XII. Elementary Function

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What is the class of arithmetic functions we use in mathematics?

Definition

The class $\mathcal{E}$ of elementary function is constructed from

1. zero, successor, projection, and
2. subtraction $x \dot{-} y$ (for defining conditionals); and is
3. closed under composition and
4. bounded sum/product (bounded recursion).

Remark.

1. Addition and multiplication can be defined using bounded sum; (hyper) exponential can be defined using bounded product.

2. Lower elementary functions are constructed by dropping (4).

3. A predicate is elementary if its characterization function is elementary.

Bounded Minimalisation is Elementary

Fact. $\mathcal{E}$ is closed under bounded minimalisation.

Proof.

Suppose $f(x,z)$ is elementary. Then $\mu z < y.f(x,z) = 0$ is

$$\sum_{v<y} \prod_{u \leq v} \mathrm{sg}(f(x,u)).$$

It is easy to see that sg is elementary. □

Logical Operation

Fact. The set of elementary predicates is closed under negation, conjunction, disjunction and bounded quantifiers.

Basic Arithmetic Functions are Elementary

1. The exponential x^y is defined by $\prod_{z<y} U_1^2(x,z)$.
2. The function p_x is defined by

$$\begin{aligned} p_x &= \mu y < 2^{2^x}.(x = 0 \text{ } or \text{ } y \text{ is the } x\text{th prime}) \\ &= \mu y < 2^{2^x}.\left(x = \sum_{z \leq y} Pr(z)\right) \\ &= \mu y < 2^{2^x}.\left(x \dot{-} \sum_{z \leq y} Pr(z) = 0\right). \end{aligned}$$

Bounded Recursion

Fact. Let $f(x)$ and $g(x,y,z)$ be elementary and $h(x,y)$ be defined from f,g via primitive recursion. If $h(x,y) \leq b(x,y)$ for some elementary function $b(x,y)$, then $h(x,y)$ is elementary.

Proof.

Observe that

$$2^{h(x,0)} 3^{h(x,1)} \ldots p_{y+1}^{h(x,y)} \leq \prod_{z \leq y} p_{z+1}^{b(x,z)}.$$

So we can define $h(x,y)$ by

$$\mu e \leq \prod_{z \leq y} p_{z+1}^{b(x,z)}.((e)_1 = f(x) \wedge \forall z < y.((e)_{z+2} = g(x,z,(e)_{z+1}))).$$

We are done. □

Gödel Encoding is Elementary

Fact. Gödel encoding functions are elementary.

Kleene's Predicate is Elementary

Fact. The Kleene's functions σ_n, c_n and j_n are elementary.

Elementary Time Functions

A computable function ϕ_e is in elementary time if $t_e(x) \leq b(x)$ almost everywhere for some elementary function $b(x)$.

Fact. The elementary time functions are elementary.

Proof.
$\phi_e(x)$ is almost everywhere computable by the elementary function

$$(c_n(e, \widetilde{x}, \mu t \leq b(\widetilde{x}).j_n(e, \widetilde{x}, t) = 0))_1,$$

which implies that $\phi_e(x)$ is elementary. □

It has been suggested that $\mathcal{E}$ contains all practical computable functions.

A computable function $f(x)$ is practically computable if it can be computed in

$$exp_k(x) = \underbrace{2^{2^{\cdots^{2^x}}}}_{k}$$

steps for some k. We let $2^{exp_k(x)}$ stand for $exp_{k+1}(x)$.

Upper Bound of Elementary Functions

Theorem. For each elementary function $f(\widetilde{x})$ there is some k such that $f(\widetilde{x}) \leq exp_k(\max\{\widetilde{x}\})$.

Proof.

The basic elementary functions satisfy the upper bound. The elementary operations preserves the upper bound. □

Corollary. $exp_x(x)$ is primitive recursive but not elementary.

Proof.

The function $exp_x(x)$ is defined by $g(x, x)$, where

$$\begin{aligned} g(x, 0) &= x, \\ g(x, y+1) &= 2^{g(x,y)}. \end{aligned}$$

We are done. □

Elementary Functions are Elementary Time

Lemma. Suppose $f(\widetilde{x})$ and $g(\widetilde{x}, y, z)$ are in elementary time and $h(\widetilde{x}, y)$ is defined from f, g via recursion. If $h(\widetilde{x}, y)$ is elementary, then $h(\widetilde{x}, y)$ is in elementary time.

Proof.

The standard program that calculates h does it in elementary time. □

Elementary Functions are Elementary Time

Theorem. If $f(\widetilde{x})$ is elementary, then there is a program P for f such that $t_P^n(\widetilde{x})$ is elementary.

Proof.

Use the above lemma. □

Complexity Theoretical Characterization

Theorem. A total function $f(\widetilde{x})$ is elementary iff it is computable in time $exp_k(\max\{\widetilde{x}\})$ for some k.

$$\mathbf{ELEMENTARY} = \mathbf{TIME}(2^n) \cup \mathbf{TIME}(2^{2^n}) \cup \mathbf{TIME}(2^{2^{2^n}}) \cup \ldots.$$