Logical Characterizations of Simulation and Bisimulation for Fuzzy Transition Systems

Hengyang Wu[1*] Yuxin Deng[2]

[1]Information Engineer College, Hangzhou Dianzi University

Hangzhou 310018, P.R. China

[2]Shanghai Key Laboratory of Trustworthy Computing, East China Normal University

Shanghai 200062, P.R. China

Abstract

Simulations and bisimulations are known to be useful for abstracting and comparing formal systems, and they have recently been introduced into fuzzy systems. In this study, we provide sound and complete logical characterizations for simulation and bisimulation, which are defined over fuzzy labeled transition systems via two variants of the Hennessy-Milner Logic. The logic for characterizing fuzzy simulation has neither negation nor disjunction, which is very different from the well-known logical characterizations of probabilistic simulations, although the completeness proofs of our characterization results are inspired by relevant research in probabilistic concurrency theory. The logic for characterizing fuzzy bisimulation also deviates from that for probabilistic bisimulations.

Keywords: Bisimulation, Completeness, Fuzzy labeled transition system, Logical characterization, Simulation, Soundness.

1 Introduction

The analysis of fuzzy systems has been the subject of active research during the last 60 years and many formalisms have been proposed for modeling them,

*Corresponding author.

Email address: wuhengy_1974@aliyun.com, yxdeng@sei.ecnu.edu.cn.

including fuzzy automata (e.g., see [2, 3, 7, 28, 29, 31, 36, 38, 44]), fuzzy Petri nets [40], fuzzy Markov processes [4], and fuzzy discrete event systems [30, 37, 39].

Recently, a new formal model for fuzzy systems called fuzzy labeled transition systems (**FLTS**s) was proposed [6, 18, 24]. **FLTS**s are a natural generalization of the classical labeled transition systems in computer science, where after performing some action, a system evolves from one state into a fuzzy set of successor states instead of a unique state. Many formal description tools for fuzzy systems, such as fuzzy Petri nets and fuzzy discrete event systems [30, 37], are not **FLTS**s. However, it is possible to translate a system's description in one of these formalisms into an **FLTS** to represent its behavior.

Bisimulation [34] has been investigated in depth in process algebras because it offers a convenient co-inductive proof technique for establishing behavioral equivalence [32]. Bisimulation has mostly been used for verifying formal systems and it is the foundation of state-aggregation algorithms, which compress models by merging bisimilar states. State aggregation is used routinely as a preprocessing step before model checking [1, 19]. Recently, bisimulation has been introduced into fuzzy systems. For example, Cao et al. [6] considered bisimulations for **FLTS**s. Bisimulation-based reasoning also appeared for fuzzy automata and fuzzy discrete event systems [10, 11, 18, 33, 42, 45].

Following a seminal study that explored the connection between bisimulation and modal logic [22], many studies have characterized various types of bisimulations using appropriate logics, e.g., [16, 17, 23, 27]. A logic characterizes a bisimulation soundly and completely when two states are bisimilar if and only if they satisfy the same set of logical formulae. The significance of logical characterizations is twofold. Based on a sound and complete logical characterization, the problem of checking whether two states are bisimilar is converted into a logical judgment of whether two states satisfy the same set of logical formulae, which can benefit from traditional logic theories and be assisted by some practical tools. A logical characterization also allows model checking to be performed based on a bisimulation quotient transition system because a logical formula holds for the quotient if and only if it holds for the original transition system.

In the present study, we provide logical characterizations of bisimulation and simulation for **FLTS**s. Often, a state or system can simulate another but not *vice versa*. For example, when we check that an implementation matches its specification, we normally do not demand that the implementation performs anything more than that required. It is usually acceptable that the implementation simulates its specification. Hence, it is also interesting to investigate simulations. Unlike other studies of fuzzy systems, we define simulation and bisimulation by virtue of closed subsets of some binary relation (Section 5 provides a detailed discussion). Moreover, some recent studies of **FLTS**s and fuzzy automata have focused mainly on simulations and bisimulations [6, 10, 13, 26, 42], whereas they did not consider logical characterizations. A logical characterization of fuzzy bisimulation was provided by [18], but the differences from the current study are as follows: (1) the logic used in [18] employs recursive formulae where it interprets a formula as a fuzzy set that gives the measure of satisfaction and unsatisfaction for the formula; and (2) we consider bisimulation and simulation, whereas [18] only considered bisimulation.

The logic used to characterize fuzzy bisimulation is very similar to Larsen and Skou's probabilistic extension of the Hennessy-Milner Logic[1]. The completeness proof for our logical characterization of fuzzy bisimulation is also inspired by [23], who characterized probabilistic bisimulation. Indeed, there is only a slight difference between the **FLTS** model and probabilistic labeled transition systems (**PLTS**s). This may create the impression that the current study is a straightforward generalization of the study of **PLTS**s, but this is not the case. For **PLTS**s, disjunction is necessary to characterize simulation, whereas it is not for **FLTS**s. For **PLTS**s, negation is not necessary for characterizing bisimulation and binary conjunction is already sufficient, whereas for **FLTS**s, both negation and infinite conjunction are necessary to characterize bisimulation for general **FLTS**s, which may be infinitely branching. Moreover, different techniques are needed to prove characterization theorems for **FLTS**s and **PLTS**s. For example, in the case of **PLTS**s, the well-known π-λ theorem holds, which greatly simplifies the completeness proof for the logical characterization of bisimulation. However, the π-λ theorem is invalid for **FLTS**s, so we adopt a different approach to prove completeness, where we try to construct a characteristic formula for each equivalence class, i.e., the formula is satisfied only by the states in that equivalence class. Sections 4.2 and 4.3 provide more details.

The remainder of this paper is organized as follows. We briefly review some of the basic concepts used in this study in Section 2. In Section 3, we describe some properties of simulations and bisimulations for **FLTS**s. In particular, similarity and bisimilarity are shown to be closed under the parallel composition of **FLTS**s. Section 4 presents the logical characterization theorems. In this section, we also analyze why the logics characterizing bisimulations for **FLTS**s and **PLTS**s are different. An extended abstract of this part of contents has appeared in [15]. We introduce some related research in Section 5. Finally,

[1]Our logic for fuzzy systems originates from computer science and it is intended to be used for reasoning about fuzzy labeled transition systems, and thus it differs from the fuzzy logic investigated in [41].

we give our conclusions in Section 6 by providing a summary of the differences between the logical characterizations of **FLTS**s and **PLTS**s, as well as discussing future reearch.

2 Preliminaries

In this section, we briefly recall some basic concepts and terminologies from set theory and fuzzy set theory, before introducing **FLTS**s.

Let S be an ordinary set. A *fuzzy set* μ of S is a function that assigns each element s of S with a value $\mu(s)$ in the real unit interval $[0,1]$. The *support* of μ is the set $supp(\mu) = \{s \in S \mid \mu(s) > 0\}$. We denote $\mathcal{F}(S)$ as the set of all fuzzy sets in S and $\mathcal{F}_f(S)$ as the set of all fuzzy sets with finite-support, i.e., $\mathcal{F}_f(S) = \{\mu \in \mathcal{F}(S) \mid supp(\mu) \text{ is finite}\}$. Whenever $supp(\mu)$ is a finite set, such as $\{s_1, s_2, \cdots, s_n\}$, then a fuzzy set μ can be written in Zadeh's notation as follows:

$$\mu = \frac{\mu(s_1)}{s_1} + \frac{\mu(s_2)}{s_2} + \cdots + \frac{\mu(s_n)}{s_n}.$$

For any $\mu \in \mathcal{F}(S)$ and $U \subseteq S$, the notation $\mu(U)$ denotes $\sup_{s\in U} \mu(s)$. With a slight abuse of notation, we sometimes write a possibility distribution as representing a fuzzy set[2].

Let S be a set. For a binary relation $R \subseteq S \times S$, we write sRt if $(s,t) \in R$. A *preorder* relation R is a reflexive and transitive relation, and an *equivalence* relation is a reflexive, symmetric, and transitive relation. An equivalence relation R partitions a set S into equivalence classes. For $s \in S$, we use $[s]_R$ to denote the unique equivalence class containing s. We drop the subscript R if the relation considered is clear from the context. Let $R(s)$ denote the set $\{s' \mid (s,s') \in R\}$. A set U is said to be R-closed if $R(s) \subseteq U$ for all $s \in U$. We let R^* be the reflexive transitive closure of R. Note that if R is a preorder, then R^* coincides with R. For any $s \in S$, the set $R^*(s)$ is clearly a R-closed set.

The following two lemmas will be useful when we prove Theorem 3.6 and 3.8, respectively. Their proofs are trivial and thus they are omitted.

Lemma 2.1 *Let $R \subseteq S \times S$ be a preorder relation and $U \subseteq S$ is R-closed. Then, $U = \cup_{x\in U} R(x)$. In particular, if R is an equivalence relation, then $U = \cup_{x\in U}[x]_R$.*

Lemma 2.2 *Let R_1 and R_2 be two binary relations on S and T, respectively. Define $R = \{((s_1,t_1),(s_2,t_2)) \mid (s_1,s_2) \in R_1, (t_1,t_2) \in R_2\}$.*

[2]Strictly speaking, a possibility distribution is different from a fuzzy set, although the former can be viewed as the generalized characteristic function of the latter. See [46] for a more detailed discussion.

(1) If R_1 and R_2 are preorder relations, then R is a preorder relation on $S \times T$. Moreover, for any $(s,t) \in S \times T$, we have $R((s,t)) = R_1(s) \times R_2(t)$;

(2) If R_1 and R_2 are equivalence relations, then R is an equivalence relation on $S \times T$. Moreover, for any $(s,t) \in S \times T$, we have $[(s,t)]_R = [s]_{R_1} \times [t]_{R_2}$.

Now, we introduce **FLTS**s.

Definition 2.3 *An* **FLTS** *is a triple $\mathcal{S} = (S, A, \alpha)$, where S is a set of states, A is a set of actions, and the transition function α is a mapping from $S \times A$ to $\mathcal{F}(S)$.*

A **PLTS**[3] *is an* **FLTS** *$\mathcal{S} = (S, A, \alpha)$, where the transition function α is a mapping from $S \times A$ to $\mathcal{D}(S)$. In this case, $\mathcal{D}(S) = \{d : S \to [0,1] \mid \sum_{s\in S} d(s) = 1\}$ is the set of all probability distributions on S.*

Sometimes, we write $s \xrightarrow{a} \mu$ and $s \xrightarrow{a[\lambda]} s'$ for $\alpha(s,a) = \mu$ and $\alpha(s,a)(s') = \lambda$, respectively. Note that if s cannot perform action a, then $\alpha(s,a)(s') = 0$ for all $s' \in S$. An **FLTS** $\mathcal{S} = (S, A, \alpha)$ is said to be *image-finite* ($\mathbf{F}_f$**LTS**) if for each state s and label a, $\alpha(s,a) \in \mathcal{F}_f(S)$.

The **FLTS**s considered in the current study are deterministic [8] in the sense that for each state s and label a, the transition function α returns at most one possibility distribution $\alpha(s,a)$. The **PLTS**s defined above are usually called reactive probabilistic systems [20] or labeled Markov chains [16] in probabilistic concurrency theory. In nondeterministic fuzzy transition systems [8], for a state and a label, more than one possibility distribution may be returned by the transition function, which is similar to simple probabilistic automata [43].

Remark 2.4 *Since determinism and nondeterminism have different meanings in different contexts, it is necessary to provide further explanations.*

In the classical concurrency theory, a labeled transition system is deterministic if the transition relation is a partial function from $S \times A$ to S, i.e., after performing an action, the system changes from the current state into at most one successor state. A system is nondeterministic if the transition relation is a partial function from $S \times A$ to $\mathcal{P}(S)$, i.e., after performing an action, the system may evolve nondeterministically into one state among a set of successor states. Similarly, in probabilistic concurrency theory, a probabilistic system is deterministic if the transition relation is a partial function from $S \times A$ to $\mathcal{D}(S)$, i.e., at most one distribution of states can be reached after one transition step. A probabilistic system is nondeterministic if the transition relation is a partial

[3] A variety of probabilistic models have been proposed in previous studies. In the present study, we define **PLTS**s as reactive probabilistic processes, as studied by [27].

function from $S \times A$ to $\mathcal{P}(\mathcal{D}(S))$. Given these concepts, it is appropriate to refer to our **FLTS***s as deterministic.*

We note that determinism has a different meaning in some studies of fuzzy automata. For example, Bělohlávek [3] stated that a fuzzy automaton is deterministic if the initial state is a singleton set, the final state is a fuzzy set on the state space, and the transition function α is a mapping from $S \times A$ to S. Essentially, the same notion was also used by Li and Pedrycz [28]. This kind of determinism was referred to as crisp-determinism by Ćirić et al. [9]. In addition, González de Mendívil and Garitagoitia proposed other types of determinism (see [21] for further details).

As an application, let us illustrate **FLTS**s using an example related to medical diagnosis and treatment, as described by Qiu [37] and Lin and Ying [30] (see also [8, 45]). We assume that there is an unknown bacterial infection. Based on their experience, the physicians consider that three drugs, i.e., $u_1,\ u_2,\ u_3$, may be useful for treating this disease. Three possible negative symptoms, i.e., $v_1,\ v_2,\ v_3$, must also be considered. The physicians consider that the patient's condition can be one of four rough types, i.e., "poor," "fair," "good," or "excellent", which are denoted by $q_1,\ q_2,\ q_3$, and q_4, respectively. A treatment (or a negative symptom) may lead to a state among multiple possible states with certain degrees. For example, the transition $q_2 \xrightarrow{u_1[0.6]} q_3$ means that the patient's condition has changed from "fair" to "good" with a possibility of 0.6 after using drug u_1, whereas $q_2 \xrightarrow{v_1[0.3]} q_1$ means that the patient's condition has changed from "fair" to "poor" with a possibility of 0.3 if the patient has negative symptom v_1. The transition possibilities of these events among states are estimated by the physicians. In this manner, we obtain **FLTS** $\mathcal{S} = (S, A, \alpha)$, where $S = \{q_1, q_2, q_3, q_4\}$ and $A = \{u_1, u_2, u_3, v_1, v_2, v_3\}$.

A patient's initial condition may be "poor" and it will become "fair," "good," or even "excellent" after a specific treatment. When a patient's health becomes "fair," we naturally hope that this will improve to become "excellent" instead of deteriorating. Analogously, if the patient's condition is "excellent," it is desirable to maintain good health and thus a supervisor is necessary to disable events $v_1,\ v_2,\ v_3$ if they are controllable. A general approach for determining whether supervisory control exists for fuzzy discrete event systems is in terms of the fuzzy language equivalence. However, this is not satisfactory because some strings (negative symptoms) are not accepted if they are controllable. Xing et al. proposed the use of bisimulation equivalence to solve this problem [45], while Deng and Qiu [13] also noted some limitations of language equivalence, where they addressed this problem using simulation equivalence.

3 Simulation and Bisimulation

In this section, we introduce our notions of simulation and bisimulation for **FLTS**s, and we discuss their properties.

Based on the idea of defining bisimulations for **PLTS**s [17], we require that if (s,t) is a pair of states in a simulation relation, then t can mimic all the stepwise behaviors of s with respect to R. Thus, if s can perform an action on a possibility distribution μ, then t can perform the same action on another possibility distribution ν such that μ and ν are related via a relation between distributions established by R-closed sets.

Definition 3.1 *Let $\mathcal{S} = (S, A, \alpha)$ be an* **FLTS**. *A relation $R \subseteq S \times S$ is a* simulation *relation if $(s,t) \in R$ implies that for any action $a \in A$, $\alpha(s,a)(U) \leq \alpha(t,a)(U)$ for any R-closed set $U \subseteq S$. For any two states $s,t \in S$, we say that s is simulated by t in $\mathcal{S}$, which is written as $s \preceq_{\mathcal{S}} t$, if a simulation relation R exists with $(s,t) \in R$. We omit the subscript $\mathcal{S}$ if the* **FLTS** *considered is clear from the context.*

A bisimulation relation links states that behave in the same way, i.e., two states are bisimilar if they can mimic each other's stepwise behavior.

Definition 3.2 *Let $\mathcal{S} = (S, A, \alpha)$ be an* **FLTS**. *A relation $R \subseteq S \times S$ is a* bisimulation *relation if $(s,t) \in R$ implies that for any action $a \in A$, $\alpha(s,a)(U) = \alpha(t,a)(U)$ for any R-closed set $U \subseteq S$. Two states $s,t \in S$ are bisimilar in $\mathcal{S}$, which is written as $s \sim_{\mathcal{S}} t$, if there is a bisimulation relation R with $(s,t) \in R$. We omit the subscript $\mathcal{S}$ if the* **FLTS** *considered is clear from the context.*

Let $R = R_1 \cup R_2$. We observe that if a set U is R-closed, then it is also R_i-closed for $i = 1, 2$. It follows that the union of two simulations (resp. bisimulations) is also a simulation (resp. bisimulation). Moreover, $\preceq$ is the largest simulation, which is called the *similarity*, because it is the union of all simulations. Similarly, $\sim$ is the largest bisimulation, which is called the *bisimilarity*.

Proposition 3.3 *Let $\mathcal{S} = (S, A, \alpha)$ be an* **FLTS**. *Then, $\sim$ is an equivalence relation and $\preceq$ is a preorder.*

Proof. We show that $\sim$ is reflexive, symmetric, and transitive.

- The reflexivity is obvious because the identity relation $Id_S = \{(s,s) \mid s \in S\}$ is a bisimulation.

- Now, we check the symmetry. Suppose that $s \sim t$. Then, a bisimulation R exists such that $(s,t) \in R$. Let $R' = (t,s) \cup R$ and we prove that R' is a bisimulation.

  Note that any R'-closed set U is also R-closed. This holds because xRy implies $xR'y$. Suppose that $(m,n) \in R'$, a is any action, and U is any R'-closed set. We can distinguish two cases, as follows.

  - If $(m,n) = (t,s)$, then $\alpha(m,a)(U) = \alpha(t,a)(U) = \alpha(s,a)(U) = \alpha(n,a)(U)$ since $(s,t) \in R$ and by assumption, R is a bisimulation.
  - If $(m,n) \in R$, we also find that $\alpha(m,a)(U) = \alpha(n,a)(U)$ because R is a bisimulation.

  Hence, R' is a bisimulation. Thus, $t \sim s$ and the symmetry holds.

- To demonstrate the transitivity, we assume that $s \sim t$ and $t \sim h$. Then, R_1 and R_2 exist, which are both bisimulations, such that $(s,t) \in R_1$ and $(t,h) \in R_2$. Let $R' = (s,h) \cup R_1 \cup R_2$ and we prove that R' is a bisimulation.

  Note that any R'-closed set U is both R_1-closed and R_2-closed. For any $(m,n) \in R'$, any action a, and any R'-closed set U, we have two cases to consider, as folllows.

  - If $(m,n) = (s,h)$, then $\alpha(s,a)(U) = \alpha(t,a)(U) = \alpha(h,a)(U)$ because $(s,t) \in R_1$, $(t,h) \in R_2$, both R_1 and R_2 are bisimulations, and U is also R_1-closed and R_2-closed. Thus, $\alpha(m,a)(U) = \alpha(n,a)(U)$.
  - If $(m,n) \in R_1$ or $(m,n) \in R_2$. It holds that $\alpha(m,a)(U) = \alpha(n,a)(U)$ because R_1, R_2 are bisimulations and U is also R_1-closed and R_2-closed.

  Hence, R' is a bisimulation. Thus, $s \sim h$ and the transitivity holds.

Consequently, R is an equivalence relation, as desired. In a similar manner, we can show that $\preceq$ is a preorder. □

Proposition 3.4 *Let $\mathcal{S} = (S, A, \alpha)$ be an* **FLTS**. *If R is a bisimulation and $t \in R^*(s)$, then for any action $a \in A$, we have $\alpha(s,a)(U) = \alpha(t,a)(U)$ for any R-closed set U.*

Proof. If $t \in R^*(s)$, then states $t_1, ..., t_n$ exist with $n \geq 0$ such that

$$sRt_1,\ t_1Rt_2,\ ...,\ t_nRt\ .$$

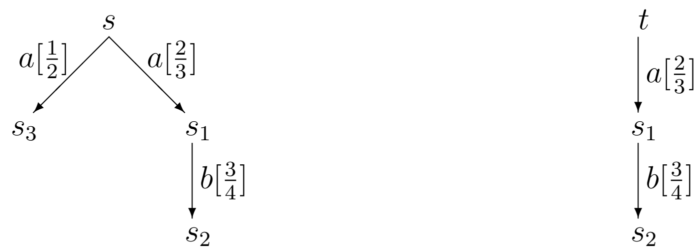

Figure 1: Bisimilarity is strictly finer than simulation equivalence.

Since R is a bisimulation, then we find that

$$\alpha(s,a)(U) = \alpha(t_1,a)(U) = ... = \alpha(t,a)(U) \ .$$

$\square$

Two states $s, t \in S$ are said to be simulation equivalent, which is denoted by $s \simeq t$, if $s \preceq t$ and $t \preceq s$. Bisimilarity implies simulation equivalence but not *vice versa.*

Example 3.5 *Consider the* **FLTS** *depicted in Figure 1. Let $S = \{s, t, s_1, s_2, s_3\}$ and $R = \{(x,x) \mid x \in S\} \cup \{(s,t), (s_3, s_1)\}$. It is easy to check that R is a simulation, and thus $s \preceq t$. Now, let $R' = \{(x,x) \mid x \in S\} \cup \{(t,s)\}$. Obviously, R' is also a simulation, and hence we have $t \preceq s$. It follows that s and t are simulation equivalent.*

Now, we assume by contradiction that s and t are bisimilar. Then, a bisimulation R exists with $(s,t) \in R$. Let $s \xrightarrow{a} \mu$ and $t \xrightarrow{a} \nu$. Then, we have $\mu(R^*(s_3)) = \nu(R^*(s_3))$. Since $\mu(R^*(s_3)) \neq 0$ and ν takes a non-zero value only at s_1, then we can infer that $s_1 \in R^*(s_3)$. By Proposition 3.4, we have $\alpha(s_3, b)(R^*(s_2)) = \alpha(s_1, b)(R^*(s_2))$, which contradicts the fact that s_1 can perform action b on a nonempty distribution whereas s_3 cannot. Hence, s and t are not bisimilar.

The following theorem is very important for proving the logical characterization theorems in the next section.

Theorem 3.6 *Let $\mathcal{S} = (S, A, \alpha)$ be an* **FLTS**.

(1) A preorder relation $R \subseteq S \times S$ is a simulation iff for all $(s,t) \in R$, $a \in A$ and $x \in S$, we have $\alpha(s,a)(R(x)) \leq \alpha(t,a)(R(x))$.

(2) An equivalence relation $R \subseteq S \times S$ is a bisimulation iff for all $(s,t) \in R$, $a \in A$ and all equivalence classes $[x]$ of R, we have $\alpha(s,a)([x]) = \alpha(t,a)([x])$.

Proof. (1) For any $x \in S$, $R(x)$ is R-closed since R is a preorder. By Lemma 2.1, any R-closed set is the union of these $R(x)$, and any possibility distribution also preserves arbitrary unions. Thus, the desired result follows.

(2) Similar to the proof of (1). □

It should be noted that Theorem 3.6(2) also holds for **PLTS**s [23]. However, Theorem 3.6(1) does not hold for **PLTS**s. This is because from $\alpha(s,a)(R(x)) \leq \alpha(t,a)(R(x))$, for any $x \in S$, we cannot conclude that $\alpha(s,a)(U) \leq \alpha(t,a)(U)$ for any R-closed set U. This is supported by the following example.

Example 3.7 *Let $S = \{s_1, s_2, s_3, s_4\}$ and $R = \{(s_i, s_i) \mid i = 1, \cdots, 5\} \cup \{(s_1, s_3), (s_2, s_3)\}$. Obviously, R is a preorder. Moreover, $R(s_1) = \{s_1, s_3\}$, $R(s_2) = \{s_2, s_3\}$, and $R(s_i) = \{s_i\}$ for $i = 3, 4$. Take two probability distributions $d_1, d_2 \in \mathcal{D}(S)$ and let d_1 assign probabilities of 0.6 and 0.4 to s_1 and s_2, respectively, and 0 to other states; d_2 assign probabilities of 0.2, 0.4, and 0.4 to states s_1, s_3 and s_4, respectively, and 0 for state s_2. Then, it is easy to verify that $d_1(R(s_i)) \leq d_2(R(s_i))$ $(i = 1, \cdots, 4)$. However, for the R-closed set $U = \{s_1, s_2, s_3\}$, we have $d_1(U) = 1 > d_2(U) = 0.6$.*

Let $\mathcal{S} = (S, A, \alpha)$ and $\mathcal{T} = (T, B, \beta)$ be two **FLTS**s. Following [6], we consider their *parallel composition*, which is defined as follows:

$$\mathcal{S} \parallel \mathcal{T} = (S \times T, A \cup B, \delta),$$

where for all $(s_1, t_1), (s_2, t_2) \in S \times T$ and action a, we have

$$\delta((s_1,t_1),a)((s_2,t_2)) = \begin{cases} \alpha(s_1,a)(s_2) \wedge \beta(t_1,a)(t_2), & \text{if } a \in A \cap B \\ \alpha(s_1,a)(s_2), & \text{if } a \in A \backslash B \text{ and } t_2 = t_1 \\ \beta(t_1,a)(t_2), & \text{if } a \in B \backslash A \text{ and } s_2 = s_1 \\ 0, & \text{otherwise.} \end{cases}$$

In the composite **FLTS** $\mathcal{S} \parallel \mathcal{T}$, all the actions in the set $A \cap B$ must be synchronized by the two systems $\mathcal{S}$ and $\mathcal{T}$, and all of the other actions may be interleaved.

The following theorem shows that simulation and bisimulation are preserved by the parallel composition operator.

Theorem 3.8 *Let $\mathcal{S} = (S, A, \alpha)$ and $\mathcal{T} = (T, B, \beta)$ be two* **FLTS***s.*

(1) If $s_1 \preceq_{\mathcal{S}} s_2$ and $t_1 \preceq_{\mathcal{T}} t_2$, then $(s_1, t_1) \preceq_{\mathcal{S}\parallel\mathcal{T}} (s_2, t_2)$.

(2) If $s_1 \sim_{\mathcal{S}} s_2$ and $t_1 \sim_{\mathcal{T}} t_2$, then $(s_1, t_1) \sim_{\mathcal{S}\parallel\mathcal{T}} (s_2, t_2)$.

Proof. As an example, let us prove (2). Define

$$R = \{((s,t),(s',t')) \mid (s,s') \in \sim_{\mathcal{S}}, (t,t') \in \sim_{\mathcal{T}}\}.$$

Clearly, we have $((s_1,t_1),(s_2,t_2)) \in R$. By Proposition 3.3, $\sim_{\mathcal{S}}$ and $\sim_{\mathcal{T}}$ are equivalence relations on S and T, respectively. Lemma 2.2 (2) implies that R is an equivalence relation on $S \times T$. To establish $(s_1,t_1) \sim_{\mathcal{S}\|\mathcal{T}} (s_2,t_2)$, it suffices to show that R is a bisimulation on $\mathcal{S} \parallel \mathcal{T}$. By Theorem 3.6(2), we only need to check that

$$\delta((s,t),a)([(s'',t'')]_R) = \delta((s',t'),a)([(s'',t'')]_R) \tag{1}$$

for any $((s,t),(s',t')) \in R$, $a \in A \cup B$, and any equivalence class $[(s'',t'')]_R$ of R. By the definition of parallel composition, three cases need to be considered.

1. $a \in A \cap B$. In this case, we have

$$\begin{aligned}
\delta((s,t),a)([(s'',t'')]_R) &= \delta((s,t),a)([s'']_{\sim_{\mathcal{S}}} \times [t'']_{\sim_{\mathcal{T}}}) \text{ (by Lemma 2.2 (2))}\\
&= \sup\nolimits_{m\in[s'']_{\sim_{\mathcal{S}}}, n\in[t'']_{\sim_{\mathcal{T}}}} \delta((s,t),a)((m,n))\\
&= \sup\nolimits_{m\in[s'']_{\sim_{\mathcal{S}}}, n\in[t'']_{\sim_{\mathcal{T}}}} \alpha(s,a)(m) \wedge \beta(t,a)(n)\\
&= \sup\nolimits_{m\in[s'']_{\sim_{\mathcal{S}}}} \sup\nolimits_{n\in[t'']_{\sim_{\mathcal{T}}}} \alpha(s,a)(m) \wedge \beta(t,a)(n)\\
&= \sup\nolimits_{m\in[s'']_{\sim_{\mathcal{S}}}} \alpha(s,a)(m) \wedge \beta(t,a)([t'']_{\sim_{\mathcal{T}}})\\
&= \alpha(s,a)([s'']_{\sim_{\mathcal{S}}}) \wedge \beta(t,a)([t'']_{\sim_{\mathcal{T}}}).
\end{aligned}$$

   Similarly, we can also prove that

$$\delta((s',t'),a)([(s'',t'')]_R) = \alpha(s',a)([s'']_{\sim_{\mathcal{S}}}) \wedge \beta(t',a)([t'']_{\sim_{\mathcal{T}}}).$$

   Note that $((s,t),(s',t')) \in R$, which means that $s \sim_{\mathcal{S}} s'$ and $t \sim_{\mathcal{T}} t'$. By Theorem 3.6(2), we have

$$\alpha(s,a)([s'']_{\sim_{\mathcal{S}}}) = \alpha(s',a)([s'']_{\sim_{\mathcal{S}}}) \text{ and } \beta(t,a)([t'']_{\sim_{\mathcal{T}}}) = \beta(t',a)([t'']_{\sim_{\mathcal{T}}}).$$

   As a consequence, the equation in (1) holds, as desired.

2. $a \in A \backslash B$. We distinguish two subcases as follows.

   - If $t \notin [t'']_{\sim_{\mathcal{T}}}$, then $t \neq n$ for any $n \in [t'']_{\sim_{\mathcal{T}}}$. It follows that

$$\delta((s,t),a)([(s'',t'')]_R) = \sup_{m\in[s'']_{\sim_{\mathcal{S}}}, n\in[t'']_{\sim_{\mathcal{T}}}} \delta((s,t),a)((m,n)) = 0.$$

     Since $t \sim_{\mathcal{T}} t'$, we must have $t' \notin [t'']_{\sim_{\mathcal{T}}}$. For the same reason, we derive that $\delta((s',t'),a)([(s'',t'')]_R) = 0$. Hence, the equation in (1) holds.

- If $t \in [t'']_{\sim_{\mathcal{T}}}$, then we can infer that

$$\begin{aligned}\delta((s,t),a)([(s'',t'')]_R) &= \sup_{m\in[s'']_{\sim_{\mathcal{S}}}, n\in[t'']_{\sim_{\mathcal{T}}}} \delta((s,t),a)((m,n)) \\ &= \sup_{m\in[s'']_{\sim_{\mathcal{S}}}} \alpha(s,a)(m) \\ &= \alpha(s,a)([s'']_{\sim_{\mathcal{S}}}).\end{aligned}$$

  Similarly, we can also prove that

$$\delta((s',t'),a)([(s'',t'')]_R) = \alpha(s',a)([s'']_{\sim_{\mathcal{S}}}).$$

  Note that $s \sim_{\mathcal{S}} s'$, which implies that $\alpha(s,a)([s'']_{\sim_{\mathcal{S}}}) = \alpha(s',a)([s'']_{\sim_{\mathcal{S}}})$. It follows that the equation in (1) holds.

3. $a \in B\backslash A$. This is analogous to the latter case, and thus it is omitted. This completes the proof of this theorem.

$\square$

4 Logical Characterizations of Bisimulation and Simulation

Specification, i.e., the description of the required properties of an implementation, is a major issue for transition systems [43]. These properties are best expressed as formulae in a logic language. In this section, we introduce a variant of the Hennessy-Milner Logic [22] to characterize bisimulation. We also show that a negation-free sub-logic is sufficient to characterize simulation.

4.1 Logic

Let A be a countable set of *actions* ranged over by $a, b, \cdots$, and let $\top$ be a propositional constant. The language $\mathcal{L}^{bi}$ of formulae is the least set generated by the following BNF grammar:

$$\varphi ::= \ \top \mid \bigwedge_{i\in I} \varphi_i \mid \neg\varphi \mid \langle a\rangle_p \varphi,$$

where I is a possibly uncountable index set and p is a real number in the unit interval $[0,1]$. This is the basic logic that we employ to establish the logical characterization of bisimulation for an **FLTS**.

Note that the formulae in $\mathcal{L}^{bi}$ are defined inductively, so we can only generate formulae of finite depth such as $\top$, $\langle a\rangle_p\top$, $\neg\langle a\rangle_p\top$, $\langle a\rangle_p\langle b\rangle_r\top$, $\langle a\rangle_p\neg\langle b\rangle_r\top$, etc.

Let us fix an **FLTS** $\mathcal{S} = (S, A, \alpha)$. The semantic interpretation of the formulae in $\mathcal{L}^{bi}$ is given by:

- $s \models_{bi} \top$, for any state s;
- $s \models_{bi} \bigwedge_{i \in I} \varphi_i$ iff for each $i \in I$, $s \models_{bi} \varphi_i$;
- $s \models_{bi} \neg\varphi$ iff $s \not\models_{bi} \varphi$;
- $s \models_{bi} \langle a \rangle_p \varphi$ iff $\exists A \subseteq S.\ (\forall s' \in A.\ s' \models \varphi) \wedge (\alpha(s,a)(A) \geq p)$.

We write $[\![\varphi]\!]$ for the set $\{s \in S \mid s \models_{bi} \varphi\}$. Then, it is immediate that $s \models_{bi} \langle a \rangle_p \varphi$ iff $\alpha(s,a)([\![\varphi]\!]) \geq p$, i.e., $\sup_{s' \in [\![\varphi]\!]} \alpha(s,a)(s') \geq p$. Thus, $s \models_{bi} \langle a \rangle_p \varphi$ means that the state s can make an a-move to a state that satisfies φ with a possibility greater than p. In the sequel, we always use this fact as the semantic interpretation of the formula $\langle a \rangle_p \varphi$ in $\mathcal{L}^{bi}$.

Again, we consider the **FLTS** depicted in Figure 1 and we see that s satisfies, among others, the formula $\langle a \rangle_{\frac{1}{2}} \neg \langle b \rangle_{\frac{3}{4}} \top$, i.e., $s \models_{bi} \langle a \rangle_{\frac{1}{2}} \neg \langle b \rangle_{\frac{3}{4}} \top$, because s can make an a-move to state s_3, which is a deadlock state, and thus it cannot perform action b with a possibility of at least $\frac{3}{4}$. In addition, since s can make an a-move to state s_1 with a possibility of $\frac{2}{3}$ followed by a b-move to state s_2 with a possibility of $\frac{3}{4}$, we can see that $s \models_{bi} \langle a \rangle_{\frac{1}{2}} \langle b \rangle_{\frac{3}{4}} \top$ also holds.

Let $Th(s) = \{\psi \mid s \models_{bi} \psi\}$ be the set of formulae satisfied by state s, which is called the *theory* of state s. For example, in Figure 1, the set $Th(t)$ includes, among others, the formulae $\langle a \rangle_p \top (p \leq \frac{2}{3})$, $\langle a \rangle_p \langle b \rangle_r \top (p \leq \frac{2}{3}, r \leq \frac{3}{4})$ and $\neg \langle a \rangle_p \langle b \rangle_r \top (p > \frac{2}{3}, r \in [0,1])$.

Two states $s, t \in S$ are logically equivalent if $Th(s) = Th(t)$. In other words, the two states cannot be distinguished by the logic $\mathcal{L}^{bi}$ because $s \models_{bi} \psi \Longleftrightarrow t \models_{bi} \psi$ for any $\psi \in \mathcal{L}^{bi}$.

In the following, we explain why the logic $\mathcal{L}^{bi}$ is selected to characterize bisimulation for **FLTS**s. In previous studies, various extensions of the Hennessy-Milner Logic have been proposed. We consider two that have been proposed in the setting of PLTSs.

Larsen and Skou [27] used the logic $\mathcal{L}^{ls}$ to characterize probabilistic bisimulation for image-finite reactive systems, which comprises the following set of formulae:

$$\varphi \ ::= \ \top \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \langle a \rangle_p \varphi,$$

where p is a rational number in the unit interval $[0,1]$. We write $\models_{ls}$ for the satisfaction relation of this logic. Then, $s \models_{ls} \langle a \rangle_p \varphi$ iff $\exists A \subseteq S.\ (\forall s' \in A.\ s' \models_{ls} \varphi) \wedge (\alpha(s,a)(A) \geq p)$, where $\alpha(s,a) \in \mathcal{D}(S)$, i.e., the state s can make an a-move to a probability distribution that evolves into a state satisfying φ with a probability of at least p. Similarly, $s \models_{ls} \langle a \rangle_p \varphi$ iff $\alpha(s,a)([\![\varphi]\!]) \geq p$, i.e., $\sum_{s' \in [\![\varphi]\!]} \alpha(s,a)(s') \geq p$.

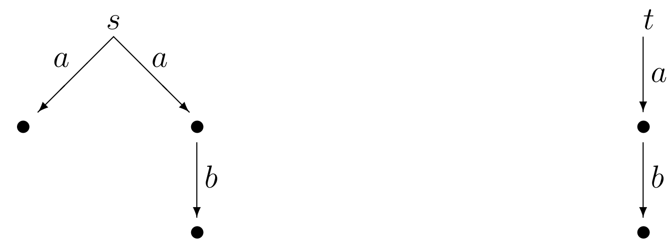

Figure 2: Two states cannot be distinguished without negation in **NLTS**s.

Desharnais et al. [16] showed that negation is not necessary to characterize bisimulations for reactive systems, and this holds even for general reactive systems that may not be image-finite. The syntax of their logic $\mathcal{L}^{dep}$ is as follows:

$$\varphi \ ::= \ \top \mid \varphi_1 \wedge \varphi_2 \mid \langle a \rangle_p \varphi,$$

where p is a rational number in the unit interval $[0, 1]$. We write $\models_{dep}$ for the satisfaction relation of this logic. In particular, $s \models_{dep} \langle a \rangle_p \varphi$ iff $s \models_{ls} \langle a \rangle_p \varphi$.

Remark 4.1 *A subtle difference between the logic $\mathcal{L}^{bi}$ and $\mathcal{L}^{dep}$ is that p is a real number in the formula $\langle a \rangle_p \varphi$ in the logic $\mathcal{L}^{bi}$, whereas it is a rational number in the logic $\mathcal{L}^{dep}$. A consequence of the restriction is that the set of all the formulae in the logic $\mathcal{L}^{dep}$ is countable, whereas the set of all the formulae in the logic $\mathcal{L}^{bi}$ is uncountable. This has an implication for the proof of Theorem 4.10.*

There is only a slight difference between an **FLTS** and a **PLTS**, which raises a natural question: can the logic $\mathcal{L}^{dep}$ characterize bisimulations for **FLTS**s? The answer to this question is no, according to the following two examples.

Example 4.2 *In this example, we compare* **FLTS***s with nondeterministic labeled transition systems (***NLTS***s) and* **PLTS***s.*

Consider the two states s and t in Figure 2. A simple formula that distinguishes them is $\langle a \rangle \neg \langle b \rangle \top$, which means that a state can perform an a action and then be in a state where it cannot perform a b action. Thus, state s satisfies this formula whereas state t does not. It is well known that they cannot be distinguished by a negation-free formula of the Hennessy-Milner logic [22].

Now, consider the **PLTS***s in Figure 3, where $m \neq 0$. It is easy to see that*

$$t \models_{dep} \langle a \rangle_1 \langle b \rangle_1 \top \quad but \quad s \not\models_{dep} \langle a \rangle_1 \langle b \rangle_1 \top.$$

Hence, s and t can be distinguished without negation in this **PLTS***.*

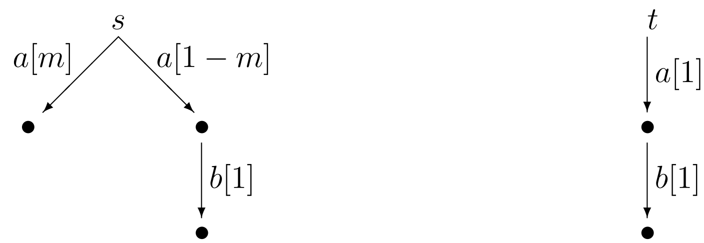

Figure 3: Two states can be distinguished without negation in **PLTS**s.

However, the case is different for **FLTS***s. In Example 3.5, we show that the two states s and t in Figure 1 are not bisimilar, but they cannot be distinguished by the logic $\mathcal{L}^{dep}$. This is explained as follows. Since $s \models_{bi} \langle a\rangle_p\varphi$ iff $\sup_{s'\in[\![\varphi]\!]}\alpha(s,a)(s') \geq p$, then the non-trivial formulae that both s and t satisfy are*

$$\langle a\rangle_p\top\ (p \leq \frac{2}{3}), \langle a\rangle_q\langle b\rangle_r\top\ (q \leq \frac{2}{3} \text{ and } r \leq \frac{3}{4})$$

and binary conjunction of these formulae. Moreover, the formulae that both s and t do not satisfy are

$$\langle a\rangle_p\top\ (p > \frac{2}{3}) \text{ and } \langle a\rangle_q\langle b\rangle_r\top\ (q > \frac{2}{3} \text{ or } r > \frac{3}{4}).$$

Hence, the logic $\mathcal{L}^{dep}$ cannot distinguish s and t. However, we can easily distinguish these two states if negation is allowed. For instance, we have

$$s \models_{bi} \langle a\rangle_{\frac{1}{2}}\neg\langle b\rangle_{\frac{3}{4}}\top \text{ but } t \not\models_{bi} \langle a\rangle_{\frac{1}{2}}\neg\langle b\rangle_{\frac{3}{4}}\top.$$

Example 4.3 *In this example, we show that finite conjunction is insufficient to characterize bisimulations for* **FLTS***s. This example has been adapted from Example 5.1 in [23].*

The only difference between the two **FLTS***s in Figure 4 is that t has a transition to itself. First, we show that s and t are not bisimilar. Let*

$$S = \{s, t, t_0, t_1, t_2, \cdots\}, t \xrightarrow{a} \mu, t_n \xrightarrow{a} \mu_n (n = 1, 2, \cdots), \text{ and } s \xrightarrow{a} \nu.$$

Now, suppose that s and t are bisimilar. Then, a bisimulation R exists with $(s,t) \in R$. It follows that

$$\nu(R^*(t)) = \mu(R^*(t)) = \frac{1}{2}.$$

Since ν takes values of 0 at s and t, $R^*(t)$ includes at least one $t_n (n = 0, 1, 2, \cdots)$. Thus, it follows from Proposition 3.4 that

$$\mu_n(R^*(t)) = \mu(R^*(t)) = \frac{1}{2}.$$

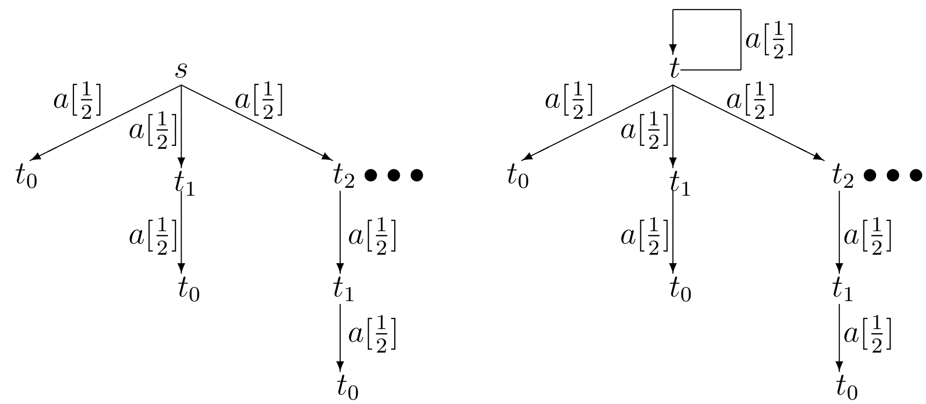

Figure 4: Two states cannot be distinguished without infinite conjunction in **FLTS**s.

Since μ_n *takes a value of* $\frac{1}{2}$ *only at* t_{n-1} *and* 0 *otherwise, then*

$$t_{n-1} \in R^*(t) \text{ and } \mu_{n-1}(R^*(t)) = \mu(R^*(t)) = \frac{1}{2}.$$

If we continue in this manner, then we finally obtain $t_0 \in R^*(t)$*. It follows that*

$$\alpha(t_0, a)(R^*(t)) = \mu(R^*(t)) = \frac{1}{2},$$

which leads to a contradiction because t *can perform an* a*-action on a nonempty distribution whereas* t_0 *cannot. Hence,* s *and* t *are not bisimilar.*

However, in the logic $\mathcal{L}^{dep}$*, we cannot find a formula that can distinguish* s *and* t *due to the following reasons. Obviously, all formulae satisfied by* s *can also be satisfied by* t*. Hence, in order to distinguish them we need to find a formula that can be satisfied by* t *but not by* s*. In the logic* $\mathcal{L}^{dep}$*, the formulae satisfied by* t *are:*

$$\top, \langle a\rangle_{p_1}\top, \langle a\rangle_{p_1}\langle a\rangle_{p_2}\top, \langle a\rangle_{p_1}\langle a\rangle_{p_2}\langle a\rangle_{p_3}\top, \cdots,$$

and binary conjunction of these formulae, where $p_i \leq \frac{1}{2}(i = 1, 2, 3, \cdots)$*. However, all these formulae can also be satisfied by* s*. Hence, the logic* $\mathcal{L}^{dep}$ *cannot distinguish* s *and* t *in this* **FLTS**.

The situation is different if infinite conjunction is allowed. Consider the formula φ_i *defined as follows:* $\varphi_0 = \top$*, and* $\varphi_i = (\langle a\rangle_{\frac{1}{2}})^{(i)}\top$*, which means that*

$$\underbrace{\langle a\rangle_{\frac{1}{2}}\langle a\rangle_{\frac{1}{2}}\cdots\langle a\rangle_{\frac{1}{2}}}_{i \text{ times}}\top.$$

Let $\varphi = \bigwedge_{i\in\mathbb{N}} \varphi_i$*. Then,* $t \models_{bi} \varphi$*. By mathematical induction, we can prove that for any* n*,*

$$t_n \models_{bi} \varphi_n \text{ but } t_n \not\models_{bi} \varphi_{n+1}.$$

Then, it follows that $t_n \not\models_{bi} \varphi$ for any n. Now, let $\psi = \langle a \rangle_{\frac{1}{2}}\varphi$. Then, $t \models_{bi} \psi$ because $t \in [\![\varphi]\!]$ and $\mu(t) = \frac{1}{2}$. However, $s \not\models_{bi} \psi$ since ν takes nonzero values only at t_n ($n = 0, 1, \cdots$) and $t_n \notin [\![\varphi]\!]$. Consequently, we find a formula with infinite conjunction to distinguish s from t.

Hence, the logic $\mathcal{L}^{bi}$ is selected instead of $\mathcal{L}^{dep}$ to characterize bisimulations for **FLTS**s.

4.2 Logical Characterization of Bisimulation

In this section, we show that two states are observationally indistinguishable or bisimilar iff they are logically indistinguishable. Symbolically,

$$s \sim t \text{ iff } Th(s) = Th(t).$$

The following two technical lemmas will be useful for proving this fact.

Lemma 4.4 [35] states that when considering logics with negation, it is impossible for the set of formulae satisfied by one state to be strictly included in the set of formulae satisfied by another state.

Lemma 4.4 *Given a logic with negation, for each pair of states s and t if $Th(s) \subseteq Th(t)$, then $Th(s) = Th(t)$.*

The second lemma states that the transition possibilities to sets of the form $[\![\psi]\!]$ are determined completely by the formulae.

Lemma 4.5 *Given an **FLTS** $\mathcal{S} = (S, A, \alpha)$ and the logic $\mathcal{L}^{bi}$, for any two states $s, t \in S$, if $Th(s) = Th(t)$, then for any formula $\psi \in \mathcal{L}^{bi}$ and $a \in A$, $\alpha(s, a)([\![\psi]\!]) = \alpha(t, a)([\![\psi]\!])$.*

Proof. Without loss of generality, let us assume that a formula ψ exists such that $\alpha(s, a)([\![\psi]\!]) < \alpha(t, a)([\![\psi]\!])$. Then, we can squeeze in a number p with $\alpha(s, a)([\![\psi]\!]) < p \leq \alpha(t, a)([\![\psi]\!])$. It follows that $t \models_{bi} \langle a \rangle_p \psi$ but $s \not\models_{bi} \langle a \rangle_p \psi$, which contradicts $Th(s) = Th(t)$. □

Theorem 4.6 *Given an **FLTS** $\mathcal{S} = (S, A, \alpha)$ and the logic $\mathcal{L}^{bi}$, for any two states $s, t \in S$, $s \sim t$ iff $Th(s) = Th(t)$.*

Proof. First we show the soundness, i.e.,

$$\forall s, t \in S.\ s \sim t \Longrightarrow Th(s) = Th(t).$$

Let $s, t \in S$, $s \sim t$, and ψ be a formula. We show that $s \models_{bi} \psi \Longleftrightarrow t \models_{bi} \psi$ by structural induction on ψ. The cases of $\top$ and conjunction are trivial. Now, we consider other cases as follows.

(1) $\psi \equiv \neg\varphi$. In this case, $s \models_{bi} \psi \Longleftrightarrow s \not\models_{bi} \varphi$. By structural induction, we have $s \not\models_{bi} \varphi \Longleftrightarrow t \not\models_{bi} \varphi$. Now, we also have $t \not\models_{bi} \varphi \Longleftrightarrow t \models_{bi} \psi$.

(2) $\psi \equiv \langle a \rangle_p \varphi$. By structural induction, $[\![\varphi]\!]$ is $\sim$-closed for any formula φ. It follows from $s \sim t$ that $\alpha(s,a)([\![\varphi]\!]) = \alpha(t,a)([\![\varphi]\!])$. Then, it is immediate that $s \models_{bi} \psi \Longleftrightarrow t \models_{bi} \psi$.

For completeness, we define $R = \{(s,t) \mid Th(s) = Th(t)\}$. It suffices to prove that R is a bisimulation. Obviously, R is an equivalence relation. Let $G = \{[s_i] \mid i \in I\}$ be the set of all equivalence classes of R. Then, by Theorem 3.6(2), it remains to show that for any $(s,t) \in R$, $a \in A$, and $i \in I$,

$$\alpha(s,a)([s_i]) = \alpha(t,a)([s_i]). \tag{2}$$

First, we claim that for any equivalence class $[s_i]$, a characteristic formula φ_i exists such that $[\![\varphi_i]\!] = [s_i]$. This can be proved as follows.

- If G contains only one equivalence class $[x]$, then $S = [x]$. Thus, we can take the characteristic formula as being $\top$ because $[\![\top]\!] = [x]$.

- If G contains more than one equivalence class, then for any $i, j \in I$ with $i \neq j$, a formula φ_{ij} exists such that $s_i \models_{bi} \varphi_{ij}$ and $s_j \not\models_{bi} \varphi_{ij}$. Otherwise, for any formula φ, $s_i \models_{bi} \varphi$ implies $s_j \models_{bi} \varphi$, which means that $Th(s_i) \subseteq Th(s_j)$. It follows from Lemma 4.4, that $Th(s_i) = Th(s_j)$. Thus, $s_j \in [s_i]$, which contradicts the fact that s_i and s_j are taken from different equivalence classes. For each $i \in I$, define $\varphi_i = \bigwedge_{j \neq i} \varphi_{ij}$, then by construction, $[\![\varphi_i]\!] = [s_i]$. Let us check the last equality. First, if $s_k \in [\![\varphi_i]\!]$ for some $k \in I$, then $s_k \models_{bi} \varphi_i$, which means that $s_k \models_{bi} \varphi_{ij}$ for all $j \neq i$. Thus, $s_k \notin [s_j]$ for all $j \neq i$, and this in turn implies that $s_k \in [s_i]$. Second, if $s_k \in [s_i]$, then $s_k \models_{bi} \varphi_i$ as $s_i \models_{bi} \varphi_i$, which means that $s_k \in [\![\varphi_i]\!]$.

This completes the proof of the claim that each equivalence has a characteristic formula.

Now, suppose that $(s,t) \in R$. For any action $a \in A$ and index $i \in I$, by the claim above and Lemma 4.5, we can infer that

$$\alpha(s,a)([s_i]) = \alpha(s,a)([\![\varphi_i]\!]) = \alpha(t,a)([\![\varphi_i]\!]) = \alpha(t,a)([s_i])$$

. Hence, the equation in (2) holds. □

In the proof given above, the idea of using characteristic formulae is inspired by [23]. We can see that the logic $\mathcal{L}^{bi}$ is highly expressive because it characterizes bisimulation as well as equivalence classes in the sense that a formula for each equivalence class necessarily exists that is satisfied only by the states in that class.

Moreover, from the construction of formula φ_i above, we can see that infinite conjunction is indeed necessary. The advantage of infinite conjunction is that it allows for a universal description of a class of states of interest. However, infinity is difficult to process in real applications. Fortunately, in most practical applications, the support of a fuzzy set is finite and the state space is at most countable. Then, the logic $\mathcal{L}^{bi}$ restricted to binary conjunction, i.e., the logic $\mathcal{L}^{ls}$, is already sufficient to characterize bisimulation for **FLTS**s, as we show in the following.

Theorem 4.7 *Given the logic $\mathcal{L}^{ls}$, and if we let the state space S be countable and $\mathcal{S} = (S, A, \alpha)$ is an $\mathbf{F}_f$**LTS**; then, for any two states $s, t \in S$, $s \sim t$ iff $Th(s) = Th(t)$.*

Proof. Theorem 4.6 implies the soundness. For the completeness, let R, G, and $\{\varphi_{ij}\}_{i,j \in I}$ be defined as given. Note that G is now countable because of our restriction to a countable state space S. We fix an arbitrary index k. For each $i \in I$, define $\Phi_i^k = \bigwedge_{j \leq k} \varphi_{ij}$. It is then easy to see that for each $i \in I$, the formula Φ_i^k only has finite conjunctions, and

$$[s_i] \subseteq [\![\Phi_i^k]\!] \subseteq [s_i] \cup \bigcup_{m \in I \wedge m > k} [s_m]. \tag{3}$$

Hence, for any $\mu \in \mathcal{F}_f(S)$, $\mu([s_i]) \leq \mu([\![\Phi_i^k]\!]) \leq \mu([s_i] \cup \bigcup_{m \in I \wedge m > k} [s_m])$, i.e.,

$$\mu([s_i]) \leq \mu([\![\Phi_i^k]\!]) \leq \mu([s_i]) \vee \mu(\bigcup_{m \in I \wedge m > k} [s_m]).$$

If we fix an arbitrary index i and take the infimum for $k \in I$, then we can obtain

$$\mu([s_i]) \leq \inf_{k \in I} \mu([\![\Phi_i^k]\!]) \leq \inf_{k \in I} [\mu([s_i]) \vee \mu(\bigcup_{m \in I \wedge m > k} [s_m])],$$

i.e.,

$$\mu([s_i]) \leq \inf_{k \in I} \mu([\![\Phi_i^k]\!]) \leq \mu([s_i]) \vee \inf_{k \in I} \mu(\bigcup_{m \in I \wedge m > k} [s_m]). \tag{4}$$

We argue that

$$\inf_{k \in I} \mu(\bigcup_{m \in I \wedge m > k} [s_m]) = 0. \tag{5}$$

In fact, since *supp*(μ) is finite, a sufficiently large number $N \in I$ exists such that for any $s \in supp(\mu)$, some $m_s \in I$ exists with $m_s < N$ and $s \in [s_{m_s}]$. Thus, we always have $\mu(\bigcup_{m\in I\wedge m>k}[s_m])=0$ when $k\geq N$. Hence, the equation in (5) holds.

By combining (4) and (5), for any $i\in I$ and any $\mu\in\mathcal{F}_f(S)$, we have

$$\mu([s_i])=\inf_{k\in I}\mu([\![\Phi_i^k]\!]). \tag{6}$$

Now let $(s,t)\in R$. It remains to show that $\alpha(s,a)([s_i])=\alpha(t,a)([s_i])$ for any $a\in A$ and $i\in I$. By the left part of (3), we have $\alpha(s,a)([\![\Phi_i^k]\!])\geq\alpha(s,a)([s_i])$ for each $i\in I$, thereby implying that $s\models_{bi}\langle a\rangle_{p_i}\Phi_i^k$ for each $i,k\in I$, where $p_i=\alpha(s,a)([s_i])$. By the definition of R, $t\models_{bi}\langle a\rangle_{p_i}\Phi_i^k$ for each $i,k\in I$. Hence, $\alpha(t,a)([\![\Phi_i^k]\!])\geq p_i$ for each $i\in I$. Again, by the right part of (3), for an arbitrary index i and any $k\in I$, we can obtain

$$p_i\leq\alpha(t,a)([\![\Phi_i^k]\!])\leq\alpha(t,a)([s_i])\vee\alpha(t,a)(\bigcup_{m\in I\wedge m>k}[s_m]).$$

It follows that

$$p_i\leq\alpha(t,a)([s_i])\vee\inf_{k\in I}\alpha(t,a)(\bigcup_{m\in I\wedge m>k}[s_m]).$$

Thus, by (5), we have $p_i\leq\alpha(t,a)([s_i])$, i.e.

$$\alpha(s,a)([s_i])\leq\alpha(t,a)([s_i])$$

for each $i\in I$. Now, suppose that an $i_0\in I$ exists such that $\alpha(s,a)([s_{i_0}])<\alpha(t,a)([s_{i_0}])$. Then, we can take $\epsilon_0>0$ such that

$$\alpha(s,a)([s_{i_0}])<\alpha(s,a)([s_{i_0}])+\epsilon_0<\alpha(t,a)([s_{i_0}]).$$

For this ϵ_0, by applying (6) to $[s_{i_0}]$, we can see that some $k_0\in I$ exists such that

$$\alpha(s,a)([\![\Phi_{i_0}^{k_0}]\!])<\alpha(s,a)([s_{i_0}])+\epsilon_0.$$

Thus, since

$$\alpha(t,a)([\![\Phi_{i_0}^{k_0}]\!])\geq\alpha(t,a)([s_{i_0}])>\alpha(s,a)([s_{i_0}])+\epsilon_0,$$

we have

$$s\not\models_{bi}\langle a\rangle_{\alpha(s,a)([s_{i_0}])+\epsilon_0}\Phi_{i_0}^{k_0}\text{ but }t\models_{bi}\langle a\rangle_{\alpha(s,a)([s_{i_0}])+\epsilon_0}\Phi_{i_0}^{k_0},$$

which contradicts $Th(s)=Th(t)$. Hence, for each $i\in I$, $\alpha(s,a)([s_i])=\alpha(t,a)([s_i])$, as desired. □

4.3 Analysis

Next, we explain why the logic $\mathcal{L}^{dep}$ can characterize bisimulations for **PLTS**s but not for **FLTS**s, which is essentially because the π-λ theorem [5] plays a crucial role in the logical characterizations for **PLTS**s whereas it does not for **FLTS**s.

Let X be a set. A family $\mathcal{P}$ of subsets of X is called a π-*class* if it is closed under finite intersection; a family $\mathcal{L}$ of subsets of X is called a λ-*class* if it is closed under complementations and countable disjoint unions; and a family $\mathcal{M}$ of subsets of X is called a σ-algebra if it contains X, and it is closed under complementations and countable unions.

Theorem 4.8 (The π-λ theorem) *Let $\mathcal{P}$ be a π-class of a set X. Then, $\sigma(\mathcal{P})$ is the smallest λ-class containing $\mathcal{P}$, where $\sigma(\mathcal{P})$ is a σ-algebra containing $\mathcal{P}$.*

The next proposition is a typical application of the π-λ theorem, which shows us that when two probability distributions agree on a π-class, then they also agree on the generated σ-algebra.

Proposition 4.9 *Let S be a state space, $\mathcal{A}_0 = \{[\![\varphi]\!] \mid \varphi \in \mathcal{L}^{dep}\}$, and $\mathcal{A} = \sigma(\mathcal{A}_0)$. For any $d, d' \in \mathcal{D}(S)$, if $d(A) = d'(A)$ for any $A \in \mathcal{A}_0$, then $d(B) = d'(B)$ for any $B \in \mathcal{A}$.*

Proof. Let

$$\mathcal{P} = \{A \in \mathcal{A} \mid d(A) = d'(A)\}.$$

Then, $\mathcal{P}$ is closed under countable disjoint unions because probability distributions are σ-additive. Furthermore, $d(S) = d'(S) = 1$ implies that if $A \in \mathcal{P}$, then $d(S \backslash A) = d(S) - d(A) = d'(S) - d'(A) = d'(S \backslash A)$, i.e. $S \backslash A \in \mathcal{P}$. Thus, $\mathcal{P}$ is also closed under complementation. It follows that $\mathcal{P}$ is a λ-class. Note that $\mathcal{A}_0$ is a π-class given the equation $[\![\varphi_1 \wedge \varphi_2]\!] = [\![\varphi_1]\!] \cap [\![\varphi_2]\!]$. Since $\mathcal{A}_0 \subseteq \mathcal{P}$, then we can apply the π-λ Theorem 4.8 to obtain $\mathcal{A} = \sigma(\mathcal{A}_0) \subseteq \mathcal{P} \subseteq \mathcal{A}$, i.e., $\mathcal{A} = \mathcal{P}$. Therefore, $d(B) = d'(B)$ for any $B \in \mathcal{A}$. □

Theorem 4.10 *Given the logic $\mathcal{L}^{dep}$ and let $\mathcal{S} = (S, A, \alpha)$ be a* **PLTS**. *Then, for any two states $s, t \in S$, $s \sim t$ iff $Th(s) = Th(t)$.*

Proof. The proof of soundness is conducted as described in Theorem 4.6. In the following, we focus on the completeness. Let $R = \{(s,t) \mid Th(s) = Th(t)\}$. It suffices to show that R is a bisimulation. Obviously, R is an equivalence relation. Moreover, for any $x \in S$, the equivalence class containing x is

$$[x] = \bigcap \{[\![\phi]\!] \mid x \models \phi\} \cap \bigcap \{S \backslash [\![\phi]\!] \mid x \not\models \phi\}. \tag{7}$$

In (7), only countable intersections are used because the set of all the formulae in the logic $\mathcal{L}^{dep}$ is countable. Let $\mathcal{A}_0$ be defined as in Proposition 4.9. Then, each equivalence class of R is a member of $\sigma(\mathcal{A}_0)$.

In addition, for any $(s,t) \in R$, $Th(s) = Th(t)$ implies that for any $a \in A$ and $\phi \in \mathcal{L}^{dep}$, $\alpha(s,a)([\![\phi]\!]) = \alpha(t,a)([\![\phi]\!])$, the proof of which is exactly the same as that for Lemma 4.5. Thus, by Proposition 4.9, we have

$$\alpha(s,a)([x]) = \alpha(t,a)([x]), \tag{8}$$

where $[x]$ is any equivalence class of R. Then, it follows from applying Theorem 3.6(2) to **PLTS**s that the equivalence relation R is a bisimulation. $\square$

Therefore, the logic $\mathcal{L}^{dep}$ can characterize a bisimulation for **PLTS**s, and this logic has neither negation nor infinite conjunction. Moreover, the result given above holds for general **PLTS**s, which are not necessarily image-finite.

However, a counterpart of Theorem 4.10 for **FLTS**s would not be valid. For possibility distributions, the family of sets $\mathcal{P}$ in the proof of Proposition 4.9 is not closed under complementation or countable intersections. Thus, we cannot show that all equivalence classes are in $\mathcal{P}$ (i.e., $\sigma(\mathcal{A}_0)$). It follows that (8) cannot be established for **FLTS**s.

4.4 Logical Characterization of Simulation

The logic that characterizes simulations for **FLTS**s is the negation-free fragment of the logic $\mathcal{L}^{bi}$ called $\mathcal{L}^{si}$, which is given as follows.

$$\varphi ::= \ \top \mid \textstyle\bigwedge_{i\in I} \varphi_i \mid \langle a\rangle_p \varphi.$$

The semantics of $\mathcal{L}^{si}$ is defined in the same manner as $\mathcal{L}^{bi}$. In this subsection, $Th(s) = \{\psi \mid s \models_{si} \psi\}$.

Theorem 4.11 *Given an* **FLTS** $\mathcal{S} = (S, A, \alpha)$ *and the logic* $\mathcal{L}^{si}$*, for any two states* $s,t \in S$*,* $s \preceq t$ *iff* $Th(s) \subseteq Th(t)$*.*

Proof. First, we show the soundness. Let $s,t \in S$, $s \preceq t$ and ψ be a formula. We can prove $s \models_{si} \psi \Longrightarrow t \models_{si} \psi$ by structural induction on ψ. The cases for $\top$ and conjunction are trivial. Now, consider $\psi \equiv \langle a\rangle_p\varphi$. By structural induction, $[\![\varphi]\!]$ is $\preceq$-closed for any formula φ. If we assume that $s \models_{si} \psi$, then $\alpha(s,a)([\![\varphi]\!]) \geq p$. Since $s \preceq t$, we obtain $\alpha(t,a)([\![\varphi]\!]) \geq \alpha(s,a)([\![\varphi]\!]) \geq p$, i.e., $t \models_{si} \psi$.

Next, we show the completeness. Let $R = \{(s,t) \mid Th(s) \subseteq Th(t)\}$ and it suffices to prove that R is a simulation. Obviously, R is a preorder. Let $S = \{s_i \mid i \in I\}$.

First, we claim that for any $i \in I$, a characteristic formula φ_i exists such that $[\![\varphi_i]\!] = R(s_i)$. This can be proved as follows.

- If for all $j \in I$, $s_j \in R(s_i)$, then $R(s_i) = S$. Thus, we can take $\top$ as being the characteristic formula because $[\![\top]\!] = R(s_i)$.

- If some $j \in I$ exists with $s_j \notin R(s_i)$, then there must be a formula φ_{ij} such that $s_i \models_{si} \varphi_{ij}$ and $s_j \not\models_{si} \varphi_{ij}$. Otherwise, for any formula φ, $s_i \models_{si} \varphi$ implies that $s_j \models_{si} \varphi$, which means that $Th(s_i) \subseteq Th(s_j)$ and thus $(s_i, s_j) \in R$, and this contradicts the assumption that $s_j \notin R(s_i)$. For each $i \in I$, define $\varphi_i = \wedge_{s_j \notin R(s_i)} \varphi_{ij}$, and then by construction, we have $[\![\varphi_i]\!] = R(s_i)$.

Now, let $(s,t) \in R$, a be any action, and define $\varphi_a = \wedge_{i \in I} \langle a \rangle_{p_i} \varphi_i$, where $p_i = \alpha(s,a)(R(s_i))$ for each $i \in I$. For each $i \in I$, we have $s \models_{si} \langle a \rangle_{p_i} \varphi_i$, and thus $s \models_{si} \varphi_a$. By the definition of R, we have $t \models_{si} \varphi_a$, which means that for each $i \in I$, $t \models_{si} \langle a \rangle_{p_i} \varphi_i$. Thus, for each $i \in I$, we have

$$\alpha(t,a)(R(s_i)) \;=\; \alpha(t,a)([\![\varphi_i]\!]) \;\geq\; p_i \;=\; \alpha(s,a)(R(s_i)).$$

From Theorem 3.6(1), it follows that R is a simulation. $\square$

In the construction of the formula φ above, we can see that only finite conjunctions are needed if the state space is finite. In this case, the logic $\mathcal{L}^{si}$ restricted to binary conjunction, i.e., the logic $\mathcal{L}^{dep}$, is already sufficient to characterize simulations for **FLTS**s. This result can be generalized to image-finite **FLTS**s.

Theorem 4.12 *Let $\mathcal{S} = (S, A, \alpha)$ be an $\mathbf{F}_f$**LTS**, where the state space S is countable. Then, for any two states $s, t \in S$, $s \preceq t$ iff $Th(s) \subseteq Th(t)$ with respect to the logic $\mathcal{L}^{dep}$.*

Proof. The soundness follows from Theorem 4.11. For the completeness, let R, $S = \{s_i \mid i \in I\}$ and $\{\varphi_{ij}\}_{s_j \notin R(s_i)}$ is defined as given in Theorem 4.11. Now, the index set I is enumerative because the state space S is countable. We fix an arbitrary index k. For each $i \in I$, define $\Phi_i^k = \bigwedge_{j \leq k \wedge s_j \notin R(s_i)} \varphi_{ij}$. Intuitively, Φ_i^k is satisfied by all states in $R(s_i)$ but it is not satisfied by any state s_j with $j \leq k$ and $s_j \notin R(s_i)$. However, since the maximal index of the finite conjunction is k, then Φ_i^k may be satisfied by some states s_j with $j > k$. Hence, for each $i \in I$,

$$R(s_i) \;\subseteq\; [\![\Phi_i^k]\!] \;\subseteq\; R(s_i) \cup \{s_j \in S \mid j \in I \wedge j > k\}. \tag{9}$$

Suppose that $(s,t) \in R$. It remains to show that for any $a \in A$, $\alpha(s,a)(R(s_i)) \leq \alpha(t,a)(R(s_i))$ for each $i \in I$. Because of the first inclusion in (9), we have

$\alpha(s,a)([\![\Phi_i^k]\!]) \geq \alpha(s,a)(R(s_i))$ for each $i,k \in I$, which implies that $s \models_{si} \langle a\rangle_{p_i}\Phi_i^k$ for each $i,k \in I$, where $p_i = \alpha(s,a)(R(s_i))$. By the definition of R, we have $t \models_{si} \langle a\rangle_{p_i}\Phi_i^k$ for each $i,k \in I$ and hence $\alpha(t,a)([\![\Phi_i^k]\!]) \geq p_i$ for each $i \in I$. By the second inclusion in (9), for any $i,k \in I$, we have

$$p_i \;\leq\; \alpha(t,a)([\![\Phi_i^k]\!]) \;\leq\; \alpha(t,a)(R(s_i)) \vee \alpha(t,a)(\{s_j \in S \mid j \in I \wedge j > k\}).$$

It follows that

$$p_i \leq \alpha(t,a)(R(s_i)) \vee \inf_{k\in I} \alpha(t,a)(\{s_j \in S \mid j \in I \wedge j > k\}).$$

Similar to the proof of Eq. (5) given in Theorem 4.7 we also find that $\inf_{k\in I}\alpha(t,a)(\{s_j \in S \mid j \in I \wedge j > k\}) = 0$, and hence $p_i \leq \alpha(t,a)(R(s_i))$, i.e., for each $i \in I$, $\alpha(s,a)(R(s_i)) \leq \alpha(t,a)(R(s_i))$. By Theorem 3.6(1), we conclude that R is a simulation. $\square$

It should be noted that the completeness proofs for Theorems 4.11 and 4.12 are inspired by (although they differ from) the corresponding proofs in [23]. In the present study, we rely greatly on Theorem 3.6(1), which holds for **FLTS**s but not for **PLTS**s, as shown at the end of Section 3. The completeness proofs for simulations in [23] employ finitely-generated R-closed sets, thereby necessitating the use of disjunctions in the logical characterizations of simulations for **PLTS**s, as described in [17].

5 Related Work

Fuzzy simulations and bisimulations have attracted much attention from researchers in the field. Next, we briefly summarize some of the recent research in this area.

Errico and Loreti [18] proposed a notion of fuzzy bisimulation and applied it to fuzzy reasoning. Kupferman and Lusting [26] defined a latticed simulation between two lattice-valued Kripke structures, which they applied to latticed games. Pan et al. [33] studied simulation for lattice-valued doubly labeled transition systems. Cao et al. [6] defined a fuzzy bisimulation relation between two different **FLTS**s by a correlational pair based on some relation. Ćirić et al. [10] introduced two types of simulations (forward and backward) and four types of bisimulations (forward, backward, forward-backward, and backward-forward) for fuzzy automata. Sun et al. [42] investigated forward and backward bisimulations for fuzzy automata. Deng and Qiu [13], and Xing et al. [45] addressed the supervisory control of fuzzy discrete event systems by using simulation equivalence and bisimulation equivalence, respectively. Damljanović et al. [12]

also studied simulation and bisimulation for weighted automata in a similar manner, as also described in [10, 11].

All of these approaches can be divided into two classes. In the first class, simulations or bisimulations are based on a crisp relation on the state space, and thus one state is either (bi)similar to another state or not. As with [6, 12, 18, 42, 45], the present study belongs to this class. In the second class, simulations or bisimulations are based on a fuzzy relation (or a lattice-valued relation) on the state space, which shows the degree to which one state is (bi)similar to another. This approach was adapted in [10, 13, 26, 33]. In addition, in [18], a bisimulation is necessarily an equivalence relation, which is not the case in [6] and the present study.

There are other approaches for defining simulations and bisimulations for **FLTS**s, which were inspired by relevant research into **PLTS**s. For example, we can use the post-fixed points of a function [23] and lift relations by weight functions [25, 14]. A fuzzy analogue of the lifting operation is given as follows [8].

Let $R \subseteq S \times S$ be any relation. The lifted relation $R_{\mathcal{F}} \subseteq \mathcal{F}(S) \times \mathcal{F}(S)$ is a relation over possibility distributions such that $\mu R_{\mathcal{F}} \nu$ iff a weight function $e : S \times S \to [0, 1]$ exists with respect to R such that the following lifting conditions hold:

- $\mu(s) = \sup_{t \in S} e(s, t)$, for any $s \in S$;
- $\nu(t) = \sup_{s \in S} e(s, t)$, for any $t \in S$;
- $e(s, t) = 0$, if $(s, t) \notin R$.

Using the lifting operation above, we can define another simulation for **FLTS**s, as follows.

Let $\mathcal{S} = (S, A, \alpha)$ be an **FLTS**. A relation $R \subseteq S \times S$ is a simulation if whenever sRt, then for any transition $s \xrightarrow{a} \mu$, some transition $t \xrightarrow{a} \nu$ exists with $\mu R_{\mathcal{F}} \nu$.

If R is a preorder, then the two approaches for defining simulations, i.e., one based on relation lifting and the other based on R-closed sets, are equivalent for **PLTS**s [23], but they are different for **FLTS**s. In fact, for any $\mu, \nu \in \mathcal{F}(S)$, $\mu R_{\mathcal{F}} \nu$ implies that $\mu(U) \leq \nu(U)$ for any R-closed set U. However, the converse does not hold in general. Next, we provide a counter-example.

Example 5.1 *Let $S = \{s_1, s_2\}$ and $R = \{(s_1, s_1), (s_2, s_2), (s_1, s_2)\}$. There are only two R-closed sets: $\{s_2\}$ and $\{s_1, s_2\}$. Let μ be the possibility distribution with $\mu(s_1) = 0$ and $\mu(s_2) = \frac{1}{2}$. Let ν be the distribution with $\nu(s_1) = \frac{1}{3}$ and $\nu(s_2) = \frac{2}{3}$. Then, we can see that for any R-closed U, $\mu(U) \leq \nu(U)$. However,* *in this case, a weight function does not exist that satisfies the above lifting conditions. Therefore, we have $(\mu, \nu) \notin R_{\mathcal{F}}$.*

Nevertheless, if R is an equivalence relation, then the two approaches for defining bisimulations are equivalent for **FLTS**s [8].

6 Conclusion and Future Work

In this study, we investigated two fuzzy variants of the Hennessy-Milner Logic and characterized bisimulations and simulations for **FLTS**s soundly and completely. Compared with the logical characterizations for **PLTS**s, the following are the main differences.

- Characterizing simulations. For **PLTS**s, disjunction is necessary and binary conjunction is already sufficient [17], whereas for **FLTS**s, infinite conjunction is generally necessary but disjunction is not, i.e., the logic $\mathcal{L}^{si}$. The logic with binary conjunction, $\mathcal{L}^{dep}$, can characterize simulation for $\mathbf{F}_f$**LTS**s.

- Characterizing bisimulations. For **PLTS**s, negation is not necessary and binary conjunction is already sufficient, i.e., the logic $\mathcal{L}^{dep}$, whereas for **FLTS**s, both negation and infinite conjunction are necessary, i.e., the logic $\mathcal{L}^{bi}$. The logic with binary conjunction, $\mathcal{L}^{ls}$, can characterize bisimulation for an $\mathbf{F}_f$**LTS**.

In future research, it would be interesting to consider a logical characterization of simulation defined by relation lifting. We will probably need to adapt the logic $\mathcal{L}^{si}$ or impose some restrictions on the possibility distributions.

Another research direction may be to investigate logical characterizations for nondeterministic fuzzy transition systems [7, 8]. We consider that the logic for nondeterministic systems may need distribution semantics [35], i.e., the semantic interpretation of the logic is given in terms of distributions. In [8], Cao et al. noted that a nondeterministic fuzzy labeled system can easily be combined into a deterministic fuzzy labeled system. However, after their combination, the transitions in the deterministic system can differ from any transition in the nondeterministic system. For instance, we again consider the example of medical diagnosis from Section 2. Suppose that based on their experience, two physicians give the following two transitions:

$$\begin{aligned} \alpha(\text{poor},\ u_1) &= \tfrac{0.3}{\text{poor}} + \tfrac{0.5}{\text{fair}} + \tfrac{0.3}{\text{good}} \\ \alpha(\text{poor},\ u_1) &= \tfrac{0.4}{\text{fair}} + \tfrac{0.4}{\text{good}} + \tfrac{0.2}{\text{excellent}}. \end{aligned}$$

By combining the transitions and taking their supremum, we obtain the following transition:

$$\beta(\text{poor},\ u_1) \ = \ \frac{0.3}{\text{poor}} + \frac{0.5}{\text{fair}} + \frac{0.4}{\text{good}} + \frac{0.2}{\text{excellent}}$$

which is completely different from any one of the two transitions above. From a semantic view point, we consider that this way of reducing nondeterministic fuzzy transition systems to deterministic ones is unsatisfactory.

At present, we are still unclear about the relationship between our (bi)simulation and the corresponding concepts defined for fuzzy automata, so we leave this as further research.

Acknowledgment

The first author was supported by the *Zhejiang Provincial Natural Science Foundation of China (LY13F020046)* and *Zhejiang Provincial Education Department Fund of China (Y201223001).* The second author was supported by the *National Natural Science Foundation of China (61173033, 61261130589) and ANR 12IS02001 PACE.* The first author would also like to thank Dr Haiyu Pan for helpful discussions and suggestions. The authors would like to thank the anonymous reviewers and the language editor for their invaluable suggestions and comments.

References

[1] P. A. Abdulla, A. Legay, J. d'Orso and A. Rezine, Tree regular model checking: A simulation-based approach, The Journal of Logic and Algebraic Programming 69 (2006) 93-121.

[2] G. Bailador and G. Triviño, Pattern recognition using temporal fuzzy automata, Fuzzy Sets and Systems 161(2010) 37-55.

[3] R. Bělohlávek, Determinism and fuzzy automata, Information Sciences 142 (2002) 205-209.

[4] M. Bhattacharyya, Fuzzy Markovian decision process, Fuzzy Sets and Systems 99 (1998) 273-282.

[5] P. Billingsley, Probability and Measure, Wiley-Interscience, New York, 1995.

[6] Y. Z. Cao, G. Q. Chen and E. E. Kerre, Bisimulations for fuzzy transition systems, IEEE Transactions on Fuzzy Systems 19 (2010) 540-552.

[7] Y. Z. Cao and Y. Ezawa, Nondeterministic fuzzy automata, Information Sciences 191 (2012) 86-97.

[8] Y. Z. Cao, S. X. Sun, H. Wang and G. Q. Chen, A behavioral distance for fuzzy-transition systems, IEEE Transactions on Fuzzy Systems 21 (2013) 735-747.

[9] M. Ćirić, M. Droste, J. Ignjatović and H. Vogler, Determinization of weighted finite automata over strong bimonoids, Information Sciences 180 (2010) 3497-3520.

[10] M. Ćirić, J. Ignjatović, N. Damljanović and M. Bašić, Bisimulations for fuzzy automata, Fuzzy Sets and Systems 186 (2012) 100-139.

[11] M. Ćirić, J. Ignjatović, I. Jančić and N. Damljanović, Computation of the greatest simulations and bisimulations between fuzzy automata, Fuzzy Sets and Systems 208 (2012) 22-42.

[12] N. Damljanović, M. Ćirić and J. Ignjatović, Bisimulations for weighted automata over an additively idempotent semiring, Theoretical Computer Science 534 (2014) 86-100.

[13] W. L. Deng and D. W. Qiu, Supervisory control of fuzzy discrete event systems for simulation equivalence, IEEE Transactions on Fuzzy Systems 23 (2015) 178-192.

[14] Y. Deng, Semantics of Probabilistic Processes: An Operational Approach, Springer, 2015.

[15] Y. Deng and H. Wu, Modal Characterisations of Probabilistic and Fuzzy Bisimulations, in Proceedings of the 16th International Conference on Formal Engineering Methods. Lecture Notes in Computer Science 8829, pages 123-138. Springer, 2014.

[16] J. Desharnais, A. Edalat and P. Panangaden, Bisimulation for labelled Markov processes, Information and Computation 179 (2002) 163-193.

[17] J. Desharnais, V. Gupta, R. Jagadeesan and P. Panangaden, Approximating labelled Markov processes, Information and Computation 184 (2003) 160-200.

[18] L. D'Errico and M. Loreti, A process algebra approach to fuzzy reasoning, in Proceedings of the Joint 2009 International Fuzzy Systems Association World Congress and 2009 European Society of Fuzzy Logic and Technology Conference (IFSA-EUSFLAT 2009), Lisbon, Portugal, 2009, pp. 1136-1141.

[19] K. Fisler and M. Y. Vardi, Bisimulation minimization and symbolic model checking, Formal Methods in System Design 21 (2002) 39-78.

[20] R. J. van Glabbeek, S. A. Smolka, B. Steffen and C. M. N. Tofts, Reactive, generative, and stratified models of probabilistic processes, in Proc. 5th Annu. IEEE Symp. Logic in Computer Science, Philadelphia, USA, 1990, pp. 130-141.

[21] J. R. González de Mendívil and J. R. Garitagoitia, Fuzzy languages with infinite range accepted by fuzzy automata: pumping lemma and determinization procedure, Fuzzy Sets and Systems 249 (2014) 1-26.

[22] M. Hennessy and R. Milner, Algebraic laws for nondeterminism and concurrency, Journal of the Association for Computing Machinery 32 (1985) 137-161.

[23] H. Hermanns, A. Parma, et al., Probabilistic logical characterization, Information and Computation 209 (2011) 154-172.

[24] J. Ignjatović, M. Ćirić and V. Simović, Fuzzy relation equations and subsystems of fuzzy transition systems, Knowledge-Based Systems 38 (2013) 48-61.

[25] B. Jonsson and K. G. Larsen, Specification and refinement of probabilistic processes, in Proc. 6th Annu. IEEE Symp. Logic in Computer Science (LICS 1991), Amsterdam, Neth, 1991, pp. 266-277.

[26] O. Kupferman and Y. Lustig, Latticed simulation relations and games, International Journal of Foundations of Computer Science 21 (2010) 167-189.

[27] K. G. Larsen and A. Skou, Bisimulation through probabilistic testing, Information and Computation 94 (1991) 1-28.

[28] Y. M. Li and W. Pedrycz, Fuzzy finite automata and fuzzy regular expressions with membership values in lattice-ordered monoids, Fuzzy Sets and Systems 156 (2005) 68-92.

[29] Y. M. Li, Fuzzy automata theory with membership values in lattices, Information Sciences 181 (2011) 1003-1017.

[30] F. Lin and H. Ying, Modeling and control of fuzzy discrete event systems, IEEE Transactions on Systems, Man, and Cybernetics-Part B: Cybernetics 32 (2002) 408-415.

[31] J. N. Mordeson and D. S. Malik, Fuzzy Automata and Languages: Theory and Applications, Boca Raton, FL: Chapman & Hall/CRC, 2002.

[32] R. Milner, A calculus for communicating systems Lecture Notes in Computer Science 92 (1980), Springer-Verlag.

[33] H. Y. Pan, Y. Z. Cao, M. Zhang and Y. X. Chen, Simulation for lattice-valued doubly labeled transition systems, International Journal of Approximate Reasoning 55 (2014) 797-811.

[34] D. Park, Concurrency and automata on infinite sequences, in Proc. 5th GI Conf. Theor. Comput. Sci (Lecture Notes in Computer Science vol. 104), G. Goos and J. Hartmanis, Eds. Berlin, Germany: Springer-Verlag, 1981, pp. 167-183.

[35] A. Parma and R. Segala, Logical characterizations of bisimulations for discrete probabilistic systems, in 10th Int Conf. Foundations of Software Science and Computational Structures (Lecture Notes in Computer Science vol. 4423), H. Seidl, Ed. Berlin, Germany: Springer-Verlag, 2007, pp. 287-301.

[36] D. W. Qiu, Characterizations of fuzzy finite automata, Fuzzy Sets and Systems 141 (2004) 391-414.

[37] D. W. Qiu, Supervisory control of fuzzy discrete event systems: a formal approach, IEEE Transactions on Systems, Man, and Cybernetics-Part B: Cybernetics 35 (2005) 72-88.

[38] D. W. Qiu, Pumping lemma in automata theory based on complete residuated lattice-valued: A note, Fuzzy Sets and Systems 157 (2006) 2128-2138.

[39] D. W. Qiu and F. C. Liu, Fuzzy discrete event systems under fuzzy observability and a test-algorithm, IEEE Transactions on Fuzzy Systems 17 (2009) 578-589.

[40] V. R. L. Shen, Knowledge representation using high-level fuzzy Petri nets, IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans 36 (2006) 1220-1227.

[41] T. J. Ross, Fuzzy Logic with Engineering Applications, Wiley, 2010.

[42] D. D. Sun, Y. M. Li and W. W. Yang, Bisimulation relations for fuzzy finite automata, Fuzzy Systems and Mathematics 23 (2009) 92-100 (in Chinese).

[43] R. Segala and N. A. Lynch, Probabilistic simulations for probabilistic process, Nordic Journal of Computing 2 (1995) 250-273.

[44] W. G. Wee and K. S. Fu, A formulation of fuzzy automata and its application as a model of learning systems, IEEE Transactions on Systems Science and Cybernetics SSC-5 (1969) 215-223.

[45] H. Y. Xing, Q. S. Zhang and K. S. Huang, Analysis and control of fuzzy discrete event systems using bisimulation equivalence, Theoretical Computer Science 456 (2012) 100-111.

[46] L. A. Zadeh, Fuzzy sets as a basis for a theory of possibility, Fuzzy Sets and Systems 1 (1978) 3-28.