Types and Programming Languages

Lecture 1. Untyped arithmetic expressions

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Course overview

Introduction; Preliminaries
The Lambda Calculus
Types; The simply typed Lambda Calculus
Extensions; derived forms
Normalization, references; exceptions
Subtyping
Imperitive objects
Recursive types
Polymorphism; Universal and existential types
System F
Type operators and kinding
Higher-order polymorphism, subtyping
Course policy

- Mid exam: 30%
- Final exam: 40%
- Homework: 30%
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- Course homepage: http://basics.sjtu.edu.cn/~xiaojuan/tap12015
Outline

Introduction

Preliminaries

Untyped arithmetic expressions
  Abstract syntax
  Induction on terms
Semantics
  Booleans
  Numbers and Booleans
Types in Computer Science

Type systems is the most popular and best established lightweight formal methods.

Definition
A type system is a tractable syntactic method for proving the absence of certain program behaviors by classifying phrases according to the kinds of values they compute.

Two branches in Computer Science:
- More practical: applications to PL
- More abstract: Connections of types in PL with logic
Brief history

Types system (type theory) refers to a much broader field.

- **1900.** Formalized, Russell’s paradox
- **1925.** Simple theory of types, Ramsey
- **1940.** Simply typed λ-calculus, Church
- **1973.** Constructive type theory, Martin Löf
- **1992.** Pure type theory, Barendregt
- ...
Some definitions

- **Static type system.** Type checking during compile-time
- **Dynamic type system.** Type checking during run-time
- Static $\Rightarrow$ Conservative $\Rightarrow$ prove the absence of bad behaviours
- Incapable of finding all undesired program behaviours, e.g. divide by zero
- Type checkers
  - automatic: no manual interaction
  - type annotations
- Simple type checker may exhibit huge type checking.
What types good for

- Detecting errors **early**.
- Maintenance tools.
- Abstracting
- Documentation
- Language safety. A safe language is one that protects its own abstractions. C/C++ are unsafe.
- Efficiency
- Applications: network security, program analysis, theorem prover, database, xml, ...
- Language design goes hand-in-hand with type system design.
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An \(n\)-place relation is a set \(R \subseteq S_1 \times S_2 \times \cdots \times S_n\).

A two-place relation \(R\) on sets \(S\) and \(T\) is called a binary relation. We often write \(s R t\) instead of \((s, t) \in R\).

The ”mixfix” concrete syntax, e.g, \(\Gamma \vdash s : T\) means “the triple \((\Gamma, s, T)\) in the typing relation”.

\(P\) is preserved by \(R\) if whenever we have \(s R t\) and \(P(s)\), we also have \(P(t)\).
Functions

- \textit{dom}(R): the domain of a relation \( R \) on sets \( S \) and \( T \) is the set of elements \( s \in S \) such that \( (s, t) \in R \) for some \( t \).

- A relation \( R \) on sets \( S \) and \( T \) is called a \textit{partial function} if, whenever \( (s, t_1) \in R \) and \( (s, t_2) \in R \), we have \( t_1 = t_2 \). If \( \text{dom}(R) = S \), then \( R \) is a \textit{total function}.

- We write \( f(x) \uparrow \) to mean “\( f \) is undefined on \( x \),” and \( f(x) \downarrow \) to mean “\( f \) is defined on \( x \).”
Ordered sets

A binary relation \( R \) on a set \( S \) is

- **Reflexive**: \( \forall x \in S. x R x. \)
- **Transitive**: \( x R y \land y R z \) implies \( x R z \).
- **Symmetric**: \( x R y \) implies \( y R x. \)
- **Antisymmetric**: \( x R y \land y R x \) implies \( x = y \).

1. **Preorder** (or Quasi order): Reflexive + Transitive
2. **Equivalence**: Preorder + Symmetric
3. **Partial order**: Preorder + Antisymmetric
4. **Total order**: Partial order + (\( \forall x, y \in S. x R y \lor y R x \))
5. **Well quasi order**: Preorder + (Any infinite sequence contains an increasing pair)
6. **Well founded order**: Preorder + (No infinite decreasing sequences)

Quiz: 1. Can Transitivity + Symmetry indicate Reflexivity?
2. Give examples to differentiate these orders.
Inductions

- **Ordinary induction on natural numbers**
  If $P(0)$
  and for all $i$, $P(i)$ implies $P(i) + 1$,
  then $P(n)$ holds for all $n$.

- **Complete induction on natural numbers**
  If, for each natural number $n$,
  given $P(i)$ for all $i < n$
  we can show $P(n)$
  then $P(n)$ holds for all $n$. 
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Untyped systems

- Untyped arithmetic expressions
- Untyped $\lambda$-calculus
- ML implementations
Introduction

$t ::= \text{true} \quad \text{constant true}
\text{false} \quad \text{constant false}
\text{if } t \text{ then } t \text{ else } t \quad \text{conditional}
0 \quad \text{constant zero}
\text{succ } t \quad \text{successor}
\text{pred } t \quad \text{predecessor}
iszero t \quad \text{zero test}

- BNF grammar
- $t$ is metavariable.
- For simplicity, we use arabic numbers, e.g. 3 stands for $(\text{succ} (\text{succ} (\text{succ} 0)))$
- Currently, $\text{if } (\text{succ } 0) \text{ then } \text{true} \text{ else } (\text{pred } 0)$ is a valid term.
Other ways to give syntax definition

The set of *terms* is the smallest set $T$ such that

- Inductively.
  - $\{\text{true}, \text{false}, 0\} \subseteq T$;
  - if $t_1 \in T$, then $\{\text{succ } t_1, \text{pred } t_1, \text{iszero } t_1\} \subseteq T$;
  - if $t_1, t_2, t_3 \in T$, then if $t_1$ then $t_2$ else $t_3 \in T$

- By inference rules

\[
\begin{align*}
\text{true} & \in T & \text{false} & \in T & \text{0} & \in T \\
\text{succ } t_1 & \in T & \text{pred } t_1 & \in T & \text{iszero } t_1 & \in T \\
\text{if } t_1 \text{ then } t_2 \text{ else } t_3 & \in T
\end{align*}
\]
Concretely.

For each natural number $i$, define $S_i$ as follows:

$$
S_0 = \emptyset
$$

$$
S_{i+1} = \{\text{true}, \text{false}, 0\}
\cup\{\text{succ } t_1, \text{pred } t_1, \text{iszero } t_1 \mid t_1 \in S_i\}
\cup\{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \mid t_1, t_2, t_3 \in S_i\}
$$

$$
S = \bigcup_i S_i
$$

Lemma. $S = T$.

Quiz. What if we change the definition of $S_{i+1}$ to

$$
S_{i+1} = \{\text{succ } t_1, \text{pred } t_1, \text{iszero } t_1 \mid t_1 \in S_i\}
\cup\{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \mid t_1, t_2, t_3 \in S_i\}
$$
Inductive structure

For any $t \in T$, one of three things must be true about $t$:

1. $t$ is constant
2. $t$ has form $\text{succ } t_1, \text{pred } t_1, \text{ or iszero } t_1$
3. $t$ has form $\text{if } t_1 \text{ then } t_2 \text{ else } t_3$.

Two ways to use this observation: inductive definition and inductive proof.
Inductive definition

\[
\begin{align*}
\text{consts}(\text{true}) & = \{\text{true}\} \\
\text{consts}(\text{false}) & = \text{false} \\
\text{consts}(0) & = \{0\} \\
\text{consts}(\text{succ } t_1) & = \text{consts}(t_1) \\
\text{consts}(\text{pred } t_1) & = \text{consts}(t_1) \\
\text{consts}(\text{iszero } t_1) & = \text{consts}(t_1) \\
\text{consts}(\text{if } t_1 \text{ then } t_2 \text{ else } t_3) & = \text{consts}(t_1) \cup \text{consts}(t_2) \cup \text{consts}(t_3)
\end{align*}
\]

Quiz. Give an inductive definition of \textit{size}, which is the size of the syntax tree of a term \( t \).
\[
\begin{align*}
\text{size(true)} & = 1 \\
\text{size(false)} & = 1 \\
\text{size(0)} & = 1 \\
\text{size(succ } t_1) & = \text{size}(t_1) + 1 \\
\text{size(pred } t_1) & = \text{size}(t_1) + 1 \\
\text{size(iszero } t_1) & = \text{size}(t_1) + 1 \\
\text{size(if } t_1 \text{ then } t_2 \text{ else } t_3) & = \text{size}(t_1) + \text{size}(t_2) + \text{size}(t_3) + 1
\end{align*}
\]

**Lemma.** \(|\text{consts}(t)| \leq \text{size}(t)|

**Proof.** By induction on the structure of \(t\).
Principles of induction on terms

- **Induction on depth:**
  If, for each term $s$,
  given $P(r)$ for all $r$ such that $\text{depth}(r) < \text{depth}(s)$,
  we can show $P(s)$,
  then $P(s)$ holds for all $s$.

- **Induction on size:**
  If, for each term $s$,
  given $P(r)$ for all $r$ such that $\text{size}(r) < \text{size}(s)$,
  we can show $P(s)$,
  then $P(s)$ holds for all $s$.

- **Structural Induction:**
  If, for each term $s$,
  given $P(r)$ for all immediate subterms $r$ of $s$,
  we can show $P(s)$,
  then $P(s)$ holds for all $s$. 
Semantics of languages

- **Operational semantics.** It specifies the behavior of PL by defining an *abstract machine*.
- **Denotational semantics.** The meaning of a term is taken to be some mathematical object (a number or a function).
- **Axiomatic semantics.** It takes the laws themselves as the definition of the language.
Evaluation

Syntax

\[ t ::= \]
\[
\text{true} \quad \text{constant true}
\]
\[
\text{false} \quad \text{constant false}
\]
\[
\text{if } t \text{ then } t \text{ else } t \quad \text{conditional}
\]

\[ v ::= \]
\[
\text{true} \quad \text{true value}
\]
\[
\text{false} \quad \text{false value}
\]
Evaluation rules for Booleans

Evaluation

\[
\begin{align*}
\text{E-IfTrue} & \quad \text{if true then } t_2 \text{ else } t_3 \rightarrow t_2 \\
\text{E-IfFalse} & \quad \text{if false then } t_2 \text{ else } t_3 \rightarrow t_3 \\
\text{E-If} & \quad t_1 \rightarrow t'_1 \\
\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \rightarrow \text{if } t'_1 \text{ then } t_2 \text{ else } t_3
\end{align*}
\]

\text{E-IfTrue} \text{ and E-IfFalse are also called computation rules and E-If is called congruence rule.}

\textbf{Quiz.} Evaluate the following terms:

\begin{itemize}
\item true
\item if true then (if false then false else false) else true
\end{itemize}
Derivation tree of One-step evaluation

\[ s \overset{\text{def}}{=} \text{if true then false else false} \]

\[ t \overset{\text{def}}{=} \text{if } s \text{ then false else false} \]

\[ u \overset{\text{def}}{=} \text{if false then false else false} \]

\[ \begin{array}{c}
\text{E-IfTrue} \\
\text{E-If} \\
\text{E-If}
\end{array} \\
\hline
\begin{array}{c}
s \rightarrow \text{false} \\
t \rightarrow u
\end{array}
\]

\[ \text{if } t \text{ then false else false } \rightarrow \text{if } u \text{ then false else false } \]

**Theorem 3.5.4** [Determinacy of one-step evaluation]: If \( t \rightarrow t' \) and \( t \rightarrow t'' \), then \( t' = t'' \).

**Proof.** By induction on the depth of the derivation tree.
Normal form and multi-step evaluation

- A term $t$ is in normal form if no evaluation rule can apply to it.

**Theorem 3.5.7:** Every value is in normal form.
**Theorem 3.5.8:** If $t$ is in normal form, then it is a value.

- The multi-step evaluation relation $\rightarrow^*$ is the reflexive, transitive closure of $\rightarrow$.

**Theorem 3.5.11 [Uniqueness of normal forms]:** If $t \rightarrow^* u$ and $t \rightarrow^* u'$ where $u, u'$ are normal forms, then $u = u'$.
**Theorem 3.5.12 [Termination of evaluation]:** For every term $t$ there is some normal form $u$ such that $t \rightarrow^* u$. 
Evaluation

Syntax

t ::=

... terms

0 constant zero
succ t successor
pred t predecessor
iszero t zero test

v ::=

... values

... numeric values

nv ::=

nv numeric value

0 zero value
succ nv successor value
Evaluation rules

Quiz. Give the definition of evaluation rules to guarantee

[Determinacy of one-step evaluation]:
If \( t \rightarrow t' \) and \( t \rightarrow t'' \), then \( t' = t'' \).

Evaluation

\[
\begin{align*}
\text{E-PredZero} & : \quad \text{pred } 0 \rightarrow 0 \quad \text{E-IszeroZero} : \quad \text{iszero } 0 \rightarrow \text{true} \\
\text{E-PredSucc} & : \quad \text{pred } (\text{succ } \text{nv}) \rightarrow \text{nv} \\
\text{E-IszeroSucc} & : \quad \text{iszero } (\text{succ } \text{nv}) \rightarrow \text{false} \\
\text{E-Pred} & : \quad \text{pred } t_1 \rightarrow \text{pred } t'_1 \\
\text{E-Iszero} & : \quad \text{iszero } t_1 \rightarrow \text{iszero } t'_1 \\
\text{E-Succ} & : \quad \text{succ } t_1 \rightarrow \text{succ } t'_1
\end{align*}
\]
Normal form and stuckness

- Note there are meaningless terms, such as if 0 then (succ true) else (iszeoro false).
- What if we change $E\text{-PRED Succ}$ to $\text{pred} (\text{succ } t) \rightarrow t$? Does it still satisfy [Determinacy of one-step transition]?
- A term $t$ is stuck if it is in normal form but not a value.
- See exercises for more about stuckness.
Types are very important for PLs.

This course will give a full view of type systems from the simplest one to full-fledged one.

Fundamental concepts for PLs:

- **syntax**, defined inductively, concretely, ...
- **inductive proofs** are very important for PLs, especially, *structural induction*.
- **operational semantics** plays more and more important roles. We define evaluation rules by using operational transitions.

Properties such as [Determinacy of one-step evaluation], [Uniqueness of normal forms], and [Termination] are important for a good language design.
Homework

- 2.2.7, 2.2.8, 3.2.5, 3.3.4, 3.5.13, 3.5.16, 3.5.17, 3.5.18.