X. Creative Set

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Quotation from Post

The terminology 'creative set' was introduced by E. Post in

Recursively Enumerable Sets of Positive Integers and their Decision Problems. *Bulletin of American Mathematical Society*, 1944.

"…every symbolic logic is incomplete and extensible relative to the class of propositions".

"The conclusion is inescapable that even for such fixed, well-defined body of mathematical propositions, mathematical thinking is, and must remain, essentially creative."

What are the Most Difficult Semi-Decidable Problems?

We know that K is the most difficult semi-decidable problem.

What is then the m-degree $d_m(K)$?

What is an r.e. set C s.t. $A \leq_m C$ for every r.e. set A?

What are the Most Difficult Semi-Decidable Problems?

An r.e. set is very difficult if it is very non-recursive.

An r.e. set is very non-recursive if its complement is very non-r.e..

A set is very non-r.e. if it is easy to distinguish it from any r.e. set.

These sets are creative respectively productive.

Synopsis

1. Productive Set
2. Creative Set
3. The Lattice of m-Degrees

1. Productive Set

Suppose $W_x \subseteq \overline{K}$. Then $x \in \overline{K} \setminus W_x$.

So x witnesses the strict inclusion $W_x \subsetneq \overline{K}$.

In other words the identity function is an effective proof that $\overline{K}$ differs from every r.e. set.

Productive Set

A set A is productive if there is a total computable function p such that whenever $W_x \subseteq A$, then $p(x) \in A \setminus W_x$.

The function p is called a productive function for A.

A productive set is not r.e. by definition.

Example

1. $\overline{K}$ is productive.
2. $\{x \mid c \notin W_x\}$ is productive.
3. $\{x \mid c \notin E_x\}$ is productive.
4. $\{x \mid \phi_x(x) \neq 0\}$ is productive.

Example

Suppose $A = \{x \mid \phi_x(x) \neq 0\}$.

By S-m-n Theorem one gets a primitive recursive function $p(x)$ such that $\phi_{p(x)}(y) = 0$ if and only if $\phi_x(y)$ is defined. Then

$$p(x) \in W_x \Leftrightarrow p(x) \notin A.$$

So if $W_x \subseteq A$ we must have $p(x) \in A \setminus W_x$.

Thus p is a productive function for A.

Productive Set

Lemma. If $A \leq_m B$ and A is productive, then B is productive.

Proof.

Suppose $r : A \leq_m B$ and p is a production function for A.

By applying S-m-n Theorem to $\phi_x(r(y))$, one gets a primitive recursive function $k(x)$ such that $W_{k(x)} = r^{-1}(W_x)$.

Then rpk is a production function for B. □

Productive Set

Theorem. Suppose that $\mathcal{B}$ is a set of unary computable functions with $f_\emptyset \in \mathcal{B}$ and $\mathcal{B} \neq \mathcal{C}_1$. Then $B = \{x \mid \phi_x \in \mathcal{B}\}$ is productive.

Proof.

Suppose $g \notin \mathcal{B}$. Consider the function f defined by

$$f(x,y) \simeq \begin{cases} g(y), & \text{if } x \in W_x, \\ \uparrow, & \text{if } x \notin W_x. \end{cases}$$

By S-m-n Theorem there is a primitive recursive function $k(x)$ such that $\phi_{k(x)}(y) \simeq f(x,y)$.

Clearly $x \notin W_x$ iff $\phi_{k(x)} = f_\emptyset$ iff $\phi_{k(x)} \in \mathcal{B}$ iff $k(x) \in B$.

Hence $k : \overline{K} \leq_m B$. □

Property of Productive Set

Lemma. Suppose that g is a total computable function. Then there is a primitive recursive function p such that for all x, $W_{p(x)} = W_x \cup \{g(x)\}$.

Proof.

Using S-m-n Theorem, take $p(x)$ to be a primitive recursive function such that

$$\phi_{p(x)}(y) \simeq \begin{cases} 1, & \text{if } y \in W_x \vee y = g(x), \\ \uparrow, & \text{otherwise.} \end{cases}$$

We are done. □

Property of Productive Set

Theorem. A productive set contains an infinite r.e. subset.

Proof.

Suppose p is a production function for A.

Take e_0 to be some index for $\emptyset$. Then $p(e_0) \in A$ by definition.

By the Lemma there is a primitive recursive function k such that for all x, $W_{k(x)} = W_x \cup \{p(x)\}$.

Apparently $\{e_0, \ldots, k^n(e_0), \ldots\}$ is r.e.

Consequently $\{p(e_0), \ldots, p(k^n(e_0)), \ldots\}$ is a r.e. subset of A, which must be infinite by the definition of k. □

Productive Function via a Partial Function

Proposition. A set A is productive iff there is a partial recursive function p such that

$$\forall x.(W_x \subseteq A \Rightarrow (p(x)\downarrow \wedge p(x) \in A \setminus W_x)). \qquad (1)$$

Proof.

Suppose p is a partial recursive function satisfying (1). Let s be a primitive recursive function such that

$$\phi_{s(x)}(y) \simeq \begin{cases} y, & p(x)\downarrow \wedge\, y \in W_x, \\ \uparrow, & \text{otherwise.} \end{cases}$$

A productive function q can be defined by running $p(x)$ and $p(s(x))$ in parallel and stops when either terminates. □

Productive Function Made Injective

Proposition. A productive set has an injective productive function.

Proof.

Suppose p is a productive function of A. Let

$$W_{h(x)} = W_x \cup \{p(x)\}.$$

Clearly

$$W_x \subseteq A \Rightarrow W_{h(x)} \subseteq A. \qquad (2)$$

Define $q(0) = p(0)$.

- If $p(x{+}1), ph(x{+}1), \ldots, ph^{x+1}(x{+}1)$ are pairwise distinct, let $q(x{+}1)$ be the smallest one not in $\{q(0), \ldots, q(x)\}$.
- Otherwise we can let $q(x{+}1)$ be $\mu y.y \notin \{q(0), \ldots, q(x)\}$. This is fine since $W_x \not\subseteq A$ due to (2).

It is easily seen that q is an injective production function for A. □

Myhill's Characterization of Productive Set

Theorem. (Myhill, 1955) A is productive iff $\overline{K} \leq_1 A$ iff $\overline{K} \leq_m A$.

$\overline{K} \leq_1 A$ implies $\overline{K} \leq_m A$, which in turn implies "A is productive".

Proof

Suppose p is a productive function for A. Define

$$f(x,y,z) \simeq \begin{cases} 0, & \text{if } z = p(x) \text{ and } y \in K, \\ \uparrow, & \text{otherwise.} \end{cases}$$

By S-m-n Theorem there is an injective primitive recursive function $s(x,y)$ such that

$$\phi_{s(x,y)}(z) \simeq f(x,y,z).$$

By definition,

$$W_{s(x,y)} = \begin{cases} \{p(x)\}, & \text{if } y \in K, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Proof

By Recursion Theorem there is an injective primitive recursive function $n(y)$ such that $W_{s(n(y),y)} = W_{n(y)}$ for all y. So

$$W_{n(y)} = \begin{cases} \{p(n(y))\}, & \text{if } y \in K, \\ \emptyset, & \text{otherwise.} \end{cases}$$

We claim that $\overline{K} \leq_m A$.

$y \in K \Rightarrow W_{n(y)} = \{p(n(y))\} \Rightarrow p(n(y)) \notin A$.

$y \notin K \Rightarrow W_{n(y)} = \emptyset \Rightarrow p(n(y)) \in A$.

By the previous theorem we may assume that p is injective. So the reduction function $p(n(\_))$ is injective. Conclude $\overline{K} \leq_1 A$.

2. Creative Set

Creative Set

A set A is creative if it is r.e. and its complement $\overline{A}$ is productive.

Intuitively a creative set A is effectively non-recursive in the sense that the non-recursiveness of $\overline{A}$, hence the non-recursiveness of A, can be effectively demonstrated.

Creative Set

1. K is creative.
2. $\{x \mid c \in W_x\}$ is creative.
3. $\{x \mid c \in E_x\}$ is creative.
4. $\{x \mid \phi_x(x) = 0\}$ is creative.

Creative Set

Theorem. Suppose that $\mathcal{A} \subseteq \mathcal{C}_1$ and let $A = \{x \mid \phi_x \in \mathcal{A}\}$. If A is r.e. and $A \neq \emptyset, \mathbb{N}$, then A is creative.

Proof.

Suppose A is r.e. and $A \neq \emptyset, \mathbb{N}$. If $f_\emptyset \in \mathcal{A}$, then A is productive by a previous theorem. This is a contradiction.

So $\overline{A}$ is productive by the same theorem. Hence A is creative. □

Creative Set

The set $K_0 = \{x \mid W_x \neq \emptyset\}$ is creative. It corresponds to the set $\mathcal{A} = \{f \in \mathcal{C}_1 \mid f \neq f_\emptyset\}$.

Creative Sets are m-Complete

Theorem. (Myhill, 1955)
C is creative iff C is m-complete iff C is 1-complete iff $C \equiv K$.

3. The Lattice of m-Degrees

What Else?

Q: In the world of recursively enumerable sets, is there anything between the recursive sets and the creative sets?

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Q: In the world of recursively enumerable sets, is there anything between the recursive sets and the creative sets?

A: There is plenty.

Trivial m-Degrees

1. $\mathbf{o} = \{\emptyset\}$.
2. $\mathbf{n} = \{\mathbb{N}\}$.
3. $\mathbf{o} \leq_m \mathbf{a}$ provided $\mathbf{a} \neq \mathbf{n}$.
4. $\mathbf{n} \leq_m \mathbf{a}$ provided $\mathbf{a} \neq \mathbf{o}$.

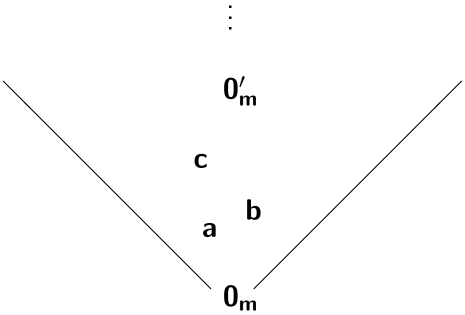
Nontrivial m-Degrees

5. The recursive m-degree $\mathbf{0}_m$ consists of all the nontrivial recursive sets.

6. An r.e. m-degree contains only r.e. sets.

7. The maximum r.e. m-degree $d_m(K)$ is denoted by $\mathbf{0}'_m$.

The Distributive Lattice of m-Degrees



The m-degrees ordered by $\leq_m$ form a distributive lattice.

Problem with m-Degree

The m-reducibility has two unsatisfactory features:

(i) The exceptional behavior of $\emptyset$ and $\mathbb{N}$.

(ii) The invalidity of $A \not\equiv_m \overline{A}$ in general.

The problem is due to the restricted use of oracles.

We shall remove this restriction in Turing reducibility.