Modal Characterisations of Probabilistic and Fuzzy Bisimulations

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Abstract. This paper aims to investigate bisimulation on fuzzy systems. For that purpose we revisit bisimulation in the model of reactive probabilistic processes with countable state spaces and obtain two findings: (1) bisimilarity coincides with simulation equivalence, which generalises a result on finite-state processes originally established by Baier; (2) the modal characterisation of bisimilarity by Desharnais et al. admits a much simpler completeness proof. Furthermore, inspired by the work of Hermanns et al. on probabilistic systems, we provide a sound and complete modal characterisation of fuzzy bisimilarity.

1 Introduction

The analysis of fuzzy systems has been the subject of active research during the last sixty years, and many formalisms have been proposed to model them: fuzzy automata (see, for example, [3, 4, 8, 24, 26, 30, 37]), fuzzy Petri nets [31, 33], fuzzy Markov processes [5] and fuzzy discrete event systems [25, 32].

Recently, a new formal model for fuzzy systems, fuzzy labelled transition systems (fLTSs, for short), has been proposed [15, 7, 21]. fLTSs are a natual generalization of the classical labelled transition systems in computer science in that after performing some action a system evolves from one state to a fuzzy set of successor states instead of a unique state. Many formal description tools for fuzzy systems, such as fuzzy Petri nets and fuzzy discrete event systems, are not fLTSs. However, it is possible to translate a system's description in one of these formalisms into an fLTS to represent its behaviour.

Bisimulation [28] has been investigated in depth in process algebras because it offers a convenient co-inductive proof technique to establish behavioural equivalence [27]. It has been mostly used for verifying formal systems and is the foundation of state-aggregation algorithms that compress models by merging bisimilar states. State aggregation is routinely used as a preprocessing step before model checking [1, 17].

Errico and Loreti [15] proposed a notion of fuzzy bisimulation and applied it to fuzzy reasoning; Kupferman and Lusting [22] defined a latticed simulation between two lattice-valued Kripke structures, and applied it to latticed games; Cao et al. [7]

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defined a fuzzy bisimulation relation between two different fLTSs by a correlational pair based on some relation; Ćirić et al. [9] introduced four types of bisimulations (forward, backward, forward-backward, and backward-forward) for fuzzy automata.

All these approaches can be divided into two classes. In the first class, bisimulations are based on a crisp relation on the state space, so one state is either bisimilar to another state or not. As in [7, 15], the current paper falls into this class. In the second class, simulations or bisimulations are based on a fuzzy relation (or a lattice-valued relation) on the state space, which shows the degree of one state being bisimilar to another. This approach was adapted in [9, 22]. In addition, in [15] a bisimulation is necessarily an equivalence relation, which is not the case for [7] and the current work.

Following the seminal paper on exploring the connection between bisimulation and modal logic [19], a great amount of work has appeared that characterizes various kinds of bisimulations by appropriate logics. A logical characterizations of fuzzy bisimulation appeared in [15], which uses recursive formulas, and interprets a formula as a fuzzy set that gives the measure, respectively, of satisfaction and unsatisfaction of the formula.

In this paper we seek a variant of the Hennessy-Milner Logic (HML) to characterise fuzzy bisimulation. Since fuzzy systems are close to probabilistic systems, we revisit probabilistic bisimulations and their logical characterisations e.g. [13, 14, 20, 23]. We find that two results in probabilistic concurrency theory can be generalised or simplified.

- For finite-state reactive probabilistic labelled transition systems (rpLTSs) it is known that bisimilarity coincides with simulation equivalence. The result was originally proved by Baier [2], using techniques from domain theory. That proof is sophisticated and later on simplified by Zhang [39]. In the current work we generalise the above coincidence result to rpLTSs with *countable* state space. Our proof is surprisingly elementary and employs only some basic concepts of set theory.
- Desharnais et al. [14] proposed a probabilistic version of the HML to capture bisimilarity on rpLTSs. Their logic has neither negation nor infinite conjunctions, but is expressive enough to work for general rpLTSs that may have continuous state spaces. Their proof of this fact uses the machinery of analytic spaces. For rpLTSs with countable state spaces, we find that the completeness proof of their modal characterisation can be greatly simplified.

By adding negation and infinite conjunctions to the logic of Desharnais et al. we obtain a fuzzy variant of the HML to characterise bisimulation on general fLTSs. Unlike [15], there is no need of recursive formulas in our logic. For fLTSs with countable state spaces and finite-support possibility distributions, infinite conjunctions can be replaced by binary conjunctions. The completeness proofs of our logical characterisations are inspired by Hermanns et al. [20]. Since the differences between the two models, fLTSs and rpLTSs, seem to be small, one is tempted to think that the current work might be a straightforward generalization of the above mentioned works for rpLTSs. However, this is not the case. For rpLTSs, negation is unnecessary to characterize bisimulation and binary conjunction is already sufficient, whereas for fLTSs both negation and infinite conjunction are necessary to characterize bisimulation for general fLTSs that may be infinitely branching. Therefore, fLTSs resemble more to nondeterministic systems rather than to deterministic systems in this aspect. Moreover, different techniques are needed for proving characterization theorems for fLTSs and rpLTSs. For example, in the case of rpLTSs the famous π-λ theorem [6] holds, which greatly simplifies the completeness proof of the logical characterization. However, the π-λ theorem is invalid for fLTSs, so we adopt a different approach to proving completeness: the basic idea is to construct a characteristic formula for each equivalence class, i.e. the formula is satisfied only by the states in that equivalence class. See Section 6 for more details.

The rest of this paper is structured as follows. We briefly review some basic concepts used in this paper in Section 2. Section 3 is devoted to showing the coincidence of bisimilarity with simulation equivalence in countable rpLTSs. Section 4 gives a modal characterization of probabilistic bisimulation for rpLTSs. Fuzzy bisimulation is introduced in Section 5 and a modal characterisation is provided in Section 6. Finally, this paper is concluded in Section 7.

2 Preliminaries

Let S be a set and $\Delta : S \rightarrow [0,1]$ a fuzzy set. The *support* of Δ is the set $supp(\Delta) = \{s \in S \mid \Delta(s) > 0\}$. We denote by $\mathcal{F}(S)$ the set of all fuzzy sets in S, and $\mathcal{F}_f(S)$ the set of all fuzzy sets with finite supports, i.e. $\mathcal{F}_f(S) = \{\Delta \in \mathcal{F}(S) \mid supp(\Delta)$ is finite$\}$. With a slight abuse of notations, we sometimes write a possibility distribution to mean a fuzzy set[3].

A probability distribution on S is a fuzzy set Δ with $\sum_{s\in S} \Delta(s) = 1$. We write $\mathcal{D}(S)$ for the set of all probability distributions on S. We use $\overline{s}$ to denote the point distribution, satisfying $\overline{s}(t) = 1$ if $t = s$, and 0 otherwise. If $p_i \geq 0$ and Δ_i is a distribution for each i in some index set I, then $\sum_{i\in I} p_i \cdot \Delta_i$ is given by

$$(\sum_{i\in I} p_i \cdot \Delta_i)(s) \quad = \quad \sum_{i\in I} p_i \cdot \Delta_i(s)$$

If $\sum_{i\in I} p_i = 1$ then this is easily seen to be a distribution in $\mathcal{D}(S)$.

Let S be a set. For a binary relation $R \subseteq S \times S$ we write $s\ R\ t$ if $(s,t) \in R$. A *preorder* relation R is a reflexive and transitive relation; an *equivalence* relation is a reflexive, symmetric and transitive relation. An equivalence relation R partitions a set S into equivalence classes. For $s \in S$ we use $[s]_R$ to denote the unique equivalence class containing s. We drop the subscript R if the relation considered is clear from the context. Let $R(s)$ denote the set $\{s' \mid (s,s') \in R\}$. A set U is said to be R-closed if $R(s) \subseteq U$ for all $s \in U$, i.e. the image of U under R, written $R(U)$, is contained in U. We let R^* be the reflexive transitive closure of R. Note that if R is a preorder then R^* coincides with R. For any $s \in S$, the set $R^*(s)$ is clearly a R-closed set.

Definition 1. *A* fuzzy labelled transition system (fLTS) *is a triple* $(S, A, \rightarrow)$ *where* S *is a countable set of states*[4]*,* A *is a set of actions, and the transition relation* $\rightarrow$ *is a*

[3] Strictly speaking a possibility distribution is different from a fuzzy set, though the former can be viewed as the generalized characteristic function of the latter. See [38] for more detailed discussion.

[4] The constraint that S is countable will be important for later development, especially for the validity of the proof of Theorem 5.

partial function from $S \times A$ to $\mathcal{F}(S)$. If the transition relation $\rightarrow$ is a partial function from $S \times A$ to $\mathcal{D}(S)$, we say the fLTS is a reactive probabilistic labelled transition system (rpLTS).

We sometimes write $s \xrightarrow{a} \Delta$ and $s \xrightarrow{a[\lambda]} s'$ for $\rightarrow (s,a) = \Delta$ and $\rightarrow (s,a)(s') = \lambda$, respectively. An fLTS $(S, A, \rightarrow)$ is said to be *image-finite* if for each state s and label a, we have $\rightarrow (s,a) \in \mathcal{F}_f(S)$.

The fLTSs defined above are deterministic in the sense that for each state s and label a, there is at most one possibility distribution Δ with $s \xrightarrow{a} \Delta$. The rpLTSs defined above are usually called reactive probabilistic systems [18] or labelled Markov chains [13] in probabilistic concurrency theory. Similar to simple probabilistic automata [36], one could also define nondeterministic fuzzy transition systems by allowing for more than one transitions labelled with a same action leaving from a state.

3 Probabilistic Bisimulation and Simulation Equivalence

Let s and t be two states in a probabilistic labelled transition system, we say t can simulate the behaviour of s if whenever the latter can exhibit action a and lead to distribution Δ then the former can also perform a and lead to a distribution, say Θ, which can mimic Δ in successor states. We are interested in a relation between two states, but it is expressed by invoking a relation between two distributions. To formalise the mimicking of one distribution by the other, we make use of a lifting operation by following [11].

Definition 2. *Given a set S and a relation $R \subseteq S \times S$, we define $R^\dagger \subseteq \mathcal{D}(S) \times \mathcal{D}(S)$ as the smallest relation that satisfies:*

1. *$s \; R \; t$ implies $\overline{s} \; R^\dagger \; \overline{t}$*
2. *$\Delta_i \; R^\dagger \; \Theta_i$ implies $(\sum_{i\in I} p_i \cdot \Delta_i) \; R^\dagger \; (\sum_{i\in I} p_i \cdot \Theta_i)$, where I is an index set and $\sum_{i\in I} p_i = 1$.*

The proposition below is immediate.

Proposition 1. *Let Δ and Θ be two distributions over S, and $R \subseteq S \times S$. Then $\Delta \; R^\dagger \; \Theta$ if and only if there are two collections of states, $\{s_i\}_{i\in I}$ and $\{t_i\}_{i\in I}$, and a collection of probabilities $\{p_i\}_{i\in I}$, for some index set I, such that $\sum_{i\in I} p_i = 1$ and Δ, Θ can be decomposed as follows:*

1. *$\Delta = \sum_{i\in I} p_i \cdot \overline{s_i}$*
2. *$\Theta = \sum_{i\in I} p_i \cdot \overline{t_i}$*
3. *For each $i \in I$ we have $s_i \; R \; t_i$.*

An important point here is that in the decomposition of Δ into $\sum_{i\in I} p_i \cdot \overline{s_i}$, the states s_i are not necessarily distinct: that is, the decomposition is not in general unique. Thus when establishing the relationship between Δ and Θ a given state s in Δ may play a number of different roles.

If R is an equivalence relation, the lifted relation $R^\dagger$ can be defined alternatively as given in the orginal work by Larsen and Skou [23].

Proposition 2. *Let Δ, Θ be two distributions over S and $R \subseteq S \times S$ be an equivalence relation. Then $\Delta\ R^\dagger\ \Theta$ if and only if $\Delta(C) = \Theta(C)$ for each equivalence class $C \in S/R$, where $\Delta(C)$ stands for the accumulation probability $\sum_{s \in C} \Delta(s)$.*

Proof. See Theorem 2.4 (2) in [10]. □

Lemma 1. *Let $\Delta, \Theta \in \mathcal{D}(S)$ and R be a binary relation on S. If $\Delta\ R^\dagger\ \Theta$ then we have $\Delta(A) \leq \Theta(R(A))$ for each set $A \subseteq S$.*

Proof. Since $\Delta\ R^\dagger\ \Theta$, by Proposition 1 we can decompose Δ and Θ as follows.

$$\Delta = \sum_{i \in I} p_i \cdot \overline{s_i} \qquad s_i\ R\ t_i \qquad \Theta = \sum_{i \in I} p_i \cdot \overline{t_i}$$

Note that $\{s_i\}_{i \in I} = supp(\Delta)$. We define an index set $J \subseteq I$ in the following way: $J = \{i \in I \mid s_i \in A\}$. Then $\Delta(A) = \sum_{j \in J} p_j$. For each $j \in J$ we have $s_j\ R\ t_j$, i.e. $t_j \in R(s_j)$. It follows that $\{t_j\}_{j \in J} \subseteq R(A)$. Therefore, we can infer that

$$\Delta(A) \ = \ \Delta(\{s_j\}_{j \in J}) \ = \ \sum_{j \in J} p_j \ = \ \Theta(\{t_j\}_{j \in J}) \ \leq \ \Theta(R(A)).$$

□

Remark 1. The converse of Lemma 1 also holds [34], though it is not used in this paper. So an alternative way of lifting relations [12] is to say that Δ is related to Θ by lifting R if $\Delta(A) \leq \Theta(R(A))$ for each set $A \subseteq S$.

Corollary 1. *Let $\Delta, \Theta \in \mathcal{D}(S)$ and R be a binary relation on S. If $\Delta\ R^\dagger\ \Theta$ then $\Delta(A) \leq \Theta(A)$ for each R-closed set $A \subseteq S$.*

Proof. Let $A \subseteq S$ be R-closed. Then we have $R(A) \subseteq A$, and thus $\Theta(R(A)) \leq \Theta(A)$. By Lemma 1, if $\Delta\ R^\dagger\ \Theta$ then $\Delta(A) \leq \Theta(R(A))$. It follows that $\Delta(A) \leq \Theta(A)$. □

Remark 2. Let $\Delta, \Theta \in \mathcal{D}(S)$ and R be a binary relation on S. A curious reader may ask if the follwoing two conditions are equivalent:

1. $\Delta(A) \leq \Theta(R(A))$ for each set $A \subseteq S$;
2. $\Delta(A) \leq \Theta(A)$ for each R-closed set $A \subseteq S$.

Obviously, item 1 implies item 2. The converse, however, is not valid in general. For example, let $S = \{s, t\}$, $R = \{(s, t)\}$, $\Delta = \frac{1}{2}\overline{s} + \frac{1}{2}\overline{t}$ and $\Theta = \frac{1}{3}\overline{s} + \frac{2}{3}\overline{t}$. There are only two non-empty R-closed sets: $\{t\}$ and S. We have both $\Delta(\{t\}) \leq \Theta(\{t\})$ and $\Delta(S) \leq \Theta(S)$. However, $\Delta(\{t\}) = \frac{1}{2} \not\leq 0 = \Theta(\emptyset) = \Theta(R(\{t\}))$.

Nevertheless, if R is a preorder, then item 2 does imply item 1. For any set $A \subseteq S$, the transitivity of R implies that $R(A)$ is R-closed and the reflexivity of R tells us that $A \subseteq R(A)$, from which, together with item 2, we have $\Delta(A) \leq \Delta(R(A)) \leq \Theta(R(A))$.

Lemma 2. *Let R be a preorder on a set S and $\Delta, \Theta \in \mathcal{D}(S)$. If $\Delta\ R^\dagger\ \Theta$ and $\Theta\ R^\dagger\ \Delta$ then $\Delta(C) = \Theta(C)$ for all equivalence classes C with respect to the kernel $R \cap R^{-1}$ of R.*

Proof. Let us write $\equiv$ for $R \cap R^{-1}$. For any $s \in S$, let $[s]_\equiv$ be the equivalence class that contains s. Let A_s be the set $\{t \in S \mid s\, R\, t \;\wedge\; t \not R\, s\}$. It holds that

$$\begin{aligned} R(s) &= \{t \in S \mid s\, R\, t\} \\ &= \{t \in S \mid s\, R\, t \;\wedge\; t\, R\, s\} \uplus \{t \in S \mid s\, R\, t \;\wedge\; t \not R\, s\} \\ &= [s]_\equiv \;\uplus\; A_s \end{aligned}$$

where $\uplus$ stands for a disjoint union. Therefore, we have

$$\Delta(R(s)) = \Delta([s]_\equiv) + \Delta(A_s) \qquad \text{and} \qquad \Theta(R(s)) = \Theta([s]_\equiv) + \Theta(A_s) \quad (1)$$

We now check that both $R(s)$ and A_s are R-closed sets, that is $R(R(s)) \subseteq R(s)$ and $R(A_s) \subseteq A_s$. Suppose $u \in R(R(s))$. Then there exists some $t \in R(s)$ such that $t\, R\, u$, which means that $s\, R\, t$ and $t\, R\, u$. As a preorder R is a transitive relation. So we have $s\, R\, u$ which implies $u \in R(s)$. Therefore we can conclude that $R(R(s)) \subseteq R(s)$.

Suppose $u \in R(A_s)$. Then there exists some $t \in A_s$ such that $t\, R\, u$, which means that $s\, R\, t$, $t \not R\, s$ and $t\, R\, u$. As a preorder R is a transitive relation. So we have $s\, R\, u$. Note that we also have $u \not R\, s$. Otherwise we would have $u\, R\, s$, which means, together with $t\, R\, u$ and the transitivity of R, that $t\, R\, s$, a contradiction to the hypothesis $t \not R\, s$. It then follows that $u \in A_s$ and then we conclude that $R(A_s) \subseteq A_s$.

We have verified that $R(s)$ and A_s are R-closed sets. Since $\Delta\; R^\dagger\; \Theta$ and $\Theta\; R^\dagger\; \Delta$, we apply Corollary 1 and obtain that $\Delta(R(s)) \le \Theta(R(s))$ and $\Theta(R(s)) \le \Delta(R(s))$, that is

$$\Delta(R(s)) = \Theta(R(s)) \quad (2)$$

Similarly, using the fact that A_s is R-closed we obtain that

$$\Delta(A_s) = \Theta(A_s) \quad (3)$$

It follows from (1)-(3) that

$$\Delta([s]_\equiv) = \Theta([s]_\equiv)$$

as we have desired. □

Remark 3. Note that in the above proof the equivalence classes $[s]_\equiv$ are not necessarily R-closed. For example, let $S = \{s, t\}$, $Id_S = \{(s, s),\, (t, t)\}$ and $R = Id_S \cup \{(s, t)\}$. Then $\equiv\; = R \cap R^{-1} = Id_S$ and $[s]_\equiv = \{s\}$. We have $R(s) = S \not\subseteq [s]_\equiv$. So a more direct attempt to apply Corollary 1 to those equivalence classes would not work.

A restricted version of Lemma 2 (by requiring the state set S to be finite) has appeared as Lemma 5.3.5 in [2], but the proof given there is much more complicated as it relies on some properties of DCPOs, which is then simplified in [39]. In this paper, we allow the state set of a rpLTS to be a countably *infinite* set. With this key technical lemma at hand, we are ready to prove the coincidence of simulation equivalence and bisimilarity, which was originally given as Theorem 5.3.6 in [2].

Definition 3. *A relation $R \subseteq S \times S$ is a* probabilistic simulation *if $s\, R\, t$ and $s \xrightarrow{a} \Delta$ implies that some Θ exists such that $t \xrightarrow{a} \Theta$ and $\Delta\; R^\dagger\; \Theta$. If both R and R^{-1} are probabilistic simulations, then R is a* probabilistic bisimulation*. The largest probabilistic bisimulation, denoted by $\sim_{\rm p}$, is called* probabilistic bisimilarity*. The largest probabilistic simulation, denoted by $\prec_{\rm p}$, is called* probabilistic similarity*. The kernel of probabilistic similarity, i.e $\prec_{\rm p} \cap \prec_{\rm p}^{-1}$, is called* simulation equivalence*, denoted by $\asymp_{\rm p}$.*

In general, simulation equivalence is coarser than bisimilarity. However, for rpLTSs, the two relations do coincide.

Theorem 1. *For rpLTSs, simulation equivalence coincides with bisimilarity.*

Proof. It is obvious that $\sim_{\mathrm{p}}$ is included in $\asymp_{\mathrm{p}}$. For the other direction, we show that $\asymp_{\mathrm{p}}$ is a bisimulation. Let $s, t \in S$ and $s \asymp t$. Suppose that $s \xrightarrow{a} \Delta$. There exists a transition $t \xrightarrow{a} \Theta$ with $\Delta\ (\prec_{\mathrm{p}})^{\dagger}\ \Theta$. Since we are considering reactive probabilistic systems, the transtion $t \xrightarrow{a} \Theta$ from t must be matched by the transiton $s \xrightarrow{a} \Delta$ from s, with $\Theta\ (\prec_{\mathrm{p}})^{\dagger}\ \Delta$. Note that $\prec_{\mathrm{p}}$ is obviously a preorder on S. It follows from Lemma 2 that $\Delta(C) = \Theta(C)$ for any $C \in S/\asymp_{\mathrm{p}}$. Since $\asymp_{\mathrm{p}}$ is clearly an equivalence relation, by Proposition 2 we see that $\Delta\ (\asymp_{\mathrm{p}})^{\dagger}\ \Theta$. Therefore, $\asymp_{\mathrm{p}}$ is indeed a bisimulation relation. □

4 Modal Characterisation of Probabilistic Bisimulation

Let A be a set of *actions* ranged over by $a, b, \cdots$, and let $\top$ be a propositional constant. The language $\mathcal{L}_{\mathrm{p}}$ of formulas is the least set generated by the following BNF grammar:

$$\varphi ::= \top \mid \varphi_1 \wedge \varphi_2 \mid \langle a \rangle_p \varphi.$$

where a is an action and p is a rational number in the unit interval $[0, 1]$. This is the basic logic with which we establish the logical characterization of bisimulation.

Let us fix a rpLTS $(S, A, \rightarrow)$. The semantic interpretation of formulas in $\mathcal{L}_{\mathrm{p}}$ is given by:

- $s \models_{\mathrm{p}} \top$, for any state s;
- $s \models_{\mathrm{p}} \varphi_1 \wedge \varphi_2$, if $s \models_{\mathrm{p}} \varphi_1$ and $s \models_{\mathrm{p}} \varphi_2$;
- $s \models_{\mathrm{p}} \langle a \rangle_p \varphi$, if $s \xrightarrow{a} \Delta$ and $\exists A \subseteq S.\ (\forall s' \in A.\ s' \models_{\mathrm{p}} \varphi) \wedge (\Delta(A) \geq p)$.

We write $[\![\varphi]\!]$ for the set $\{s \in S \mid s \models_{\mathrm{p}} \varphi\}$. Then it is immediate that $s \models_{\mathrm{p}} \langle a \rangle_p \varphi$ iff $s \xrightarrow{a} \Delta$ and $\Delta([\![\varphi]\!]) \geq p$, i.e. $\sum_{s' \in [\![\varphi]\!]} \Delta(s') \geq p$. Thus $s \models_{\mathrm{p}} \langle a \rangle_p \varphi$ says that the state s can make an a-move to a distribution that evolves into a state satisfying φ with probability at least p. In the sequel we always use this fact as the semantic interpretation of the formula $\langle a \rangle_p \varphi$ in $\mathcal{L}_{\mathrm{p}}$.

The logic above induces a logical equivalence relation between states.

Definition 4. *Let s and t be two states in a rpLTS. We write $s =_{\mathrm{p}} t$ if $s \models_{\mathrm{p}} \varphi \Leftrightarrow t \models_{\mathrm{p}} \varphi$ for all $\varphi \in \mathcal{L}_{\mathrm{p}}$.*

The following lemma says that the transition probabilities to sets of the form $[\![\psi]\!]$ are completely determined by the formulas. It has appeared as Lemma 7.7.6 in [35].

Lemma 3. *Let s and t be two states in a rpLTS. If $s =_{\mathrm{p}} t$ and $s \xrightarrow{a} \Delta$, then some Θ exists with $t \xrightarrow{a} \Theta$, and for any formula $\psi \in \mathcal{L}_{\mathrm{p}}$ we have $\Delta([\![\psi]\!]) = \Theta([\![\psi]\!])$.*

Proof. First of all, the existence of Θ is obvious because otherwise the formula $\langle a \rangle_1 \top$ would be satisfied by s but not by t.

Let us assume, without loss of generality, that there exists a formula ψ such that $\Delta([\![\psi]\!]) < \Theta([\![\psi]\!])$. Then we can squeeze in a rational number p with $\Delta([\![\psi]\!]) < p \leq \Theta([\![\psi]\!])$. It follows that $t \models_{\mathtt{p}} \langle a \rangle_p \psi$ but $s \not\models_{\mathtt{p}} \langle a \rangle_p \psi$, which contradicts the hypothesis that $s =_{\mathtt{p}} t$. □

We will show that the logic $\mathcal{L}_{\mathtt{p}}$ can characterise bisimulation. The completeness proof of the characterisation crucially relies on the π-λ theorem [6]. Let $\mathcal{P}$ be a family of subsets of a set X. We say $\mathcal{P}$ is a π-class if it is closed under finite intersections; $\mathcal{P}$ is a λ-class if it is closed under complementations and countable disjoint unions.

Theorem 2 (The π-λ theorem). *If $\mathcal{P}$ is a π-class, then $\sigma(\mathcal{P})$ is the smallest λ-class containing $\mathcal{P}$, where $\sigma(\mathcal{P})$ is a σ-algebra containing $\mathcal{P}$.*

The next proposition is a typical application of the π-λ theorem [16], which tells us that when two probability distributions agree on a π-class they also agree on the generated σ-algebra.

Proposition 3. *Let S be a state space, $\mathcal{A}_0 = \{[\![\varphi]\!] \mid \varphi \in \mathcal{L}_{\mathtt{p}}\}$, and $\mathcal{A} = \sigma(\mathcal{A}_0)$. For any $\Delta, \Theta \in \mathcal{D}(S)$, if $\Delta(A) = \Theta(A)$ for any $A \in \mathcal{A}_0$, then $\Delta(B) = \Theta(B)$ for any $B \in \mathcal{A}$.*

Proof. Let $\mathcal{P} = \{A \in \mathcal{A} \mid \Delta(A) = \Theta(A)\}$. Then $\mathcal{P}$ is closed under countable disjoint unions because probability distributions are σ-additive. Furthermore, $\Delta(S) = \Theta(S) = 1$ implies that if $A \in \mathcal{P}$ then $\Delta(S \backslash A) = \Delta(S) - \Delta(A) = \Theta(S) - \Theta(A) = \Theta(S \backslash A)$, i.e. $S \backslash A \in \mathcal{P}$. Thus $\mathcal{P}$ is closed under complementation as well. It follows that $\mathcal{P}$ is a λ-class. Note that $\mathcal{A}_0$ is a π-class in view of the equation $[\![\varphi_1 \wedge \varphi_2]\!] = [\![\varphi_1]\!] \cap [\![\varphi_2]\!]$. Since $\mathcal{A}_0 \subseteq \mathcal{P}$, we can apply the π-λ Theorem to obtain that $\mathcal{A} = \sigma(\mathcal{A}_0) \subseteq \mathcal{P} \subseteq \mathcal{A}$, i.e. $\mathcal{A} = \mathcal{P}$. Therefore, $\Delta(B) = \Theta(B)$ for any $B \in \mathcal{A}$. □

Theorem 3. *Let s and t be two states in a rpLTS. Then $s \sim_{\mathtt{p}} t$ iff $s =_{\mathtt{p}} t$.*

Proof. The proof of soundness is carried out by a routine induction on the structure of formulas. Below we focus on the completeness. It suffices to show that $=_{\mathtt{p}}$ is a bisimulation. Note that $=_{\mathtt{p}}$ is clearly an equivalence relation. For any $u \in S$ the equivalence class in $S/_{=_{\mathtt{p}}}$ that contains u is

$$[u] = \bigcap \{[\![\varphi]\!] \mid u \models_{\mathtt{p}} \varphi\} \cap \bigcap \{S \backslash [\![\varphi]\!] \mid u \not\models_{\mathtt{p}} \varphi\}. \tag{4}$$

In (4) only countable intersections are used because the set of all the formulas in the logic $\mathcal{L}_{\mathtt{p}}$ is countable. Let $\mathcal{A}_0$ be defined as in Proposition 3. Then each equivalence class of $S/_{=_{\mathtt{p}}}$ is a member of $\sigma(\mathcal{A}_0)$.

On the other hand, $s =_{\mathtt{p}} t$ and $s \xrightarrow{a} \Delta$ implies that some distribution Θ exists with $t \xrightarrow{a} \Theta$ and for any $\varphi \in \mathcal{L}_{\mathtt{p}}$, $\Delta([\![\varphi]\!]) = \Theta([\![\varphi]\!])$ by Lemma 3. Thus by Proposition 3 we have

$$\Delta([u]) = \Theta([u]) \tag{5}$$

where $[u]$ is any equivalence class of $S/_{=_{\mathtt{p}}}$. Then it follows from Proposition 2 that $\Delta \ (=_{\mathtt{p}})^{\dagger} \ \Theta$. Symmetrically, any transition of t can be mimicked by a transition from s. Therefore, the relation $=_{\mathtt{p}}$ is a bisimulation. □

Theorem 3 tells us that $\mathcal{L}_p$ can characterize bisimulation for rpLTSs, and this logic has neither negation nor infinite conjunction. Moreover, the above result holds for general rpLTSs which are not necessarily finitely branching.

Remark 4. The proof of Theorem 3 does not carry over to fLTSs. For possibility distributions the family of sets $\mathcal{P}$ in the proof of Proposition 3 is closed under neither complementations nor countable intersections. So we cannot show that all equivalence classes are in $\mathcal{P}$ (i.e. $\sigma(\mathcal{A}_0)$). It follows that (5) cannot be established for fLTSs.

5 Fuzzy Simulation and Bisimulation

In this section we introduce our notions of simulation and bisimulation for fLTSs and discuss their properties.

In line with probabilistic simulation and bisimulation (Definition 3), we require that if (s,t) is a pair of states in a fuzzy simulation then t can mimic all stepwise behavior of s with respect to R that is lifted to compare possibility distributions. For probability distributions, we use $\Delta(U)$ to mean accumulation probabilities $\sum_{s\in U}\Delta(s)$, which no longer makes sense for possibility distributions. Now we replace the summation by a supremum. That is, for any $\Delta \in \mathcal{F}(S)$ and $U \subseteq S$, the notation $\Delta(U)$ stands for $\sup_{s\in U}\Delta(s)$, the supremum of all the possibilities in U. Possibility distributions are then compared by using R-closed sets.

Definition 5. *A relation $R \subseteq S\times S$ is a* fuzzy simulation *relation if $(s,t)\in R$ implies that, for any action $a\in A$, if $s \xrightarrow{a} \Delta$ then there exists some Θ such that $t \xrightarrow{a} \Theta$ and $\Delta(U)\le\Theta(U)$ for any R-closed set $U\subseteq S$. If both R and R^{-1} are fuzzy simulations, then R is a* fuzzy bisimulation. *The largest fuzzy simulation, denoted by $\prec_{\mathtt{f}}$, is called* fuzzy similarity*; the largest fuzzy bisimulation, written $\sim_{\mathtt{f}}$, is called* fuzzy bisimilarity.

The following proposition follows easily from the above definition.

Proposition 4. *1. $\sim_{\mathtt{f}}$ is an equivalence relation and $\prec_{\mathtt{f}}$ is a preorder.*
2. If R is a bisimulation and $t\in R^*(s)$, then for any action $a\in A$ we have $s\xrightarrow{a}\Delta$ implies $t\xrightarrow{a}\Theta$ for some Θ with $\Delta(U)=\Theta(U)$ for any R-closed set U.

The kernel of fuzzy similarity, $\prec_{\mathtt{f}} \cap \prec_{\mathtt{f}}^{-1}$, is called fuzzy simulation equivalence. Different from rpLTSs, in fLTSs bisimilarity is strictly finer than simulation equivalence.

Fig. 1. Fuzzy bisimilarity is strictly finer than fuzzy simulation equivalence

Example 1. Consider the fLTS depicted in Figure 1. We let $S = \{s, t, s_1, s_2, s_3, t_1, t_2\}$ and $R = \{(s, t), (s_1, t_1), (s_2, t_2), (s_3, t_1)\}$. It is easy to check that R is a simulation, thus $s \prec_{\mathtt{f}} t$. Now let $R' = \{(t, s), (t_1, s_1), (t_2, s_2)\}$. Obviously R' is also a simulation, and hence we have $t \prec_{\mathtt{f}} s$. It follows that s and t are simulation equivalent.

Now assume by contradiction that s and t are bisimilar. Then there exists a bisimulation R with $(s, t) \in R$. Let $s \xrightarrow{a} \Delta$ and $t \xrightarrow{a} \Theta$. Then we have $\Delta(R^*(s_3)) = \Theta(R^*(s_3))$. Since $\Delta(R^*(s_3)) \neq 0$ and Θ takes a non-zero value only at t_1, we infer that $t_1 \in R^*(s_3)$. By Proposition 4 (2), s_3 and t_1 can mimic the behaviour of each other, which contradicts the fact that t_1 can perform the action b to a nonempty distribution while s_3 cannot. Hence s and t are not bisimilar.

The following property is a counterpart of Proposition 2, by replacing accumulation probabilities by suprema of possibilities.

Proposition 5. *Let $R \subseteq S \times S$ be an equivalence relation. It is a fuzzy bisimulation iff for all $(s, t) \in R$ and $a \in A$, $s \xrightarrow{a} \Delta$ implies $t \xrightarrow{a} \Theta$ with $\Delta(U) = \Theta(U)$ for any equivalence class $U \in S/R$.*

6 Modal Characterisation of Fuzzy Bisimulation

Since the differences between the models fLTSs and rpLTSs do not seem to be big, one is tempted to think that a logic more or less the same as $\mathcal{L}_{\mathtt{p}}$ might characterise fuzzy bisimulation. However, as we have hinted in Remark 4, the modal characterisation of bisimulation for rpLTSs cannot be simply transplanted to fLTSs. In this section, we introduce another variant of the Hennessy-Milner Logic by adding negation to $\mathcal{L}_{\mathtt{p}}$, and by allowing infinite conjunctions.

The language $\mathcal{L}_{\mathtt{f}}$ of formulas is the least set generated by the following BNF grammar:

$$\varphi ::= \top \mid \textstyle\bigwedge_{i \in I} \varphi_i \mid \neg\varphi \mid \langle a \rangle_p \varphi.$$

where I is a countable index set and p is a rational number in the unit interval $[0, 1]$.

Let us fix an fLTS $(S, A, \rightarrow)$. The semantic interpretation of formulas in $\mathcal{L}_{\mathtt{f}}$ is similar to that of $\mathcal{L}_{\mathtt{p}}$ (cf. Section 4). We have $s \models_{\mathtt{f}} \langle a \rangle_p \varphi$ iff $s \xrightarrow{a} \Delta$ and $\Delta([\![\varphi]\!]) \geq p$, i.e. $\sup_{s' \in [\![\varphi]\!]} \Delta(s') \geq p$. Thus $s \models_{\mathtt{f}} \langle a \rangle_p \varphi$ says that the state s can make an a-move to a state that satisfies φ with possibility greater than p. The interpretation of negation is standard. We denote by $=_{\mathtt{f}}$ the logical equivalence induced by $\mathcal{L}_{\mathtt{f}}$. That is, $s =_{\mathtt{f}} t$ iff $s \models_{\mathtt{f}} \varphi \Longleftrightarrow t \models_{\mathtt{f}} \varphi$ for any $\varphi \in \mathcal{L}_{\mathtt{f}}$.

Consider again the fLTS depicted in Figure 1, we see that s satisfies, among others, the formula $\langle a \rangle_{\frac{1}{2}} \neg \langle b \rangle_{\frac{3}{4}} \top$ because s can make an a-move to state s_3 which is a deadlock state and thus cannot perform action b with possibility at least $\frac{3}{4}$.

Example 2. We give an example to show the importance of negation in $\mathcal{L}_{\mathtt{f}}$ but not in $\mathcal{L}_{\mathtt{p}}$.

Consider the rpLTSs in Figure 2, where $m \neq 0$. It is easy to see that $t \models_{\mathtt{p}} \langle a \rangle_1 \langle b \rangle_1 \top$ but $s \not\models_{\mathtt{p}} \langle a \rangle_1 \langle b \rangle_1 \top$. Hence s and t can be distinguished without negation in this rpLTS.

However, the case is different for fLTSs. In Example 1 we have shown that the two states s and t in Figure 1 are not bisimilar. But they cannot be distinguished by

Fig. 2. Two states can be distinguished without negation in rpLTSs

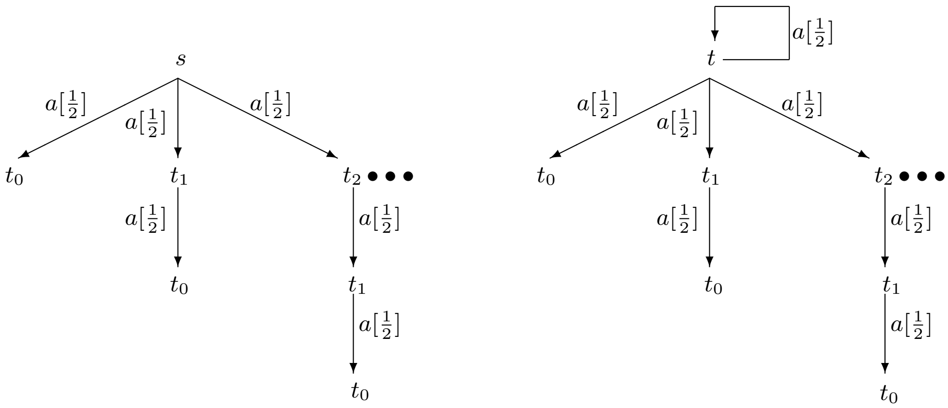

Fig. 3. Two states cannot be distinguished without infinite conjunction in fLTSs

the logic $\mathcal{L}_{\mathtt{f}}$ if negation is removed. The reason is as follows. Since $s \models_{\mathtt{f}} \langle a\rangle_p\varphi$ iff $s \xrightarrow{a} \Delta$ for some Δ with $\sup_{s'\in[\![\varphi]\!]} \Delta(s') \geq p$, the non-trivial formulas that both s and t satisfy are $\langle a\rangle_p\top$ ($p \leq \frac{2}{3}$), $\langle a\rangle_q\langle b\rangle_r\top$ ($q \leq \frac{2}{3}$ and $r \leq \frac{3}{4}$) and binary conjunction of these formulas. Moreover the non-trivial formulas that both s and t do not satisfy are $\langle a\rangle_p\top$ ($p > \frac{2}{3}$) and $\langle a\rangle_q\langle b\rangle_r\top$ ($q > \frac{2}{3}$ or $r > \frac{3}{4}$). Hence the logic $\mathcal{L}_{\mathtt{f}}$ without negation cannot differentiate s from t. However, we can easily tell apart these two states if negation is allowed. For instance, we have $s \models_{\mathtt{f}} \langle a\rangle_{\frac{1}{2}}\neg\langle b\rangle_{\frac{3}{4}}\top$ but $t \not\models_{\mathtt{f}} \langle a\rangle_{\frac{1}{2}}\neg\langle b\rangle_{\frac{3}{4}}\top$.

Example 3. This example shows that finite conjunctions are insufficient to characterize bisimulations for fLTSs. It is adapted from Example 5.1 in [20].

The only difference between the two fLTSs in Figure 3 is that t has a transition to itself. We first show that s and t are not bisimilar. Let $S = \{s, t, t_0, t_1, t_2, \cdots\}$, $t \xrightarrow{a} \Delta$, $t_n \xrightarrow{a} \Delta_n (n = 1, 2, \cdots)$, and $s \xrightarrow{a} \Theta$. Now suppose that s and t are bisimilar. Then there exists a bisimulation R with $(s, t) \in R$. It follows that $\Theta(R^*(t)) = \Delta(R^*(t)) \geq \frac{1}{2}$. Since Θ takes value 0 at s and t, $R^*(t)$ includes at least a $t_n(n = 0, 1, 2, \cdots)$ besides s and t. Thus it follows from Proposition 4 that $\Delta_n(R^*(t)) = \Theta(R^*(t))$. Since Δ_n takes value $\frac{1}{2}$ only at t_{n-1} and 0 otherwise, $t_{n-1} \in R^*(t)$ and $\Delta_{n-1}(R^*(t)) = \Delta(R^*(t)) = \frac{1}{2}$. Continue this way and we obtain $t_0 \in R^*(t)$ in the end, which leads to a contradiction because t can perform an a-action to a nonempty distribution while t_0 cannot. Hence s and t are not bisimilar.

However, if only finite conjunctions are used in the logic $\mathcal{L}_{\mathtt{f}}$ then we cannot find a formula which can differentiate s from t. The reason is as follows. For each formula φ, the maximum number of nested diamond connectives in the formula is finite, thus the satisfiability of φ is determined by some state t_n for a finite number n. Such a state t_n can also be reached from s immediately after one step of transition.

The situation is different if infinite conjunction is allowed. Consider the formula φ_i defined as follows: $\varphi_0 = \top$, and $\varphi_i = (\langle a \rangle_{\frac{1}{2}})^{(i)}\top$ for $i > 0$, which means

$$\underbrace{\langle a \rangle_{\frac{1}{2}} \langle a \rangle_{\frac{1}{2}} \cdots \langle a \rangle_{\frac{1}{2}}}_{i \text{ times}} \top.$$

Let $\varphi = \bigwedge_{i\in\mathbb{N}} \varphi_i$. Then $t \models_{\mathtt{f}} \varphi$. By mathematical induction one can prove that for any n, $t_n \models_{\mathtt{f}} \varphi_n$ but $t_n \not\models_{\mathtt{f}} \varphi_{n+1}$. It follows that $t_n \not\models_{\mathtt{f}} \varphi$ for any n. Now let $\psi = \langle a \rangle_{\frac{1}{2}}\varphi$. Then $t \models_{\mathtt{f}} \psi$ because $t \in [\![\varphi]\!]$ and $\Delta(t) = \frac{1}{2}$. However, $s \not\models_{\mathtt{f}} \psi$ since Θ takes nonzero values only at t_n ($n = 0, 1, \cdots$) and $t_n \notin [\![\varphi]\!]$. Consequently, we find a formula with infinite conjunction to distinguish s from t.

We see from the above examples that fLTSs resemble nondeterministic systems rather than deterministic systems, so they are fairly different from rpLTSs.

Below we show that two states in an fLTS are observationally indistinguishable or bisimilar if and only if they are logically indistinguishable.

We first observe that for fLTSs we have a counterpart of Lemma 3.

Lemma 4. *Let s and t be two states in an fLTS. If $s =_{\mathtt{f}} t$ and $s \xrightarrow{a} \Delta$, then there is some Θ such that $t \xrightarrow{a} \Theta$ and $\Delta([\![\varphi]\!]) = \Theta([\![\varphi]\!])$ for any $\varphi \in \mathcal{L}_{\mathtt{f}}$.*

Proof. Similar to the arguments for proving Lemma 3. □

Theorem 4. *Let s and t be two states in an fLTS. Then $s \sim_{\mathtt{f}} t$ iff $s =_{\mathtt{f}} t$.*

Proof. First we show soundness. Suppose that $s \sim_{\mathtt{f}} t$. For any $\psi \in \mathcal{L}_{\mathtt{f}}$, we show $s \models_{\mathtt{f}} \psi \Longleftrightarrow t \models_{\mathtt{f}} \psi$ (meaning that $[\![\psi]\!]$ is $\sim_{\mathtt{f}}$-closed) by structural induction on ψ. The cases of $\top$ and conjunction are trivial. Now consider other cases:

- $\psi \equiv \neg\varphi$. In this case $s \models_{\mathtt{f}} \psi \Longleftrightarrow s \not\models_{\mathtt{f}} \varphi$. By structural induction we have $s \not\models_{\mathtt{f}} \varphi \Longleftrightarrow t \not\models_{\mathtt{f}} \varphi$. Notice that we also have $t \not\models_{\mathtt{f}} \varphi \Longleftrightarrow t \models_{\mathtt{f}} \psi$.
- $\psi \equiv \langle a \rangle_p \varphi$. If $s \models_{\mathtt{f}} \psi$ then $s \xrightarrow{a} \Delta$ for some Δ with $\Delta([\![\varphi]\!]) \geq p$. By induction, $[\![\varphi]\!]$ is $\sim_{\mathtt{f}}$-closed. It follows from $s \sim_{\mathtt{f}} t$ that $t \xrightarrow{a} \Theta$ for some Θ such that $\Theta([\![\varphi]\!]) = \Delta([\![\varphi]\!])$. Then it is immediate that $t \models_{\mathtt{f}} \psi$. By symmetry, if $t \models_{\mathtt{f}} \psi$ then we have $s \models_{\mathtt{f}} \psi$.

For completeness, it suffices to prove that $=_{\mathtt{f}}$ is a bisimulation. Obviously $=_{\mathtt{f}}$ is an equivalence relation. Let $E = \{U_i \mid i \in I\}$ be the set of all equivalence classes of $=_{\mathtt{f}}$. Then by Proposition 5 it remains to show that, for any s, t with $s =_{\mathtt{f}} t$, if $s \xrightarrow{a} \Delta$ then $t \xrightarrow{a} \Theta$ for some Θ with

$$\Delta(U_i) = \Theta(U_i) \text{ for any } i \in I. \tag{6}$$

We first claim that, for any equivalence class U_i, there exists a characteristic formula φ_i in the sense that $[\![\varphi_i]\!] = U_i$. This can be proved as follows:

- If E contains only one equivalence class U_1, then $U_1 = S$. So we can take the characteristic formula to be $\top$ because $[\![\top]\!] = S$.
- If E contains more than one equivalence class, then for any $i, j \in I$ with $i \neq j$, there exists a formula φ_{ij} such that $s_i \models_{\mathtt{f}} \varphi_{ij}$ and $s_j \not\models_{\mathtt{f}} \varphi_{ij}$ for any $s_i \in U_i$ and $s_j \in U_j$. Otherwise, for any formula φ, $s_i \models_{\mathtt{f}} \varphi$ implies $s_j \models_{\mathtt{f}} \varphi$. Since the negation connective exists in the logic $\mathcal{L}_{\mathtt{f}}$, we also have $s_i \models_{\mathtt{f}} \neg\varphi$ implies $s_j \models_{\mathtt{f}} \neg\varphi$, which means $s_j \models_{\mathtt{f}} \varphi$ implies $s_i \models_{\mathtt{f}} \varphi$. Then $s_i \models_{\mathtt{f}} \varphi \Leftrightarrow s_j \models_{\mathtt{f}} \varphi$ for any $\varphi \in \mathcal{L}_{\mathtt{f}}$, which contradicts the fact that s_i and s_j are taken from different equivalence classes. For each $i \in I$, define $\varphi_i = \bigwedge_{j \neq i} \varphi_{ij}$, then by construction $[\![\varphi_i]\!] = U_i$. Let us check the last equality. On one hand, if $s' \in [\![\varphi_i]\!]$, then $s' \models_{\mathtt{f}} \varphi_i$ which means that $s' \models_{\mathtt{f}} \varphi_{ij}$ for all $j \neq i$. That is, $s' \notin U_j$ for all $j \neq i$, and this in turn implies that $s' \in U_i$. On the other hand, if $s' \in U_i$ then $s' \models_{\mathtt{f}} \varphi_i$ as $s_i \models_{\mathtt{f}} \varphi_i$, which means that $s' \in [\![\varphi_i]\!]$.

This completes the proof of the claim that each equivalence U_i has a characteristic formula φ_i.

Now suppose $s =_{\mathtt{f}} t$. By Lemma 4, if $s \xrightarrow{a} \Delta$ then there exists some Θ such that $t \xrightarrow{a} \Theta$ and $\Delta([\![\varphi_i]\!]) = \Theta([\![\varphi_i]\!])$ for all $i \in I$. Using the above claim, we can infer that

$$\Delta(U_i) = \Delta([\![\varphi_i]\!]) = \Theta([\![\varphi_i]\!]) = \Theta(U_i)$$

for each $i \in I$. Hence the equation in (6) holds. $\square$

In the above proof, the idea of using characteristic formulas is inspired by [20]. We can see that the logic $\mathcal{L}_{\mathtt{f}}$ is very expressive, since it characterizes not only bisimulation but also equivalence classes in the sense that there does necessarily exist a formula for each equivalence class which is satisfied only by the states in that class.

Moreover, from the construction of formula φ_i above we can see that infinite conjunctions are indeed necessary. The advantage of infinite conjunctions lie in the fact that they allow for a universal description of a class of states of interest. However, infinity is difficult to process in real applications. Fortunately, in most practical applications the supports of fuzzy sets are finite and then the logic $\mathcal{L}_{\mathtt{f}}$ restricted to binary conjunctions is already sufficient to characterize fuzzy bisimulation. Let us write $=_{\mathtt{f}'}$ for the logical equivalence induced by $\mathcal{L}_{\mathtt{f}}$ restricted to binary conjunctions.

Theorem 5. *Let $(S, A, \rightarrow)$ be an image-finite fLTS. Then for any two states $s, t \in S$, $s \sim_{\mathtt{f}} t$ iff $s =_{\mathtt{f}'} t$.*

Proof. Theorem 4 implies the soundness. For the completeness, let E and $\{\varphi_{ij}\}_{i,j \in I}$ be defined as in the proof of that theorem. Note that E is countable because the state space S is countable. We fix an arbitrary index k. For each $i \in I$, define $\Phi_i^k = \bigwedge_{j \leq k} \varphi_{ij}$. It is then easy to see that for each $i \in I$, the formula Φ_i^k only has finite conjunctions, and

$$U_i \subseteq [\![\Phi_i^k]\!] \subseteq U_i \cup \bigcup_{m \in I \wedge m > k} U_m.$$

Hence for any $\Gamma \in \mathcal{F}_f(S)$, $\Gamma(U_i) \leq \Gamma([\![\Phi_i^k]\!]) \leq \Gamma(U_i \cup \bigcup_{m \in I \wedge m > k} U_m)$. Recall that for any $U \subseteq S$, the notation $\Gamma(U)$ means the maximal possibility assigned by Γ to a state in U. Therefore, we have

$$\Gamma(U_i) \leq \Gamma([\![\Phi_i^k]\!]) \leq \Gamma(U_i) \sqcup \Gamma(\bigcup_{m \in I \wedge m > k} U_m) \tag{7}$$

for any $\Gamma \in \mathcal{F}_f(S)$, where we use the notation $p_1 \sqcup p_2$ to mean $\max(p_1, p_2)$. Fix an arbitrary index i and then take the infimum for $k \in I$, we can get

$$\Gamma(U_i) \leq \inf_{k \in I} \Gamma([\![\Phi_i^k]\!]) \leq \inf_{k \in I} [\Gamma(U_i) \sqcup \Gamma(\bigcup_{m \in I \wedge m > k} U_m)],$$

i.e.

$$\Gamma(U_i) \leq \inf_{k \in I} \Gamma([\![\Phi_i^k]\!]) \leq \Gamma(U_i) \sqcup \inf_{k \in I} \Gamma(\bigcup_{m \in I \wedge m > k} U_m). \tag{8}$$

We argue that

$$\inf_{k \in I} \Gamma(\bigcup_{m \in I \wedge m > k} U_m) = 0. \tag{9}$$

As a matter of fact, since $supp(\Gamma)$ is finite, there exists a sufficiently large number $N \in I$ such that for any $s \in supp(\Gamma)$, there exists some $m_s \in I$ with $m_s < N$ and $s \in U_{m_s}$. Thus we always have $\Gamma(\bigcup_{m \in I \wedge m > k} U_m) = 0$ when $k \geq N$. Hence the equation in (9) holds.

By combining (8) and (9), for any $i \in I$ and any $\Gamma \in \mathcal{F}_f(S)$, we have

$$\Gamma(U_i) = \inf_{k \in I} \Gamma([\![\Phi_i^k]\!]). \tag{10}$$

Now assume that $s =_{\mathtt{f}'} t$ and $s \xrightarrow{a} \Delta$. Then s satisfies the formula $\langle a \rangle_1 \top$. In order for t to satisfy that formula, there must exist a transition $t \xrightarrow{a} \Theta$ for some Θ. It remains to show that $\Delta(U_i) = \Theta(U_i)$ for any $i \in I$. By the left part of (7) applied to Δ we have $\Delta([\![\Phi_i^k]\!]) \geq \Delta(U_i)$ for each $i \in I$, implying $s \models_{\mathtt{f}} \langle a \rangle_{p_i} \Phi_i^k$ for each $i, k \in I$ where $p_i = \Delta(U_i)$. Then $t \models_{\mathtt{f}} \langle a \rangle_{p_i} \Phi_i^k$ for each $i, k \in I$. Hence $\Theta([\![\Phi_i^k]\!]) \geq p_i$ for each $i \in I$. By the right part of (7) applied to Θ, for an arbitrary index i and any $k \in I$ we can get

$$p_i \leq \Theta([\![\Phi_i^k]\!]) \leq \Theta(U_i) \sqcup \Theta(\bigcup_{m \in I \wedge m > k} U_m).$$

It follows that $p_i \leq \Theta(U_i) \sqcup \inf_{k \in I} \Theta(\bigcup_{m \in I \wedge m > k} U_m)$. Thus, by (9) we have $p_i \leq \Theta(U_i)$. That is, $\Delta(U_i) \leq \Theta(U_i)$ for each $i \in I$. Now suppose that there exists an $i_0 \in I$ such that $\Delta(U_{i_0}) < \Theta(U_{i_0})$. Then we can take $\epsilon_0 > 0$ such that $\Delta(U_{i_0}) < \Delta(U_{i_0}) + \epsilon_0 < \Theta(U_{i_0})$. For this ϵ_0, by (10) applied to U_{i_0} we can see that there exists some $k_0 \in I$ such that $\Delta([\![\Phi_{i_0}^{k_0}]\!]) < \Delta(U_{i_0}) + \epsilon_0$. Thus $s \not\models_{\mathtt{f}} \langle a \rangle_{\Delta(U_{i_0})+\epsilon_0} \Phi_{i_0}^{k_0}$ but $t \models_{\mathtt{f}} \langle a \rangle_{\Delta(U_{i_0})+\epsilon_0} \Phi_{i_0}^{k_0}$ since $\Theta([\![\Phi_{i_0}^{k_0}]\!]) \geq \Theta(U_{i_0}) > \Delta(U_{i_0}) + \epsilon_0$, which contradicts the assumption that $s =_{\mathtt{f}'} t$. Hence for each $i \in I$, $\Delta(U_i) = \Theta(U_i)$ as desired. □

7 Conclusions and Future Work

We have shown that on reactive probabilistic processes with countable state-space bisimilarity coincides with simulation equivalence, and the proof is very elementary. We have also simplified the modal characterisation of bisimilarity proposed by Desharnais et al.; our completeness proof does not invole advanced machinery on analytic spaces. For fuzzy labelled transition systems, we have presented a variant of the Hennessy-Milner Logic to capture bisimilarity. If the systems are image-finite, it is possible to use binary conjunctions instead of infinite conjunctions in the fuzzy logic, but negation has to be kept.

As future work, it would be interesting to investigate logical characterizations for nondeterministic fuzzy transition systems [8]. We believe that the logic for nondeterministic systems may need distribution semantics [29], i.e. semantic interpretation of the logic is given in terms of distributions.

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