XII. Elementary Function

Yuxi Fu

BASICS, Shanghai Jiao Tong University
What is the class of arithmetic functions we use in mathematics?
Definition

The class $\mathcal{E}$ of elementary function is constructed from

1. zero, successor, projection, and
2. subtraction $x - y$ (for defining conditionals); and is
3. closed under composition and
4. bounded sum/product (bounded recursion).

Remark.

1. Addition and multiplication can be defined using bounded sum; (hyper) exponential can be defined using bounded product.
2. Lower elementary functions are constructed by dropping (4).
3. A predicate is elementary if its characterization function is elementary.
**Fact.** $\mathcal{E}$ is closed under bounded minimalisation.

**Proof.**
Suppose $f(x, z)$ is elementary. Then $\mu z < y. f(x, z) = 0$ is

$$\sum \prod_{v < y \atop u \leq v} \text{sg}(f(x, u)).$$

It is easy to see that $\text{sg}$ is elementary.
Fact. The set of elementary predicates is closed under negation, conjunction, disjunction and bounded quantifiers.
1. The exponential $x^y$ is defined by $\prod_{z < y} U_1^2(x, z)$.

2. The function $p_x$ is defined by

$$p_x = \mu y < 2^{2^x} \cdot \left( x = 0 \text{ or } y \text{ is the xth prime} \right)$$

$$= \mu y < 2^{2^x} \cdot \left( x = \sum_{z \leq y} Pr(z) \right)$$

$$= \mu y < 2^{2^x} \cdot \left( x - \sum_{z \leq y} Pr(z) = 0 \right).$$
Fact. Let $f(x)$ and $g(x, y, z)$ be elementary and $h(x, y)$ be defined from $f, g$ via primitive recursion. If $h(x, y) \leq b(x, y)$ for some elementary function $b(x, y)$, then $h(x, y)$ is elementary.

Proof.
Observe that

$$2^{h(x,0)}3^{h(x,1)}\cdots p_{y+1}^{h(x,y)} \leq \prod_{z \leq y} p_{z+1}^{b(x,z)}.$$ 

So we can define $h(x, y)$ by

$$\mu e \leq \prod_{z \leq y} p_{z+1}^{b(x,z)} \cdot ((e)_1 = f(x) \land \forall z < y . ((e)_{z+2} = g(x, z, (e)_{z+1}))).$$

We are done.
Fact. Gödel encoding functions are elementary.
Kleene’s Predicate is Elementary

**Fact.** The Kleene’s functions $\sigma_n$, $c_n$ and $j_n$ are elementary.
A computable function $\phi_e$ is in elementary time if $t_e(x) \leq b(x)$ almost everywhere for some elementary function $b(x)$.

**Fact.** The elementary time functions are elementary.

**Proof.**

$\phi_e(x)$ is almost everywhere computable by the elementary function

$$(c_n(e, \tilde{x}, \mu t \leq b(\tilde{x}).j_n(e, \tilde{x}, t) = 0))_1,$$

which implies that $\phi_e(x)$ is elementary.
It has been suggested that $\mathcal{E}$ contains all practical computable functions.
A computable function $f(x)$ is practically computable if it can be computed in

$$\exp_k(x) = 2^{2^{\ldots^{2^x}}}_k$$

steps for some $k$. We let $2^{\exp_k(x)}$ stand for $\exp_{k+1}(x)$. 
Upper Bound of Elementary Functions

**Theorem.** For each elementary function $f(\tilde{x})$ there is some $k$ such that $f(\tilde{x}) \leq \exp_k(\max\{\tilde{x}\})$.

**Proof.**
The basic elementary functions satisfy the upper bound. The elementary operations preserves the upper bound. □
Corollary. \( \exp_x(x) \) is primitive recursive but not elementary.

Proof. The function \( \exp_x(x) \) is defined by \( g(x, x) \), where

\[
\begin{align*}
g(x, 0) &= x, \\
g(x, y + 1) &= 2g(x, y).
\end{align*}
\]

We are done.
Lemma. Suppose $f(\tilde{x})$ and $g(\tilde{x}, y, z)$ are in elementary time and $h(\tilde{x}, y)$ is defined from $f, g$ via recursion. If $h(\tilde{x}, y)$ is elementary, then $h(\tilde{x}, y)$ is in elementary time.

Proof.
The standard program that calculates $h$ does it in elementary time.

\qed
**Theorem.** If $f(\tilde{x})$ is elementary, then there is a program $P$ for $f$ such that $t^n_P(\tilde{x})$ is elementary.

**Proof.**
Use the above lemma.
Theorem. A total function $f(\tilde{x})$ is elementary iff it is computable in time $exp_k(\max\{\tilde{x}\})$ for some $k$. 
\[ \text{ELEMENTARY} = \text{TIME}(2^n) \cup \text{TIME}(2^{2n}) \cup \text{TIME}(2^{22n}) \cup \ldots. \]