The description given here of this alternative approach is oversimplified, but it does give an outline. Since this idea sounds so nice, why don't we just go with it? Well, it has some problems. For one, when a leaf is reached, we really need to know which variable it represents. If we cannot come up with some nice answers, we are back to square one.

We tend to believe that there is no fully abstract encoding of the full λ-calculus in π^M. Our intuition is that in the absence of the match and mismatch operators there is no way to produce an encoding that is robust enough to withstand all the attacks from the environment. Proving such a negative conjecture is a much harder challenge. The expressiveness of π^M is one of the most important issues in process theory – see Fu (2010a) for more discussion of this issue.

It is tempting to think that better encodings of the full λ-calculus can be produced using more 'advanced' variants of the π-calculus. For example, the target model can be one of the typed π-calculi (Sangiorgi and Walker 2001). But, types can only refuse syntactically wrong data, and can never rule out tree structures that are type safe but logically wrong. So even in the presence of a strong type system, additional programming is still necessary to achieve the full abstraction property. One may also try to encode the full λ-calculus in the higher order π-calculus (Sangiorgi 1993b). After all, higher order communication is closer to β-reduction than first-order communication is. However, it is unlikely that an exercise of this kind would pay off. The higher order prefix is a sequential operator, whereas the lambda binding is not. The best one can do with the higher order π-calculus is to mimic the (first-order) π-calculus when encoding the low-level activities. So a kind of double encoding is present. The point we are making is that the encoding presented in this paper brings out the intrinsic difficulties of modelling the β-reduction in a process algebraic framework. Additional language features may well complicate rather than simplify the picture.

Appendix A. Proofs of the Lemmas in Section 5.3

Viewed as a program, the encoding given in Figure 4 is small, but it is not small when we want to prove some serious properties about the encoding. The interpretation of a λ-term may engage in complex interleaving activities. The important thing to notice is that the main cause of interleaving is the instantiation of variables by terms. These interleaving activities are conducted in a completely parallel way. The proofs given in this appendix are examples of the type of process theory proof that must be given at an appropriate level of detail, since otherwise the size of the proofs would become unnecessarily large. So, in the following proofs, a number of statements will be made without proofs: the validity of these statements can be checked by routine calculations, but at the expense of occupying much more space.

A.1. *Proofs of Lemmas 4 and 5*

We prove Lemma 8 stated below. It is easy to see that, apart from clauses (2d) and (4d), Lemma 8 is precisely Lemma 5. It is also easy to see that Lemma 4 is a corollary of Lemma 8 (1, 2d, 3a, 4d, 5a).

In the following proofs, we will make use of some abbreviations defined in Section 5, which should be consulted for the details.

Lemma 8. Suppose $M, \lambda x.M'$ are closed λ-terms. Then:

(1) There is no infinite τ-action sequence

$$P_0 \xrightarrow{\tau} P_1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} P_i \xrightarrow{\tau} \cdots$$

in any of $\mathscr{T}_M$, $\mathscr{B}_M$, $\mathscr{LT}^u_{\lambda x.M'}$, $\mathscr{LB}^u_{\lambda x.M'}$ and $\mathscr{L}^{LT}_{\lambda x.M'}$.

(2) Suppose $P \in \mathscr{T}_M$. Then:

(a) If $P \xrightarrow{\tau} P'$, then either $P' \in \mathscr{T}_M$ or $\exists M_1.M \rightarrow M_1 \wedge P' \in \mathscr{B}_{M_1}$.

(b) If $P \xrightarrow{\lambda u} P'$, then M is an abstraction term and $P' \in \mathscr{LT}^u_M$; if M is an abstraction term, then $\exists P'.P \Longrightarrow \xrightarrow{\lambda u} P' \in \mathscr{LT}^u_M$.

(c) If $P \xrightarrow{\mu} P'$, then $\mu = \tau \vee \exists u.\mu = \lambda z$.

(d) If $M \rightarrow M_1$, then $\exists P'.P \overset{\tau}{\Longrightarrow} P' \in \mathscr{B}_{M_1}$.

(3) Suppose $P \in \mathscr{B}_M$. Then:

(a) If $P \xrightarrow{\tau} P'$, then $P' \in \mathscr{B}_M \cup \mathscr{T}_M$.

(b) If $P \xrightarrow{\lambda u} P'$, then M is an abstraction term and $P' \in \mathscr{LB}^u_M$; if M is an abstraction term, then $\exists P'.P \Longrightarrow \xrightarrow{\lambda u} P' \in \mathscr{LB}^u_M$.

(c) If $P \xrightarrow{\mu} P'$, then $\mu = \tau \vee \exists u.\mu = \lambda u$.

(4) Suppose $P \in \mathscr{LT}^u_{\lambda x.M'}$. Then:

(a) If $P \xrightarrow{\tau} P'$, then either $P' \in \mathscr{LT}^u_{\lambda x.M'} \cup \mathscr{L}^{\mathbf{1}_u}_{\lambda x.M'}$ or $\exists M_1.M' \rightarrow M_1 \wedge P' \in \mathscr{LB}^u_{\lambda x.M_1}$.

(b) $\exists P'.P \xrightarrow{\tau} P' \in \mathscr{LT}^u_{\lambda x.M'} \cup \mathscr{L}^{\mathbf{1}_u}_{\lambda x.M'}$.

(c) P can and can only perform τ-actions.

(d) If $\lambda x.M' \rightarrow \lambda x.M'_1$, then $\exists P'.P \overset{\tau}{\Longrightarrow} P' \in \mathscr{LB}_{M'_1}$.

(5) Suppose P in $\mathscr{LB}^u_{\lambda x.M'}$. Then:

(a) If $P \xrightarrow{\tau} P'$, then $P' \in \mathscr{LB}^u_{\lambda x.M'} \cup \mathscr{LT}^u_{\lambda x.M'}$.

(b) P can and can only perform τ-actions.

(6) Suppose $P \in \mathscr{L}^{LT}_{\lambda x.M'}$. Then:

(a) If $P \xrightarrow{\tau} P'$, then either $P' \in \mathscr{L}^{LT}_{\lambda x.M'}$ or $P' \in \mathscr{T}_{M'\{N/x\}}$ for some closed λ-term N.

(b) If $P \xrightarrow{\tau} P' \in \mathscr{T}_{M'\{N/x\}}$ for some closed λ-term N, then for every closed λ-term $\lambda x.M''$ and every $P_1 \in \mathscr{L}^{LT}_{\lambda x.M''}$, some P_1' exists such that $P_1 \overset{\tau}{\Longrightarrow} P_1' \in \mathscr{T}_{M''\{N/x\}}$.

(c) If $P \xrightarrow{n(p,l,r,v,f)} P'$, then $P' \in \mathscr{L}^{LT\cdot(n,l,r,v)}_{\lambda x.M'}$. Moreover for every closed λ-term $\lambda x.M''$ and every $P_1 \in \mathscr{L}^{LT}_{\lambda x.M''}$, some P_1' exists such that $P_1 \overset{n(p,l,r,v,f)}{\Longrightarrow} P_1' \in \mathscr{L}^{LT\cdot(n,l,r,v)}_{\lambda x.M''}$.

(d) P can and can only do either τ-actions or input actions at names in $I(TL)$.

Proof. Suppose the elements of $\mathscr{N}$ are enumerated by

$$u_0, u_1, \ldots, u_j, \ldots$$

and the elements of **LT**, the set of all the labelled trees, are enumerated by

$$LT_0, LT_1, \ldots, LT_k, \ldots.$$

Let h be a set of the sets of the π-processes having the shape

$$\{t_M, b_M, lb^u_M, lt^u_M, l^{LT}_M \mid M \in \Lambda^0, u \in \mathscr{N}, LT \in \mathbf{LT}\}.$$

An element of h is a set indexed by: a closed λ-term; a pair of a closed λ-term and a name; or a pair of a closed λ-term and a labelled tree. The set of all such h is denoted by H. The function $F : H \to H$ is defined inductively as follows. For each

$$h = \{t_M, b_M, lb^u_M, lt^u_M, l^{LT}_M \mid M \in \Lambda^0, u \in \mathscr{N}, LT \in \mathbf{LT}\},$$

$F(h)$ is

$$\{t'_M, b'_M, lb'^u_M, lt'^u_M, l'^{LT}_M \mid M \in \Lambda^0, u \in \mathscr{N}, LT \in \mathbf{LT}\}$$

whose elements are constructed by the following inductions on all $M \in \Lambda^0$, $u \in \mathscr{N}$ and $LT \in \mathbf{LT}$:

(1) $t'_M = t_M \cup \{[\![M]\!]_\lambda\}$, $b'_M = b_M$, $lt'^u_M = lt^u_M$, $lb'^u_M = lb^u_M$ and $l'^{LT}_M = l^{LT}_M$.

(2) Suppose $P \in t_M$. Then:

(a) If $P \xrightarrow{\tau} P' \equiv (s)(Sem \mid Q)$ for some Q and s, then $t'_M = t'_M \cup \{P'\}$.

(b) If $P \xrightarrow{\tau} P' \equiv (s)(Sem^- \mid Q)$ for some Q and s, and if there exists some Q' such that $Q \Longrightarrow Q' \not\rightarrow$ and $(s)(Sem^- \mid Q') \xrightarrow{\tau} (s)(Sem \mid Q'') \equiv [\![M']\!]_\lambda$ for some M', then $b'_{M'} = b'_{M'} \cup \{P'\}$.

(c) If $M \equiv \lambda x.M'$ and $P \xrightarrow{\lambda u} P'$, then $lt'^u_M = lt'^u_M \cup \{P'\}$.

(3) Suppose $P \in b_M$. Then:

(a) If $P \xrightarrow{\tau} P' \equiv (s)(Sem^- \mid Q)$ for some Q and s, then $b'_M = b'_M \cup \{P'\}$.

(b) If $P \xrightarrow{\tau} P' \equiv (s)(Sem \mid Q)$ for some Q and s, then $t'_M = t'_M \cup \{P'\}$.

(c) If $M \equiv \lambda x.M'$ and $P \xrightarrow{\lambda u} P'$, then $lb'^u_M = lb'^u_M \cup \{P'\}$.

(4) Suppose $P \in lt^u_M$. Then:

(a) If $P \xrightarrow{\tau} P' \equiv (s)(Sem \mid Q)$ for some Q and s, then $lt'^u_M = lt'^u_M \cup \{P'\}$.

(b) If $P \xrightarrow{\tau} P' \equiv (s)(Sem^- \,|\, Q)$ and if there exist some P'', P''' such that $P' \xrightarrow{\tau} P'' \xrightarrow{u(p,l,r,v,f)} P'''$, then $l'^{\mathbf{1}_u}_M = l'^{\mathbf{1}_u}_M \cup \{P'\}$.

(5) Suppose $P \in lb^u_M$. Then:

If $P \xrightarrow{\tau} P' \equiv (s)(Sem^- \,|\, Q)$ for some Q and s, then $lb'^u_M = lb'^u_M \cup \{P'\}$.

(6) Suppose $P \in l^{LT}_M$. Then:

(a) If $P \xrightarrow{\tau} P' \equiv (s)(Sem^- \,|\, Q)$ for some Q and s, then $l'^{LT}_M = l'^{LT}_M \cup \{P'\}$.

(b) If $P \xrightarrow{n(p,l,r,v,f)} P'$, then $l'^{LT\cdot(n,l,r,v)}_M = l'^{LT\cdot(n,l,r,v)}_M \cup \{P'\}$.

(c) If $P \xrightarrow{\tau} P' \equiv (s)(Sem \,|\, Q)$ for some Q and s, and if there exist some Q' and some closed λ-term N such that $Q \Longrightarrow Q' \nrightarrow$ and $(s)(Sem \,|\, Q') \equiv [\![N]\!]_\lambda$, then $t'_N = t'_N \cup \{P'\}$.

By the above definition, the function F is monotone with respect to the subset relation, so we can construct an increasing sequence:

$$\begin{aligned} h^0 &\overset{\text{def}}{=} \{\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \cdots\} \\ &\vdots \\ h^{i+1} &\overset{\text{def}}{=} F(h^i) \\ &\vdots \end{aligned}$$

The least fixed point h^ω is precisely the set

$$\{\mathscr{T}_M, \mathscr{B}_M, \mathscr{LB}^u_M, \mathscr{LT}^u_M, \mathscr{L}^{LT}_M \mid M \in \Lambda^0, u \in \mathscr{N}, LT \in \mathbf{LT}\}.$$

We denote the sets in h^i by $(t_M)^i, (b_M)^i, \cdots$. Each process P in $\mathscr{T}_M$, for example, is in $(t_M)^k$ for some k. Intuitively, $(t_M)^{i+1}, (b_M)^{i+1}, (lb^u_M)^{i+1}, (lt^u_M)^{i+1}, (l^{LT}_M)^{i+1}$ contain the processes that can be reached from $[\![M]\!]_\lambda$ after at most i actions.

We shall check that the lemma holds for $(t_M)^0$ and $(t_M)^1$. We then show that the lemma holds for $(t_M)^{i+1}$, for $i \geqslant 1$, under the assumption that it holds for $(t_M)^i$. The properties of the other sets can be checked similarly. We begin by making the following observations, which can all be proved by induction:

— All the processes in $\mathscr{T}_M$, $\mathscr{LT}^u_M$ are of the form $(s)(Sem \,|\, Q)$; and all the processes in $\mathscr{B}_M$, $\mathscr{LB}^u_M$ and $\mathscr{L}^{LT}_M$ are of the form $(s)(Sem^- \,|\, Q)$.

— If a process performs some action and evolves into a process in another set, then either it changes the state of the semaphore or it performs an input action.

— If M is not an abstraction term, then

$$\forall u \in \mathscr{N}. \forall LT \in \mathbf{LT}. (\mathscr{LT}^u_M = \mathscr{LB}^u_M = \mathscr{L}^{LT}_M = \emptyset).$$

In the following proof we assume that M is an abstraction term when we write $\mathscr{LT}^u_M$, $\mathscr{LB}^u_M$ or $\mathscr{L}^{LT}_M$ for some u, LT.

— For each $P \in \mathscr{T}_M \cup \mathscr{B}_M$, after the reduction stage and all the replication stages are finished, P evolves into the encoding of M. Hence $P \Longrightarrow [\![M]\!]_\lambda$.

Since all the elements of h^0 are $\emptyset$, the lemma holds for h^0. For h^1 we need to check that the only element $[\![M]\!]_\lambda$ of $(t_M)^1$ satisfies the properties of the lemma. For each closed λ-term M, we have $[\![M]\!]_\lambda \equiv (s)(Sem \,|\, (\widetilde{v}\widetilde{n})T_M^{n,\lambda,\perp})$, where $\widetilde{v} = \{v_x \mid x \text{ is free in } M)\}$, and $\widetilde{n}$ is the set of the internal node names of the tree $T_M^{n,\lambda,\perp}$.

(1) If $[\![M]\!]_\lambda \xrightarrow{\tau} P'$, since there are no interactions between any pair of components of $T_M^{n,\lambda,\perp}$, there must exist a redex node $L(P_i, n_i, m_i, m_i, v_i, \top) \in T_M^{n,\lambda,\perp}$ and

$$L(p_i, n_i, m_i, m_i, v_i, \top) \xrightarrow{s} L_1$$
$$Sem \xrightarrow{\overline{s}} Sem^-.$$

So P' is α-convertible to $(s)(Sem^- \,|\, Q)$ for some Q, and it is clear that $P' \notin \mathscr{T}_M$.

(2) (a) If $[\![M]\!]_\lambda \xrightarrow{\tau} P'$, we have already shown that there must be some β-redex in M. Without loss of generality, we may assume that $M \equiv C[(\lambda x.N)L]$ for some $C[\;], N, L$, and the node performing $\xrightarrow{s}$ is the redex node λx in $C[(\lambda x.N)L]$. We can rearrange the bound names in $T_M^{n,\lambda,\perp}$ by applying the structural congruence rules:

$$\begin{aligned} T_M^{n,\lambda,\perp} &\equiv T_{C[\_]}^{n,\lambda,\perp} \,|\, L(n', p', l, r, \perp, f') \,|\, L(l, n', m, m, v_x, \top) \,|\, T_N^{m,l,\perp} \,|\, T_L^{r,n',\perp} \\ P' &\equiv (s)(\widetilde{v})(\widetilde{n})(Sem^- \,|\, T_{C[\_]}^{n,\lambda,\perp} \,|\, L(n', p', l, r, \perp, f') \,|\, L_1 \,|\, T_N^{m,l,\perp} \,|\, T_L^{r,n',\perp}), \end{aligned}$$

where $L(l, n', m, m, v_x, \top) \xrightarrow{s} L_1$. Let Q' be defined by

$$Q' \overset{\text{def}}{=} (\widetilde{v}\widetilde{n})(T_{C[\_]}^{n,\lambda,\perp} \,|\, L(n', p', l, r, \perp, f') \,|\, L_1 \,|\, T_N^{m,l,\perp} \,|\, T_L^{r,n',\perp}).$$

After Q' has completed the reduction and replication stages, it will evolve into

$$\begin{aligned} Q' &\Longrightarrow Q'' \\ &\equiv (\widetilde{v'}\widetilde{n'})(T_{C[N\{L/x\}]}^{n,\lambda,\perp} \,|\, \overline{s} \,|\, [\![v_x := L]\!]_\perp) \\ &\equiv (\widetilde{v''}\widetilde{n'})(T_{C[N\{L/x\}]}^{n,\lambda,\perp} \,|\, \overline{s}). \end{aligned}$$

It is obvious that Q'' has no τ-actions and $(s)(Sem^- \,|\, Q'') \xrightarrow{\tau} [\![C[N\{L/x\}]]\!]_\lambda$. Hence, $M \to M' \equiv C[N\{L/x\}]$ and $P' \in (b_{M'})^2 \subseteq \mathscr{B}_{M'}$.

(b) If $M \equiv \lambda x.M'$ for some x, M', then

$$[\![\lambda x.M']\!]_\lambda \equiv (s)(Sem \,|\, (\widetilde{v}\widetilde{n})(L(n, \lambda, m, m, v_x, \perp) \,|\, T_M^{m,n,\perp})).$$

Therefore $[\![M]\!]_\lambda$ can perform $\xrightarrow{\lambda u}$, and according to the construction, we have $[\![M]\!]_\lambda \xrightarrow{\lambda u} P' \in (lt_M^u)^2 \subseteq \mathscr{L}\mathscr{T}_M^u$. Moreover, if $[\![M]\!]_\lambda \xrightarrow{\lambda u} P'$, then M must have the form of an abstraction, and it follows from the construction that $P' \in (lt_M^u)^2 \subseteq \mathscr{L}\mathscr{T}_M^u$.

(c) The encoding $[\![M]\!]_\lambda$ can either perform a τ-action to begin a simulation of β-reduction or, if M is an abstraction term, perform $\xrightarrow{\lambda u}$ for each u.

(d) If $M \to M'$, then, like in case (a), there must exist some P' such that $[\![M]\!]_\lambda \xrightarrow{\tau} P' \in (b_{M'})^2 \subseteq \mathscr{B}_{M'}$.

So the lemma holds for h^1.

Now assume that the sets in h^i, for $i \geqslant 1$, satisfy the properties of the lemma. Then for each closed λ-term M, consider the sets in h^{i+1}:

(1) $(b_M)^{i+1}$.

For each $P \in (b_M)^{i+1} \setminus (b_M)^i$, P must be of the form $(s)(Sem^- \,|\, Q)$ for some s, Q.

— By construction, there must be some P^* such that $P^* \xrightarrow{\tau} P$ and P^* is either in $(b_M)^i$ or in $(t_{M'})^i$ for some closed λ-term M'.

(a) If $P^* \in (b_M)^i$, there is no infinite τ-action sequence

$$P_0 \equiv P \xrightarrow{\tau} P_1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} P_i \xrightarrow{\tau} \cdots$$

such that for any i, $P_i \in \mathscr{B}_M$, otherwise P^* would have an infinite τ-action sequence with all the processes in $\mathscr{B}_M$, which would contradict the induction hypothesis.

(b) If $P^* \in (t_{M'})^i$, then for some Q^*, we have $P^* \equiv (s)(Sem \,|\, Q^*)$ and $Q^* \xrightarrow{s} Q$. Moreover, $M' \rightarrow M$. After performing a finite number of τ-actions, Q will complete the reduction and replication stages that simulate $M' \rightarrow M$. So if there was an infinite τ-action sequence

$$P \equiv P_0 \xrightarrow{\tau} P_1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} P_i \xrightarrow{\tau} \cdots$$

such that for every i, $P_i \in \mathscr{B}_M$, there would be an infinite τ-action sequence from P^*, which would contradict the induction hypothesis.

— If $P \xrightarrow{\tau} P'$, there are two cases:

– The transition is caused by $Q \xrightarrow{\tau} Q'$. So, by construction,

$$P' \equiv (s)(Sem^- \,|\, Q') \in (b_M)^{i+2} \subseteq \mathscr{B}_M.$$

– The transition is induced by $Sem^- \xrightarrow{s} Sem$ and $Q \xrightarrow{\bar{s}} Q'$. So, by construction,

$$P' \equiv (s)(Sem \,|\, Q') \in (t_M)^{i+2} \subseteq \mathscr{T}_M.$$

— If $M \equiv \lambda x.M'$ for some x, M' and $P \equiv (s)(Sem^- \,|\, Q)$, the process Q can first perform the τ-action j times, for $j \geqslant 0$, to get the abstraction node λx ready. In other words, there are some n, m and R such that

$$Q \underbrace{\xrightarrow{\tau} \cdots \xrightarrow{\tau}}_{j} Q' \equiv (nmv_x)(L(n, \lambda, m, m, v_x, \perp) \,|\, R).$$

Therefore

$$P \underbrace{\xrightarrow{\tau} \cdots \xrightarrow{\tau}}_{j} P' \equiv (s)(Sem^- \,|\, Q') \xrightarrow{\lambda u} P''.$$

It follows that $P' \in (b_M)^{i+j+1} \subseteq \mathscr{B}_M$ and $P'' \in (lb^u_M)^{i+j+2} \subseteq \mathscr{LB}^u_M$.

— If $P \xrightarrow{\lambda u} P'$, there must be an abstraction node with parent λ. By construction, this is only possible if M has the form of an abstraction. Thus $P' \in (lb^u_M)^{i+2} \subseteq \mathscr{LB}^u_M$ from the construction.

— The only public channel of P is λ. So P can only do either $\xrightarrow{\tau}$ or $\xrightarrow{\lambda u}$.

(2) $(t_M)^{i+1}$.

For each $P \in (t_M)^{i+1} \setminus (t_M)^i$, P must have the form $(s)(Sem \,|\, Q)$ for some s, Q.

— By construction, there must be some P^* in $(t_M)^i$ or $(b_M)^i$, or in $(l_N^{LT})^i$ for some λ-term N and some labelled tree LT such that $P^* \xrightarrow{\tau} P$.

(a) If $P^* \in (t_M)^i$, the case is easy.

(b) If $P^* \in b_M^i$, then $P^* \equiv (s)(Sem^- \,|\, \bar{s} \,|\, Q)$. If P has infinite τ-action sequence

$$(s)(Sem \,|\, Q) \xrightarrow{\tau} (s)(Sem \,|\, Q_1) \xrightarrow{\tau} \cdots$$

with all the processes in $\mathscr{T}_M$, then P^* also has an infinite τ-action sequence

$$(s)(Sem^- \,|\, \bar{s} \,|\, Q) \xrightarrow{\tau} (s)(Sem^- \,|\, \bar{s} \,|\, Q_1) \xrightarrow{\tau} \cdots$$

with all the processes belonging to $\mathscr{B}_M$, which would contradict the induction hypothesis.

(c) If $P^* \in (l_N^{LT})^i$, some s, m, R and $\tilde{n}$ exist such that

$$\begin{aligned} P^* &\equiv (s)(Sem^- \,|\, (\tilde{n})(\bar{s}.L(m, \cdots) \,|\, R)) \\ P &\equiv (s)(Sem \,|\, (\tilde{n})(L(m, \cdots) \,|\, R). \end{aligned}$$

If P has an infinite τ-action sequence with all the processes in $\mathscr{T}_M$, it must be caused by R because all interactions between $L(m, \cdots)$ and R should happen after turning off the semaphore. It follows that P^* would also have an infinite τ-action sequence with all the processes in $\mathscr{L}_M^{LT}$, which would contradict the induction hypothesis.

— Suppose $P \in (t_M)^{i+1} \setminus (t_M)^i$ and $P \xrightarrow{\tau} P'$. Then:

(a) If $P' \equiv (s)(Sem \,|\, Q')$ for some Q' with $Q \xrightarrow{\tau} Q'$, then $P' \in (t_M)^{i+2} \subseteq \mathscr{T}_M$.

(b) If $P' \equiv (s)(Sem^- \,|\, Q')$ for some Q' with $Q \xrightarrow{s} Q'$, there must exist some redex node, and M' such that $M \to M'$. As discussed in Section 5.1, after Q' completes this reduction stage and all the unfinished replication stages, it evolves into Q'' such that it has no more τ-actions and $(s)(Sem^- \,|\, Q'') \xrightarrow{\tau} [\![M']\!]_\lambda$. Therefore, $P' \in (b_{M'})^{i+2} \subseteq \mathscr{B}_{M'}$.

— If $M \equiv \lambda x.M'$ for some x, M' and $P \equiv (s)(Sem \,|\, Q)$, the process Q can first perform the τ-action j times, for $j \geqslant 0$, to get the abstraction node λx ready. In other words, there are some n, m and R such that

$$Q \underbrace{\xrightarrow{\tau} \cdots \xrightarrow{\tau}}_{j} Q' \equiv (nmv_x)(L(n, \lambda, m, m, v_x, \bot) \,|\, R).$$

So $P' \equiv (s)(Sem \,|\, Q') \xrightarrow{\lambda u} P''$. By the construction, we have $P' \in (t_M)^{i+j+1} \subseteq \mathscr{T}_M$ and $P'' \in (lt_M^u)^{i+j+2} \subseteq \mathscr{L}\mathscr{T}_M^u$.

— If $P \xrightarrow{\lambda u} P'$, there must be an abstraction node with parent λ. So M must have the form of an abstraction, and we get $P' \in (lt_M^u)^{i+2} \subseteq \mathscr{L}\mathscr{T}_M^u$ from the construction.

— The only public channel of P is λ. So P can only do either $\xrightarrow{\tau}$ or $\xrightarrow{\lambda u}$.

— Since $P \in (t_M)^{i+1}$, P can evolve into the encoding of M by finishing all the unfinished replication stages. If $M \to M'$, some P' exists such that $P \Longrightarrow [\![M]\!]_\lambda \xrightarrow{\tau} P'$. As in the proof for h^1, we get $P' \in \mathscr{B}_{M'}$.

(3) $(lb^u_M)^{i+1}$.

There must exist some x, M_1 such that $M \equiv \lambda x.M_1$. By the construction, for each process P in $(lb^u_M)^{i+1} \setminus (lb^u_M)^i$, there exists some P^* in $(b_M)^i$ such that $P^* \xrightarrow{\lambda u} P$, where P and P^* have the following forms:

$$\begin{aligned} P^* &\equiv (s)(Sem^- \,|\, (\widetilde{n})(L(n, \lambda, m, m, v_x, \bot) \,|\, R)) \\ P &\equiv (s)(Sem^- \,|\, (\widetilde{n})(s.Q \,|\, R)), \end{aligned}$$

where $L(n, \lambda, m, m, v_x, \bot) \xrightarrow{\lambda u} s.Q$ by definition.

— If P has an infinite τ-action sequence with all the processes in $\mathscr{LB}^u_M$, it must be induced by an infinite τ-action sequence of R, which contradicts the induction hypothesis that P^* has no infinite τ-action sequence with all the processes in $\mathscr{B}_M$.

— If $P \xrightarrow{\tau} P'$, there are two cases, either

$$R \xrightarrow{\tau} R_1 \qquad \text{and} \qquad P' \equiv (s)(Sem^- \,|\, (\widetilde{n})(s.Q \,|\, R_1))$$

or

$$R \xrightarrow{\overline{s}} R_2 \qquad \text{and} \qquad P' \equiv (s)(Sem \,|\, (\widetilde{n})(s.Q \,|\, R_2)).$$

In either case, there is some Q^* such that $P^* \xrightarrow{\tau} Q^* \xrightarrow{\lambda u} P'$. Correspondingly, there are two cases for Q^*:

$$\begin{aligned} Q^* &\equiv (s)(Sem^- \,|\, (\widetilde{n})(L(n, \lambda, m, m, v_x, \bot) \,|\, R_1)) \\ Q^* &\equiv (s)(Sem \,|\, (\widetilde{n})(L(n, \lambda, m, m, v_x, \bot) \,|\, R_2)). \end{aligned}$$

The process Q^* is in $(b_M)^{i+1}$ or $(t_M)^{i+1}$. It then follows that

$$P' \in (lb^u_M)^{i+2} \cup (lt^u_M)^{i+2} \subseteq \mathscr{LB}^u_M \cup \mathscr{LT}^u_M.$$

— Since P has not finished the reduction stage, it can perform τ-actions. Because the input action on channel u is blocked by the semaphore, P can only perform τ-actions.

(4) $(lt^u_M)^{i+1}$.

There must exist some x, M_1 such that $M \equiv \lambda x.M_1$. If $P \in (lt^u_M)^{i+1} \setminus (lt^u_M)^i$, by construction, there exists $P^* \in (t_M)^i$ such that $P^* \xrightarrow{\lambda u} P$. Here

$$\begin{aligned} P^* &\equiv (s)(Sem \,|\, (\widetilde{n})(L(n, \lambda, m, m, v_x, \bot) \,|\, R)) \\ P &\equiv (s)(Sem \,|\, (\widetilde{n})(s.Q \,|\, R)), \end{aligned}$$

where $L(n, \lambda, m, m, v_x, \bot) \xrightarrow{\lambda u} s.Q$ by definition.

— Like in the $(lb^u_M)^{i+1}$ case, for each process in $(lt^u_M)^{i+1}$, there is no infinite τ-action sequence with all the processes in $\mathscr{LT}^u_M$.

— If $P \xrightarrow{\tau} P'$, then P' has one of the following forms:

$$\begin{aligned} P' &\equiv (s)(Sem \,|\, (\widetilde{n})(s.Q \,|\, R_1)) \\ P' &\equiv (s)(Sem^- \,|\, (\widetilde{n})(s.Q \,|\, R_2)) \\ P' &\equiv (s)(Sem^- \,|(\widetilde{n})(Q \,|\, R)), \end{aligned}$$

where $R \xrightarrow{\tau} R_1$ and $R \xrightarrow{s} R_2$. If $P' \equiv (s)(Sem^- \,|\, (\widetilde{n})(Q \,|\, R))$, there exist Q', R' such that

$$\begin{aligned} &Q \xrightarrow{m(p_1,l_1,r_1,v_1,f_1)} \xrightarrow{u(p,l,r,v,f)} Q' \\ &R \xrightarrow{\overline{m}\langle p_1,l_1,r_1,v_1,f_1\rangle} R'. \end{aligned}$$

So $P' \in (l^{\mathbf{1}_u}_M)^{i+2} \subseteq \mathscr{L}^{\mathbf{1}_u}_M$. For the other two cases, there exists some Q^* such that $P^* \xrightarrow{\tau} Q^* \xrightarrow{\lambda u} P'$ as in case (3). Correspondingly, there are two cases for Q^*:

$$\begin{aligned} Q^* &\equiv (s)(Sem \,|\, (\widetilde{n})(L(n,\lambda,m,m,v_x,\perp) \,|\, R_1)) \\ Q^* &\equiv (s)(Sem^- \,|\, (\widetilde{n})(L(n,\lambda,m,m,v_x,\perp) \,|\, R_2)). \end{aligned}$$

According to the argument we made for case (2), the process Q^* is in $(t_M)^{i+1}$ or there exists some M' such that $M \to M'$ and $Q^* \in (b_{M'})^{i+1}$. It follows that $P' \in (lt^u_M)^{i+2} \cup (lb^u_{M'})^{i+2} \subseteq \mathscr{L}\mathscr{T}^u_M \cup \mathscr{L}\mathscr{B}^u_M$.

— Since $s.Q \xrightarrow{s}$, we have $P' \in (l^{\mathbf{1}_u}_M)^{i+2} \subseteq \mathscr{L}^{\mathbf{1}_u}_M$ for some P'.

— Because the input action on channel u is blocked by the semaphore, P can only perform τ-actions.

(5) $(l^{LT}_M)^{i+1}$.

$M \equiv \lambda x.M_1$ for some x, M_1. There are three major cases according to the types of LT:

(a) $LT = \mathbf{1}_u$ for some u and $P \in (l^{\mathbf{1}_u}_M)^{i+1} \setminus (l^{\mathbf{1}_u}_M)^i$.

— We first prove that there is no infinite τ-action sequence with all the processes in $\mathscr{L}^{\mathbf{1}_u}_M$. First, according to the construction, there must be a P^* in $(lt^u_M)^i$ or $(l^{\mathbf{1}_u}_M)^i$ such that $P^* \longrightarrow P$. If $P^* \in (lt^u_M)^i$, it must have the form

$$(s)(Sem \,|\, (\widetilde{n})(s.m(\cdots).Q \,|\, R))$$

for some $s, \widetilde{n}, Q, R$, where

$$L(n,\lambda,m,m,v_x,\perp) \xrightarrow{\lambda u} s.m(\cdots).Q$$

and $n, m \in \widetilde{n}$. From the definition given in Figure 4, the only action of Q is the input $u(p,l,r,v,f)$. So if P has an infinite τ-action sequence in which all the processes are in $\mathscr{L}^{\mathbf{1}_u}_M$, then P^* would also have an infinite τ-action sequence with all the processes in $\mathscr{L}\mathscr{T}^u_M$, which would contradict the induction hypothesis.

— If $P^* \in (l^{\mathbf{1}_u}_M)^i$ and $P \xrightarrow{\tau} P'$, then, since the semaphore will not be turned off until the import of a λ-term is finished, P' must have the form $(s)(Sem^- \,|\, Q)$. By construction, we then have $P' \in (l^{\mathbf{1}_u}_M)^{i+2} \subseteq \mathscr{L}^{\mathbf{1}_u}_M$.

— If $P^* \in (l_M^{\mathbf{1}_u})^i$ and $P \xrightarrow{u(p,l,r,v,f)} P'$, then, by the construction and the definition given in Figure 4, we have

$$P' \in (l_M^{\mathbf{1}_u \cdot (u,l,r,v)})^{i+2} \subseteq \mathscr{L}_M^{\mathbf{1}_u \cdot (u,l,r,v)}.$$

For every λ-term M' of the form $\lambda x.M''$ and every process $O \in (l_{\lambda x.M''}^{\mathbf{1}_u})^{i+1}$, according to the construction, there is some $O^* \in (l_M^{\mathbf{1}_u})^i$ such that $O^* \Longrightarrow O$ and O^* is of the form

$$(s)(Sem^- \,|\, (\widetilde{n})(m(\cdots).Q \,|\, R))$$

for some $s, \widetilde{n}, Q, R$, where

$$L(n, \lambda, m, m, v_x, \perp) \xrightarrow{\lambda u} \xrightarrow{s} m(\cdots).Q.$$

Since Q can perform an input on channel u, say $Q \xrightarrow{u(p,l,r,v,f)} Q'$, there must exist some k, O' such that

$$O \underbrace{\xrightarrow{\tau} \cdots \xrightarrow{\tau}}_{k} \xrightarrow{u(p,l,r,v,f)} O',$$

and, consequently, $O' \in (l_{\lambda x.M''}^{\mathbf{1}_u \cdot (u,l,r,v)})^{i+k+1} \subseteq \mathscr{L}_{\lambda x.M''}^{\mathbf{1}_u \cdot (u,l,r,v)}$.

— As discussed above, the process P can do either a τ action or an input at name u, and there are no other public channels in any process $P \in (l_{\lambda x.M'}^{\mathbf{1}_u})^{i+1}$. Since $I(\mathbf{1}_u) = \{u\}$, we get that the process P can and can only perform either τ-actions or input actions at the name in $I(\mathbf{1}_u)$.

(b) $I(LT) \neq \emptyset$ and $P \in (l_M^{LT})^{i+1} \setminus (l_M^{LT})^i$.

— We prove that there is no infinite τ-action sequence with all the processes in $\mathscr{L}_M^{LT}$. First, according to the construction, there must be a P^* such that either $P^* \in (l_M^{LT})^i$ and $P^* \xrightarrow{\tau} P$, or $P^* \in (l_M^{LT'})^i$ and $P^* \xrightarrow{n(p,l,r,v,f)} P$ and $LT = LT' \cdot (n,l,r,v)$. The case of $P^* \in (l_M^{LT})^i$ is easy. If $P^* \in (l_M^{LT'})^i$, then, since $I(LT) \neq \emptyset$, P has not finished the import. The new input on P^* will not introduce infinite τ-actions, since if P had an infinite τ-action sequence in which all the processes are in $\mathscr{L}_M^{LT}$, then P^* would also have an infinite τ-action sequence with all the processes in $\mathscr{L}_M^{LT'}$, which would contradict the induction hypothesis.

— If $P \xrightarrow{\tau} P'$, then, since the semaphore will not be turned off until the import of a λ-term is finished, P' must have the form $(s)(Sem^- \,|\, Q)$. By construction, we have $P' \in (l_M^{LT})^{i+2} \subseteq \mathscr{L}_M^{LT}$.

— Since the process P has not finished the import, it will continue to read from the environment. The set of the public channels in P is $I(LT)$, meaning that if $P \xrightarrow{n(p,l,r,v,f)} P'$, then $n \in I(LT)$. According to the construction, $P' \in (l_M^{LT \cdot (n,l,r,v)})^{i+2} \subseteq \mathscr{L}_M^{LT \cdot (n,l,r,v)}$. For every closed λ-term M' of the form $\lambda x.M''$ and every process $O \in (l_{\lambda x.M''}^{LT})^{i+1}$, we have O can perform an input action at a name

n if and only if $n \in I(LT)$. So we must also have

$$O \underbrace{\xrightarrow{\tau} \cdots \xrightarrow{\tau}}_{k} \xrightarrow{n(p,l,r,v,f)} O',$$

and, therefore, $O' \in (l_{\lambda x.M''}^{LT\cdot(n,l,r,v)})^{i+k+1} \subseteq \mathscr{L}_{\lambda x.M''}^{LT\cdot(n,l,r,v)}$.

— As we have just said, the process P can either do a τ-action or an input at names in $I(LT)$, and there are no other public channels in any process $P \in (l_{\lambda x.M'}^{LT})^{i+1}$. So the process P can and can only perform either τ-actions or input actions at names in $I(LT)$.

(c) If $I(LT) = \emptyset$, the process $P \in (l_M^{LT})^{i+1} \setminus (l_M^{LT})^i$ has completed the import from the environment and LT must be a labelled tree of some λ-term N.

— First we argue that there is no infinite τ-action sequence with all the processes in $\mathscr{L}_M^{LT}$. According to the construction, there must be a P^* such that either $P^* \in (l_M^{LT})^i$ and $P^* \xrightarrow{\tau} P$, or $P^* \in (l_M^{LT'})^i$ and $P^* \xrightarrow{n(p,l,r,v,f)} P$ and $LT = LT' \cdot (n,l,r,v)$. The case of $P^* \in (l_M^{LT})^i$ is easy. If $P^* \in (l_M^{LT'})^i$, then since P has finished the import, after at most a finite number of steps of reduction and replication, it would have no more τ-actions except those turning off the semaphore. Otherwise, P^* would have an infinite τ-action sequence with all the processes in $\mathscr{L}_M^{LT'}$, which would contradict the induction hypothesis.

— If $P \equiv (s)(Sem^- \mid Q)$ and $P \xrightarrow{\tau} P'$, there are two cases.

– The transition is caused by $Q \xrightarrow{\tau} Q'$ and $P' \equiv (s)(Sem^- \mid Q')$. Then $P' \in (l_M^{LT})^{i+2} \subseteq \mathscr{L}_M^{LT}$.

– The transition is caused by $Q \xrightarrow{\bar{s}} Q'$ and $P' \equiv (s)(Sem \mid Q')$. According to the definition of the encoding in Figure 4 and the discussion in Section 5.2, the encoding of some closed λ-term N' has been imported at this stage. Then after Q' completes the reduction stage and all the replication stages, it would evolve into some Q'' such that $Q' \overset{\tau}{\Longrightarrow} Q'' \not\rightarrow$. Then $(s)(Sem \mid Q') \equiv [\![M'\{N'/x\}]\!]_\lambda$ and $P' \in (t_{M'\{N'/x\}})^{i+2} \subseteq \mathscr{T}_{M'\{N'/x\}}$, where M' is such that $M \equiv \lambda x.M'$. For every closed λ-term $\lambda x.M''$ and every process $O \in (l_{\lambda x.M''}^{LT})^{i+1}$, O must also have finished the import. Thus $O \overset{\tau}{\Longrightarrow} O' \in \mathscr{T}_{M''\{N'/x\}}$ for some O'.

— P can perform τ-actions since the semaphore has not been turned off. In the definition of $RAbs(n, \lambda, m, v_x, f)$, the only public channel λ in P is blocked by $\bar{s}$. So P can only perform τ-actions. □

A.2. *Proof of Lemma 6*

Proof. First note that part (2) can be deduced from Lemma 6 (1, 3a, 3b, 4a, 4b, 5a, 5b) and Lemma 4 (1). So we only need to prove Lemma 6 (1, 3, 4, 5). Here is the case analysis:

(1) $P \in \mathscr{CT}^N_{\lambda x.M}$.

So $P \equiv (u)(Q \mid T(u,N))$ for some u, where $Q \in \mathscr{LT}^u_{\lambda x.M}$.

— The interactions between Q and $T(u,N)$ can only contribute a finite number of τ-actions. If P had an infinite τ-action sequence with all the processes in $\mathscr{CT}^N_{\lambda x.M}$, then Q would have an infinite τ-action sequence with all the processes in $\mathscr{LT}^u_{\lambda x.M}$, which would contradict Lemma 5 (1).

— If $P \xrightarrow{\tau} P'$, the transition must be caused by some τ-action of Q because $T(u,N)$ has no τ-actions and the interaction between Q and $T(u,N)$ cannot happen right now. Assume $P' \stackrel{\text{def}}{=} (u)(Q' \mid T(u,N))$, where $Q \xrightarrow{\tau} Q'$. By Lemma 5 (4a), $Q' \in \mathscr{LT}^u_{\lambda x.M} \cup \mathscr{L}^{\mathbf{1}_u}_{\lambda x.M}$ or $\exists M'.M \to M' \wedge P' \in \mathscr{LB}^u_{\lambda x.M'}$. Accordingly, we have $P' \in \mathscr{CT}^N_{\lambda x.M} \cup \mathscr{C}^N_{\lambda x.M}$ or $\exists M'.M \to M' \wedge P' \in \mathscr{CB}^N_{\lambda x.M'}$.

— By Lemma 5 (4b), there exists Q' such that $Q \xrightarrow{\tau} Q' \in \mathscr{L}^{\mathbf{1}_u}_{\lambda x.M}$. Therefore some P' exists such that $P \xrightarrow{\tau} P' \in \mathscr{C}^N_{\lambda x.M} \subseteq \mathscr{CT}^N_{\lambda x.M} \cup \mathscr{C}^N_{\lambda x.M}$.

— It follows from Lemma 5 (4c) that Q can and can only do τ-actions. The process $T(u,N)$ has only output actions at private channels. An interaction between Q and its descendants and $T(u,N)$ cannot happen before the process jumps out of $\mathscr{CT}^N_{\lambda x.M}$. Therefore P can only perform τ-actions.

(2) $P \in \mathscr{CB}^N_{\lambda x.M}$.

So $P \equiv (u)(Q \mid T(u,N))$, where $Q \in \mathscr{LB}^u_{\lambda x.M}$. The following arguments are very much like those in the previous case.

— If P has an infinite τ-action sequence with all the processes in $\mathscr{CT}^N_{\lambda x.M}$, then Q would have an infinite τ-action sequence with all the processes in $\mathscr{LB}^u_{\lambda x.M}$, which would contradict Lemma 5 (1).

— If $P \xrightarrow{\tau} P'$, this transition must be caused by some τ-action of Q. We can assume $P' \stackrel{\text{def}}{=} (u)(Q' \mid T(u,N))$, where $Q \xrightarrow{\tau} Q'$. By Lemma 5 (5a), $Q' \in \mathscr{LB}^u_{\lambda x.M} \cup \mathscr{LT}^u_{\lambda x.M}$. Consequently, $P' \in \mathscr{CB}^N_{\lambda x.M} \cup \mathscr{CT}^N_{\lambda x.M}$.

— It follows from Lemma 5 (5b) that Q can and can only do τ-actions. The process $T(u,N)$ has only output actions at private channels. An interaction between Q and its descendants and $T(u,N)$ cannot happen before the process jumps out of $\mathscr{CT}^N_{\lambda x.M}$. Therefore, P can only perform τ-actions.

(3) $P \in \mathscr{C}^N_{\lambda x.M}$. There are two cases:

— $P \equiv (u)(Q \mid T(u,N))$ where $Q \in \mathscr{L}^{\mathbf{1}_u}_{\lambda x.M}$.

By Lemma 5, Q has no infinite τ-action sequences. So an infinite τ-action sequence of P must be caused by the interactions between Q and $T(u,N)$. Without loss of generality, we may assume that $T(u,N)$ has the transitions

$$T(u,N) \xrightarrow{\overline{u}\langle p,l,r,v,f\rangle} \xrightarrow{\overline{l}\langle p',l',r',v',f'\rangle} \cdots \xrightarrow{\overline{n}\langle p_n,\perp,\perp,v_n,f_n\rangle} \mathbf{0}.$$

Correspondingly, Q interacts in the following manner:

$$Q \underbrace{\Longrightarrow Q_1}_{\mathscr{L}^{\mathbf{1}_u}_{\lambda x.M}} \xrightarrow{u(p,l,r,v,f)} \underbrace{\Longrightarrow Q_2}_{\mathscr{L}^{\mathbf{1}_u\cdot(u,l,r,v)}_{\lambda x.M}} \cdots Q_{n-1} \xrightarrow{n(p_n,\perp,\perp,v_n,f_n)} \underbrace{\Longrightarrow Q_n}_{\mathscr{L}^{LT}_{\lambda x.M}} \underbrace{\Longrightarrow Q_{n+1}}_{\mathscr{T}_{M\{N/x\}}}.$$

By Lemma 5 (1, 6), the τ-actions between Q and Q_{n+1} are finite. Meanwhile, $T(u, N)$ can only do a finite number of actions. So P does not have any infinite τ-action sequences in $\mathscr{C}^N_{\lambda x.M}$. Another possibility is that there exists some $P^* \equiv (u)(Q_0 \,|\, T(u, N)) \in \mathscr{C}^N_{\lambda x.M}$ such that $P^* \overset{\tau}{\Longrightarrow} P$. Since P^* does not have any infinite τ-action sequences with all the processes in $\mathscr{C}^N_{\lambda x.M}$, neither does P.

— According to the above discussion, a process in $\mathscr{C}^N_{\lambda x.M}$ has the form

$$P \equiv (\widetilde{u_i})(Q_i \,|\, \Pi_{k\in\{1,2,\cdots,n_i\}} T(u_{i_k}, N_{i_k})),$$

where $N_{i_1}, N_{i_2}, \cdots, N_{i_{n_i}}$ are sub-terms of N, $Q_i \in \mathscr{L}^{LT_i}_{\lambda x.M}$, and

$$I(LT_i) = \widetilde{u_i} = \{u_{i_1}, u_{i_2}, \cdots, u_{i_{n_i}}\}.$$

Let LT_N be the labelled tree of the λ-term N. Then $LT_i \Subset LT_N$. If $P \xrightarrow{\tau} P'$, there are two subcases:

– The transition is induced by $Q_i \xrightarrow{\tau} Q'_i$.
It follows from Lemma 5 (6a) that either $Q'_i \in \mathscr{L}^{LT_i}_{\lambda x.M}$ or $Q'_i \in \mathscr{T}_{M\{N'/x\}}$ for some closed λ-term N'. If $Q'_i \in \mathscr{L}^{LT_i}_{\lambda x.M}$, it is clear that $P' \in \mathscr{C}^N_{\lambda x.M}$. If $Q'_i \in \mathscr{T}_{M\{N'/x\}}$ for some closed λ-term N', we know from the proof of Lemma 8 that $I(LT_i)$ must be $\emptyset$. As $LT_i \Subset LT_N$ and $I(LT_i) = \emptyset$, we have $LT_i = LT_N$. Consequently, $N' \equiv N$ and $P' \equiv Q'_i \in \mathscr{T}_{M\{N/x\}}$.

– The transition is induced by an interaction between Q_i and $T(u_{i_k}, N_{i_k})$ for some $i_k \in \{1, 2, \cdots, n_i\}$.
Then

$$Q_i \xrightarrow{u_{i_k}(p,l,r,v,f)} Q_j$$

and

$$T(u_{i_k}, N_{i_k}) \xrightarrow{\overline{u_{i_k}}\langle p,l,r,v,f\rangle} T',$$

where T' is of the shape $\Pi_{k\in\{1,2,\cdots,n_j\}} T(u_{j_k}, N_{j_k})$. Therefore $P \xrightarrow{\tau} P' \equiv (\widetilde{u_j})(Q_j \,|\, T')$. By Lemma 5 (6c), the process Q_j is in $\mathscr{L}^{LT_j}_{\lambda x.M}$ where $LT_j = LT_i \cdot (u_{i_k}, l, r, v)$ and $\widetilde{u_j} = I(LT_j)$. So we have $P' \in \mathscr{C}^N_{\lambda x.M}$.

— By Lemma 5 (6d), Q_i can and can only do either τ-actions or input actions at names in $I(LT_i)$. As names in $I(LT_i)$ have been restricted in P, P can and can only perform τ-actions. □

A.3. *Proof of Lemma 7*

Proof. Recall that in the proof of Proposition 1 we constructed three relations $\mathscr{R}_1, \mathscr{R}_2$ and $\mathscr{R}_3$. We now define $\mathscr{R}'_1$ to be the relation

$$\left\{ (P, Q) \;\middle|\; \begin{array}{l} \exists M, N \in \Lambda^0.(M =_a N \\ \wedge\ P \in \mathscr{T}_M \cup \mathscr{B}_M \cup (\bigcup_{L\{L'/x\}\equiv M} \mathscr{C}\mathscr{T}^{L'}_{\lambda x.L} \cup \mathscr{C}\mathscr{B}^{L'}_{\lambda x.L} \cup \mathscr{C}^{L'}_{\lambda x.L}) \\ \wedge\ Q \in \mathscr{T}_N \cup \mathscr{B}_N \cup (\bigcup_{L\{L'/x\}\equiv N} \mathscr{C}\mathscr{T}^{L'}_{\lambda x.L} \cup \mathscr{C}\mathscr{B}^{L'}_{\lambda x.L} \cup \mathscr{C}^{L'}_{\lambda x.L})) \end{array} \right\}.$$

So if

$$P, Q \in \mathscr{T}_M \cup \mathscr{B}_M \cup (\bigcup_{L\{L'/x\} \equiv M} \mathscr{CT}^{L'}_{\lambda x.L} \cup \mathscr{CB}^{L'}_{\lambda x.L} \cup \mathscr{C}^{L'}_{\lambda x.L}),$$

then $(P, Q) \in \mathscr{R}'_1$ because $M =_a M$. As in the proof of Proposition 1, we need to check that the elements in $\mathscr{R} \stackrel{\text{def}}{=} \mathscr{R}'_1 \cup \mathscr{R}_2 \cup \mathscr{R}_3$ satisfy the following bisimulation property:

(1) If $Q\mathscr{R}P \xrightarrow{\tau} P'$, then $\exists Q'.Q \Longrightarrow Q'\mathscr{R}P'$.
(2) If $Q\mathscr{R}P \xrightarrow{a(\widetilde{n})} P'$, then $\exists Q'.Q \stackrel{a(\widetilde{n})}{\Longrightarrow} Q'\mathscr{R}P'$.

Since $\mathscr{R}_1 \subseteq \mathscr{R}'_1$, we only need to check for the pairs (P, Q) in $\mathscr{R}'_1 \setminus \mathscr{R}_1$:

— Suppose $M, N \in \Lambda^0$, $\lambda x.L, L' \in \Lambda^0$, $M =_a N$, $N \equiv L\{L'/x\}$, $P \in \mathscr{T}_M$ and $Q \in \mathscr{CT}^{L'}_{\lambda x.L} \cup \mathscr{CB}^{L'}_{\lambda x.L} \cup \mathscr{C}^{L'}_{\lambda x.L}$. By Lemma 5 (2), there are three cases:

(1) If $P \xrightarrow{\tau} P' \in \mathscr{T}_M$, then $P'\mathscr{R}Q$.

(2) If $P \xrightarrow{\tau} P'$ and $\exists M'.M \to M' \wedge P' \in \mathscr{B}_{M'}$, then $P'\mathscr{R}Q$ as $M' =_a M =_a N$.

(3) If $P \xrightarrow{\lambda u} P' \in \mathscr{LT}^u_M$, then by Lemma 5 (2b) there exist x, M' such that $M \equiv \lambda x.M'$. Since $M =_a N$, we must have $N \to^* \lambda x.N'$ and $M =_a \lambda x.N'$. According to Lemma 6 (2), we have $Q \Longrightarrow Q' \in \mathscr{T}_{\lambda x.N'}$. It follows from Lemma 5 (2b) that $Q' \Longrightarrow\xrightarrow{\lambda u} Q'' \in \mathscr{LT}^u_{\lambda x.N'}$. So $Q \stackrel{\lambda u}{\Longrightarrow} Q''$ and $P'\mathscr{R}Q''$.

— Suppose $M, N \in \Lambda^0$, $\lambda x.L, L' \in \Lambda^0$, $M =_a N$, $N \equiv L\{L'/x\}$, $P \in \mathscr{B}_M$ and $Q \in \mathscr{CT}^{L'}_{\lambda x.L} \cup \mathscr{CB}^{L'}_{\lambda x.L} \cup \mathscr{C}^{L'}_{\lambda x.L}$. According to Lemma 5 (3), there are three cases:

(1) If $P \xrightarrow{\tau} P' \in \mathscr{B}_M$, then $P'\mathscr{R}Q$.

(2) If $P \xrightarrow{\tau} P' \in \mathscr{T}_M$, then $P'\mathscr{R}Q$.

(3) If $P \xrightarrow{\lambda u} P' \in \mathscr{LB}^u_M$, then, similarly, $Q' \Longrightarrow\xrightarrow{\lambda u} Q'' \in \mathscr{LT}^u_{\lambda x.N'}$, where $N \to^* \lambda x.N'$. Hence $P'\mathscr{R}Q''$.

— Suppose $M, N \in \Lambda^0$, $\lambda x.L, L' \in \Lambda^0$, $M =_a N$, $M \equiv L\{L'/x\}$, $P \in \mathscr{CT}^{L'}_{\lambda x.L}$ and $Q \in \mathscr{T}_N \cup \mathscr{B}_N$. By Lemma 6 (3), there are three cases:

(1) If $P \xrightarrow{\tau} P' \in \mathscr{CT}^{L'}_{\lambda x.L}$, then $P'\mathscr{R}Q$.

(2) If $P \xrightarrow{\tau} P' \in \mathscr{C}^{L'}_{\lambda x.L}$, then $P'\mathscr{R}Q$.

(3) If $P \xrightarrow{\tau} P'$ and $\exists L''.L \to L'' \wedge P' \in \mathscr{CB}^{L'}_{\lambda x.L''}$, then $P'\mathscr{R}Q$ since $L''\{L'/x\} =_a L\{L'/x\} =_a N$.

— Suppose $M, N \in \Lambda^0$, $\lambda x.L, L' \in \Lambda^0$, $M =_a N$, $M \equiv L\{L'/x\}$, $P \in \mathscr{CB}^{L'}_{\lambda x.L}$ and $Q \in \mathscr{T}_N \cup \mathscr{B}_N$. By Lemma 6 (4), there are two cases:

(1) If $P \xrightarrow{\tau} P' \in \mathscr{CB}^{L'}_{\lambda x.L}$, then $P'\mathscr{R}Q$.

(2) If $P \xrightarrow{\tau} P' \in \mathscr{CT}^{L'}_{\lambda x.L}$, then $P'\mathscr{R}Q$.

— Suppose $M, N \in \Lambda^0$, $\lambda x.L, L' \in \Lambda^0$, $M =_a N$, $M \equiv L\{L'/x\}$, $P \in \mathscr{C}^{L'}_{\lambda x.L}$ and $Q \in \mathscr{T}_N \cup \mathscr{B}_N$. By Lemma 6 (5), there are two cases:

(1) If $P \xrightarrow{\tau} P' \in \mathscr{C}^{L'}_{\lambda x.L}$, then $P'\mathscr{R}Q$;

(2) If $P \xrightarrow{\tau} P' \in \mathscr{T}_{L\{L'/x\}}$, then $P'\mathscr{R}Q$ since $L\{L'/x\} \equiv M =_a N$.

We can now conclude that for every $M \in \Lambda^0$ and all P, Q in

$$\mathscr{T}_M \cup \mathscr{B}_M \cup \Big(\bigcup_{L\{N/x\} \equiv M} \mathscr{C}\mathscr{T}^N_{\lambda x.L} \cup \mathscr{C}\mathscr{B}^N_{\lambda x.L} \cup \mathscr{C}^N_{\lambda x.L} \Big),$$

that $P \approx Q$. □

Acknowledgements

We would like to thank the members of BASICS for their interest in this work. The encoding described in this paper was announced in BASICS 2009, and we would like to thank the other participants for their questions and comments. We are also grateful to the two anonymous referees for their detailed comments on a previous version of this paper.

References

Abadi, M. and Gordon, A. (1999) A Calculus for Cryptographic Protocols: The Spi Calculus. *Information and Computation* **148** (1) 1–70.

Abramsky, S. (1990) The Lazy Lambda Calculus. *Research Topics in Functional Programming* 65–116.

Amadio, R. and Prasad, S. (2000) Modelling IP Mobility. *Formal Methods in System Design* **17** (1) 61–99.

Baldamus, M., Parrow, J. and Victor, B. (2004) Spi Calculus Translated to π-calculus Preserving May-Tests. In: *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science (LICS'04)*, IEEE Computer Society 22–31.

Barendregt, H. (1984) *The Lambda Calculus: its Syntax and Semantics*, North Holland.

Boudol, G. (1992) Asynchrony and the π-calculus. Technical Report 1702, INRIA Sophia-Antipolis.

Fu, Y. (1997) A Proof Theoretical Approach to Communication. In: Degano, P. Gorrieri, R. and Marchetti-Spaccamela, A. (eds.) Automata, Languages and Programming: Proceedings of the 24th International Colloquium, ICALP'97. *Springer-Verlag Lecture Notes in Computer Science* **1256** 325–335.

Fu, Y. (1999) Variations on Mobile Processes. *Theoretical Computer Science* **221** 327–368.

Fu, Y. (2010a) Theory of Interaction. Working Paper.

Fu, Y. (2010b) The Value-Passing Calculus. Working Paper.

Fu, Y and Lu, H. (2010) On the Expressiveness of Interaction. *Theoretical Computer Science* **411** 1387–1451.

Fu, Y. and Zhu, H. (2010) The Name-Passing Calculus. Working Paper.

Honda, K. and Tokoro, M. (1991) An Object Calculus for Asynchronous Communications. In: America, P. (ed.) Proceedings ECOOP '91: European Conference on Object-Oriented Programming. *Springer-Verlag Lecture Notes in Computer Science* **512** 133–147.

Honda, K. and Tokoro, M. (1991) On Asynchronous Communication Semantics. In: Madsen, O. L. (ed.) Proceedings ECOOP '92: European Conference on Object-Oriented Programming. *Springer-Verlag Lecture Notes in Computer Science* **615** 21–51.

Merro, M. and Sangiorgi, D. (2004) On Asynchrony in Name-Passing Calculi. *Mathematical Structures in Computer Science* **14** (5) 715–767.

Milner, R. (1992) Functions as Processes. *Mathematical Structures in Computer Science* **2** (2) 119–146.

Milner, R. (1997) The Polyadic π-calculus: a Tutorial. *Theoretical Computer Science* **198** 239–249.

Milner, R., Parrow, J. and Walker, D. (1992) A Calculus of Mobile Processes, Part I and Part II. *Information and Computation* **100** (1) 1–77.

Milner, R. and Sangiorgi, D. (1992) Barbed Bisimulation. In: Kuich, W. (ed.) Automata, Languages and Programming, Proceedings 19th International Colloquium ICALP'92. *Springer-Verlag Lecture Notes in Computer Science* **623** 685–695.

Nestmann, U. and Pierce, B. (2000) Decoding Choice Encodings. *Information and Computation* **163** (1) 1–59.

Palamidessi, C. (2003) Comparing the Expressive Power of the Synchronous and Asynchronous π-calculi. *Mathematical Structures in Computer Science* **13** (5) 685–719.

Parrow, J. and Victor, B. (1997) The Update Calculus. In: Johnson, M. (ed.) Algebraic Methodology and Software Technology, Proceedings 6th International Conference, AMAST'97. *Springer-Verlag Lecture Notes in Computer Science* **1349** 409–423.

Parrow, J. and Sangiorgi, D. (1995) Algebraic Theories for Name-Passing Calculi. *Information and Computation* **120** 174–197.

Plotkin, G. (1975) Call-by-Name, Call-by-Value and the λ-calculus. *Theoretical Computer Science* **1** (2) 125–159.

Sangiorgi, D. (1993a) *Expressing Mobility in Process Algebras: First-Order and Higher-Order Paradigms*, Ph.D. thesis, University of Edinburgh.

Sangiorgi, D. (1993b) From π-calculus to Higher-Order π-calculus – and Back. In: Gaudel, M.-C. and Jouannaud, J.-P. (eds.) Proceedings TAPSOFT '93: Theory and Practice of Software Development. *Springer-Verlag Lecture Notes in Computer Science* **668** 151–166.

Sangiorgi, D. (1994) The Lazy λ-calculus in a Concurrency Scenario. *Information and Computation* **111** 120–153.

Sangiorgi, D. (1995) Lazy Functions and Mobile Processes. Technical Report 2515, INRIA Sophia-Antipolis.

Sangiorgi, D. (1996) π-calculus, Internal Mobility, and Agent-Passing Clculi. *Theoretical Computer Science* **167** (1-2) 235–274.

Sangiorgi, D. and Walker, D. (2001) *The π Calculus: A Theory of Mobile Processes*, Cambridge University Press.

Thomsen, B. (1993) Plain CHOCS – A Second Generation Calculus for Higher Order Processes. *Acta Informatica* **30** (1) 1–59.

Thomsen, B. (1995) A Theory of Higher Order Communicating Systems. *Information and Computation* **116** (1) 38–57.

Walker, D. (1991) π-calculus Semantics for Object-Oriented Programming Languages. In: Ito, T. and Meyer, A. R. (eds.) Theoretical Aspects of Computer Software: Proceedings International Conference TACS '91. *Springer-Verlag Lecture Notes in Computer Science* **526** 532–547.

Walker, D. (1995) Objects in the π-calculus. *Information and Computation* **116** (2) 253–271.