# VIII. Recursive Set

Yuxi Fu

BASICS, Shanghai Jiao Tong University

# Decision Problem, Predicate, Number Set

The following emphasizes the importance of number set:

$$\begin{array}{rcl} \text{Decision Problem} & \Leftrightarrow & \text{Predicate on Number} \\ & \Leftrightarrow & \text{Set of Number} \end{array}$$

A central theme of recursion theory is to look for sensible classification of number sets.

Classification is often done with the help of reduction.

# Synopsis

1. Reduction
2. Recursive Set
3. Undecidability
4. Rice Theorem

# 1. Reduction

# Reduction between Problems

A reduction is a way of defining a solution to a problem with the help of a solution to another problem.

In recursion theory we are only interested in reductions that are computable.

# Reduction

There are several ways of reducing a problem to another.

The differences between different reductions from $A$ to $B$ consists in the manner and the extent to which information about $B$ is allowed to settle questions about $A$.

# Many-One Reduction

The set $A$ is many-one reducible, or m-reducible, to the set $B$ if there is a total computable function $f$ such that

$$x \in A \text{ iff } f(x) \in B$$

for all $x$. We shall write $A \leq_m B$ or more explicitly $f : A \leq_m B$.

If $f$ is injective, then it is a one-one reducibility, denoted by $\leq_1$.

# An Example

Suppose $G$ is a finite graph and $k$ is a natural number.

1. The Independent Set Problem (IndSet) asks if there are $k$ vertices of $G$ with every pair of which unconnected.

2. The Clique Problem asks if there is a $k$-complete subgraph of $G$.

There is a simple one-one reduction from IndSet to Clique.

# Many-One Reduction

1. $\leq_m$ is reflexive and transitive.
2. $A \leq_m B$ iff $\overline{A} \leq_m \overline{B}$.
3. $A \leq_m \omega$ iff $A = \omega$; $A \leq_m \emptyset$ iff $A = \emptyset$.
4. $\omega \leq_m A$ iff $A \neq \emptyset$; $\emptyset \leq_m A$ iff $A \neq \omega$.

# m-Degree

1. $A \equiv_m B$ if $A \leq_m B \leq_m A$. (many-one equivalence)
2. $A \equiv_1 B$ if $A \leq_1 B \leq_1 A$. (one-one equivalence)
3. $d_m(A) = \{B \mid A \equiv_m B\}$ is the m-degree represented by $A$.

# m-Degree

The set of m-degrees is ranged over by $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$.

$\mathbf{a} \leq_m \mathbf{b}$ iff $A \leq_m B$ for some $A \in \mathbf{a}$ and $B \in \mathbf{b}$.

$\mathbf{a} <_m \mathbf{b}$ iff $\mathbf{a} \leq_m \mathbf{b}$ and $\mathbf{b} \not\leq_m \mathbf{a}$.

# The Structure of m-Degree

**Proposition**. The m-degrees form a distributive lattice.

# Recursive Permutation

A recursive permutation is one-one recursive function.

$A$ is recursively isomorphic to $B$, written $A \equiv B$, if there is a recursive permutation $p$ such that $p(A) = B$.

# Recursive Invariance

A property of sets is recursively invariant if it is invariant under all recursive permutations.

- '$A$ is infinite' is a recursively invariant property.
- '$2 \in A$' is not recursively invariant.

# Myhill Isomorphism Theorem

**Myhill Isomorphism Theorem** (1955). $A \equiv B$ iff $A \equiv_1 B$.

Proof.

The idea is to construct effectively the graph of an isomorphic function $h$ by two simultaneous symmetric inductions:

$$h_0 \subseteq h_1 \subseteq h_2 \subseteq h_4 \subseteq \ldots \subseteq h_i \subseteq \ldots$$

such that $h = \bigcup_{i \in \omega} h_i$.

At stage $z + 1 = 2x + 1$, if $h_z(x)$ is defined, do nothing.

Otherwise enumerate $\{f(x), f(h_z^{-1}(f(x))), \ldots\}$ until a number $y$ not in $rng(h_z)$ is found. Let $h_{z+1}(x) = y$. □

# The Restriction of m-Reduction

Suppose $G$ is a finite directed weighted graph and $m$ is a number.

- The Hamiltonian Circle Problem (HC) asks if there is a circle in $G$ whose overall weight is no more than $m$.
- The Traveling Sales Person Problem TSP asks for the overall weight of a circle with minimum weight if there are circles.

TSP can be reduced to HC. The reduction is not m-reduction.

# 2. Recursive Set

# Definition of Recursive Set

Let $A$ be a subset of $\omega$. The characteristic function of $A$ is given by

$$c_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

$A$ is recursive if $c_A(x)$ is computable.

# Fact about Recursive Set

**Fact**. If $A$ is recursive then $\overline{A}$ is recursive.

**Fact**. If $A$ is recursive and $B \neq \emptyset, \omega$, then $A \leq_m B$.

**Fact**. If $A, B$ are recursive and $A, B, \overline{A}, \overline{B}$ are infinite then $A \equiv B$.

**Fact**. If $A \leq_m B$ and $B$ is recursive, then $A$ is recursive.

**Fact**. If $A \leq_m B$ and $A$ is not recursive, then $B$ is not recursive.

**Theorem**. An infinite set is recursive iff it is the range of a total increasing computable function.

Proof.

Suppose $A$ is recursive and infinite. Then $A$ is range of the increasing function $f$ given by

$$\begin{aligned} f(0) &= \mu y(y \in A), \\ f(n+1) &= \mu y(y \in A \text{ and } y > f(n)). \end{aligned}$$

The function is total, increasing and computable.

Conversely suppose $A$ is the range of a total increasing computable function $f$. Obviously $y = f(n)$ implies $n \leq y$.

Hence $y \in A \Leftrightarrow y \in Ran(f) \Leftrightarrow \exists n \leq y(f(n) = y)$. □

# 3. Undecidability

# Unsolvable Problem

A decision problem $f : \omega \rightarrow \{0, 1\}$ is solvable if it is computable.

It is unsolvable if it is not solvable.

# Undecidable Predicate

A predicate $M(\widetilde{x})$ is decidable if its characteristic function $c_M(\widetilde{x})$ given by

$$c_M(\widetilde{x}) = \begin{cases} 1, & \text{if } M(\widetilde{x}) \text{ holds,} \\ 0, & \text{if } M(\widetilde{x}) \text{ does not hold.} \end{cases}$$

is computable.

It is undecidable if it is not decidable.

Non-recursive $\Leftrightarrow$ Unsolvable $\Leftrightarrow$ Undecidable

# Some Important Undecidable Sets

$$
\begin{aligned}
K &= \{x \mid x \in W_x\}, \\
K_0 &= \{\pi(x,y) \mid x \in W_y\}, \\
K_1 &= \{x \mid W_x \neq \emptyset\}, \\
Fin &= \{x \mid W_x \text{ is finite}\}, \\
Inf &= \{x \mid W_x \text{ is infinite}\}, \\
Con &= \{x \mid \phi_x \text{ is total and constant}\}, \\
Tot &= \{x \mid \phi_x \text{ is total}\}, \\
Cof &= \{x \mid W_x \text{ is cofinite}\}, \\
Rec &= \{x \mid W_x \text{ is recursive}\}, \\
Ext &= \{x \mid \phi_x \text{ is extensible to a total recursive function}\}.
\end{aligned}
$$

**Fact**. $K$ is undecidable.

Proof.

If $K$ were recursive, the characteristic function

$$c(x) = \begin{cases} 1, & \text{if } x \in W_x, \\ 0, & \text{if } x \notin W_x, \end{cases}$$

would be computable. Let $m$ be an index for

$$g(x) = \begin{cases} 0, & \text{if } c(x) = 0, \\ \uparrow, & \text{if } c(x) = 1. \end{cases}$$

Then $m \in W_m$ iff $c(m) = 0$ iff $m \notin W_m$. □

$K$ is often used to prove undecidability result.

- To show that $A$ is undecidable, it suffices to construct an m-reduction from $K$ to $A$.

**Fact**. There is a computable function $h$ such that both $Dom(h)$ and $Ran(h)$ are undecidable.

Proof.
Define

$$h(x) = \begin{cases} x, & \text{if } x \in W_x, \\ \uparrow, & \text{if } x \notin W_x. \end{cases}$$

Clearly $x \in Dom(h)$ iff $x \in W_x$ iff $x \in Ran(h)$. $\square$

**Fact**. Both *Tot* and $\{x \mid \phi_x \simeq \lambda z.0\}$ are undecidable.

Proof.

Consider the function $f$ defined by

$$f(x,y) = \begin{cases} 0, & \text{if } x \in W_x, \\ \uparrow, & \text{if } x \notin W_x. \end{cases}$$

By S-m-n Theorem there is an injective primitive recursive function $k(x)$ such that $\phi_{k(x)}(y) \simeq f(x,y)$.

It is clear that $k : K \leq_1 Tot$ and $k : K \leq_1 \{x \mid \phi_x \simeq \lambda.0\}$. □

**Fact**. Both $\{x \mid c \in W_x\}$ and $\{x \mid c \in E_x\}$ are undecidable.

Proof.

Consider the function $f$ defined by

$$f(x,y) = \begin{cases} y, & \text{if } x \in W_x, \\ \uparrow, & \text{if } x \notin W_x. \end{cases}$$

By S-m-n Theorem there is some injective primitive recursive function $k(x)$ such that $\phi_{k(x)}(y) \simeq f(x,y)$.

It is clear that $k$ is a one-one reduction from $K$ to both $\{x \mid c \in W_x\}$ and $\{x \mid c \in E_x\}$. □

**Fact**. The predicate '$\phi_x(y)$ is defined' is undecidable.

**Fact**. The predicate '$\phi_x \simeq \phi_y$' is undecidable.

# 4. Rice Theorem

Henry Rice

Classes of Recursively Enumerable Sets and their Decision Problems. Transactions of the American Mathematical Society, 77:358-366, 1953.

**Rice Theorem** (1953).

If $\emptyset \subsetneq \mathcal{B} \subsetneq \mathcal{C}$, then $\{x \mid \phi_x \in \mathcal{B}\}$ is not recursive.

Proof.

Suppose $f_\emptyset \notin \mathcal{B}$ and $g \in \mathcal{B}$. Let $f$ be defined by

$$f(x,y) = \begin{cases} g(y), & \text{if } x \in W_x, \\ \uparrow, & \text{if } x \notin W_x. \end{cases}$$

By S-m-n Theorem there is some injective primitive recursive function $k(x)$ such that $\phi_{k(x)}(y) \simeq f(x,y)$.

It is clear that $k$ is a one-one reduction from $K$ to $\{x \mid \phi_x \in \mathcal{B}\}$. □

# Applying Rice Theorem

Assume that $f(x) \simeq \phi_x(x) + 1$ could be extended to a total computable function say $g(x)$. Let $e$ be an index of $g(x)$. Then $\phi_e(e) = g(e) = f(e) = \phi_e(e) + 1$. Contradiction.

So we may use Rice Theorem to conclude that

$$Ext = \{x \mid \phi_x \text{ is extensible to a total recursive function}\}$$

is not recursive.

**Comment**: Not every partial recursive function can be obtained by restricting a total recursive function.

# Remark on Rice Theorem

Rice Theorem deals with programme independent properties.

It talks about classes of computable functions rather than classes of programmes.

All non-trivial semantic problems are algorithmically undecidable.