Advanced Algorithms (X)

Yijia Chen
Fudan University
Main Theorem
Theorem

Let $k \in \mathbb{N}$. Then the $k$-clique problem on $n$-vertex graphs requires constant depth circuits of size

$$\omega\left(n^{k/9}\right).$$
Proof

Trivial for $k \leq 2$. So let $k \geq 3$ and $\mathcal{C} = (C_n)_{n \in \mathbb{N}}$ be a sequence of circuits on $\binom{n}{2}$ inputs of constant depth and size $O(n^{t/2})$ for some

$$t := \frac{2k}{9} > \frac{2}{3}.$$ 

We define

$$\alpha := \frac{4}{9t - 2.5} < \frac{4}{9t - 3},$$

which implies $\alpha = 2/(k - 1.25) > 2/(k - 1)$. Hence

$$n^{-\alpha} \ll n^{-2/(k-1)},$$

and $G \in \text{ER}(n, n^{-\alpha})$ almost surely contains no $k$-clique.

On the other hand, the previous theorem guarantees

$$\lim_{n \to \infty} \Pr_{G \in \text{ER}(n, n^{-\alpha}), A \in \binom{[n]}{k}} [C_n(G) = C_n(G \cup K_A)] = 1.$$
Proof of the Main Technical Lemma
Definition
Let $I$ be a set, e.g., $[n]$ and $\binom{[n]}{2}$, and let $f$ be a function with domain $\{0, 1\}^I$. Then the sensitive inputs of $f$ is

$$S(f) := \{ i \in I \mid \text{there exist } x, y \in \{0, 1\}^I \text{ with } f(x) \neq f(y) \text{ and } x_j = y_j \text{ for all } j \in I \setminus \{i\} \}.$$
Definition
A restriction on $I$ is a function

$$\rho : I \to \{0, 1, \star\},$$

which fixes a certain subset of the input bits (assigning either 0 or 1) and leaves the remaining bits unrestricted (assigning $\star$).

Let $f$ be a function with domain $\{0, 1\}^I$. Then the function $f \upharpoonright \rho$ has domain

$$\{0, 1\}^{\rho^{-1}(\star)}$$

and is defined by

$$f \upharpoonright \rho (x) = f(y),$$

where $y \in \{0, 1\}^I$ with $y(i) = \rho(i)$ if $i \in \rho^{-1}(\{0, 1\})$ and $y(i) = x(i)$ if $i \in \rho^{-1}(\star)$. 

Definition
Let $I$ be an index set and $p, q \in \mathbb{R}$ with $0 \leq p, q \leq 1$. Then $R_{I}^{p,q}$ is the random restriction $\rho : I \rightarrow \{0, 1, \star\}$, where the values $\rho(i)$ are independent and identically distributed (i.i.d.) with

\[
\begin{align*}
\Pr[\rho(i) = \star] &= p, \\
\Pr[\rho(i) = 1] &= q(1 - p), \\
\Pr[\rho(i) = 0] &= (1 - q)(1 - p).
\end{align*}
\]

In case $I = [n]$, we write $R_{n}^{p,q}$ instead of $R_{[n]}^{p,q}$.
Lemma

Let \( f_n : \{0, 1\}^n \rightarrow \{0, 1\} \) for \( n \in \mathbb{N} \) be functions computed by circuits of constant depth \( d \) and size \( O(n^t) \). Then for every (small) \( \delta > 0 \) and (large) \( \ell \in \mathbb{N} \), there is a constant

\[
c = c(d, t, \delta, \ell)
\]

such that

\[
\Pr_{\rho \in \mathbb{R}_n^{\frac{1}{2} - \delta}} \left[ |S(f_n \mid \rho)| > c \right] = O\left(\frac{1}{n^{\ell}}\right)
\]
Lemma

Let \( f_n : \{0, 1\}^n \to \{0, 1\} \) for \( n \in \mathbb{N} \) be functions computed by circuits of constant depth \( d \) and size \( O(n^t) \). Then for every \( \alpha, \delta > 0 \) and \( \ell > 0 \), there is a constant

\[
c = c(d, t, \alpha, \delta, \ell)
\]

such that

\[
\Pr_{\rho \in \mathcal{R}_{n-(\alpha+\delta), n-\alpha}^{\binom{n}{2}}} \left[ |\mathbb{S}(f_n \mid \rho)| > c \right] = O\left(\frac{1}{n^\ell}\right)
\]
**Definition**
Let \( n \in \mathbb{N}, f \) be a function with domain \( G_n, \) i.e., \( \{0, 1\}^{\binom{n}{2}} \), and \( \rho : \binom{[n]}{2} \to \{0, 1, \star\} \) a graph-restriction. Recall the sensitive inputs of \( f \upharpoonright \rho \) is

\[
\mathbb{S}(f \upharpoonright \rho) := \{ i^1 \in \rho^{-1}(\star) \mid \text{there exist } x, y \in \{0, 1\}^{\rho^{-1}(\star)} \text{ with } f \upharpoonright \rho (x) \neq f \upharpoonright \rho (y) \text{ and } x_j = y_j \text{ for all } j \in \rho^{-1}(\star) \setminus \{i\} \}.
\]

Every element in \( \mathbb{S}(f \upharpoonright \rho) \) is a **sensitive edge** of \( f \upharpoonright \rho \).

Then every element in

\[
\mathbb{V}(f \upharpoonright \rho) := \{ i \in [n] \mid \text{there exists a } j \in [n] \text{ with } \{i, j\} \in \mathbb{S}(f \upharpoonright \rho) \}
\]

is a **sensitive vertex** of \( f \upharpoonright \rho \). Note

\[
|\mathbb{V}(f \upharpoonright \rho)| \leq 2|\mathbb{S}(f \upharpoonright \rho)|.
\]

\(^1\)Recall that we identify \( [\binom{n}{2}] \) with \( \binom{[n]}{2} \).
Definition
Let $H$ be a graph, $q \in \mathbb{R}$ with $0 \leq q \leq 1$, and $n \geq |V(H)|$. Then the random graph-restriction $\mathcal{GR}_n^q(H)$ is a graph-restriction $\rho : \binom{[n]}{2} \to \{0, 1, \star\}$ is defined as follow.

1. Choose an injective $w : V(H) \to [n]$ uniformly at random.
2. For every edge $\{i, j\} \in E(H)$, the element $\{w(i), w(j)\}$ is mapped to $\star$ by $\rho$.
3. All the remaining $e \in \binom{[n]}{2} \setminus w(E(H))$ is mapped to 1 with probability $q$ and to 0 with probability $1 - q$. 
Proposition

Let the functions $f_n : G_n \rightarrow \{0, 1\}^{\lceil n^\beta \rceil}$ be AC$^0$-computable. And let $H$ be a graph and $\alpha < \text{threshold}(H)$. Then

$$\Pr_{\rho \in \mathcal{R}_n^{-\alpha}(H)} \left[ |V(f_n \upharpoonright \rho)| = |V(H)| \right] \leq n^\alpha |E(H)| + (\beta - 1)|V(H)| + o(1).$$
Definition
Let $G$ be a graph. Then we define the threshold exponent of $G$ is

$$\text{threshold}(G) := \min_{H \subseteq G} \frac{|V(H)|}{|E(H)|}.$$
Lemma (Janson, Luczak, and Rucinski, 1990)

Let $H$ be a graph and $\alpha > 0$.

1. If $\alpha > \text{threshold}(H)$, then

$$\Pr_{G \in \text{ER}(n, n^{-\alpha})} \left[ \text{indsub}(G, H) > 0 \right] = \exp \left( -n^{\Omega(1)} \right).$$

2. If $\alpha < \text{threshold}(H)$, then for every $\varepsilon > 0$

$$\Pr_{G \in \text{ER}(n, n^{-\alpha})} \left[ \text{indsub}(G, H) < n^{|V(H)| - \alpha |E(H)| - \varepsilon} \right] = \exp \left( -n^{\Omega(1)} \right).$$

Here, $\text{indsub}(G, H)$ is the number of induced subgraphs of $G$ isomorphic to $H$. 

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Threshold exponent (cont'd)
Proof of the proposition (1)

Fix $\varepsilon > 0$ and let

$$\delta := \min \left\{ \frac{\varepsilon}{2|E(H)|}, \frac{\text{threshold}(H) - \alpha}{2} \right\} > 0.$$  

We also pick a random restriction

$$\xi : \binom{[n]}{2} \to \{0, 1, \star\}$$

from the distribution

$$\mathcal{R}_{n-(\alpha+\delta), n-\alpha}^{n-(\alpha+\delta), n-\alpha}.$$
Proof (2)

Let \( f = f_n : \{0, 1\}^{\binom{n}{2}} \to \{0, 1\}^{\lceil n^\beta \rceil} \) and

\[
f^1, f^2, \ldots, f^{\lceil n^\beta \rceil} : \{0, 1\}^{\binom{n}{2}} \to \{0, 1\}
\]

be the coordinate-functions of \( f \).

Recall Bootstrapping Lemma,

Let \( f_n : \{0, 1\}^n \to \{0, 1\} \) for \( n \in \mathbb{N} \) be computable by circuits of constant depth \( d \) and size \( O(n^t) \). Then for every \( \alpha, \delta > 0 \) and \( \ell > 0 \), there is a constant \( c = c(d, t, \alpha, \delta, \ell) \) such that

\[
\Pr_{\delta \in \mathcal{R}_n^{n-(\alpha+\delta), n-\alpha}} \left[ |S(f_n |_{\rho})| > c \right] = O\left(\frac{1}{n^\ell}\right).
\]

By taking \( \ell := |V(H)| \) there is a constant \( c \) such that

\[
\Pr_{\xi \in \mathcal{R}_n^{n-(\alpha+\delta), n-\alpha}} \left[ |S(f^j |_{\xi})| > c \right] = O\left(\frac{1}{n^\ell}\right)
\]

for all \( j \in \lceil n^\beta \rceil \).
Consider the event

\[ \sqrt{1} := |\nabla (f \mid \xi)| \leq 2c \lfloor n^\beta \rfloor. \]

We deduce

\[
\Pr \left[ \neg \sqrt{1} \right] \leq \Pr \left[ |\mathcal{S}(f \mid \xi)| > c \lfloor n^\beta \rfloor \right] \quad \text{(by } |\nabla (f \mid \xi)| \leq 2|\mathcal{S}(f \mid \xi)|) \\
\leq \Pr \left[ \bigvee_{j \in \lfloor \lfloor n^\beta \rfloor \rfloor} |\mathcal{S}(f_j \mid \xi)| > c \right] \quad \text{(by } \mathcal{S}(f \mid \xi) = \bigcup_{j \in \lfloor n^\beta \rfloor} \mathcal{S}(f_j \mid \xi)) \\
\leq \lfloor n^\beta \rfloor \cdot O(1/n^{|\nabla(H)|}) = O(1/n^{(1-\beta)|\nabla(H)|}).
\]
Let $G_\xi$ be the graph with

$$V(G_\xi) := [n],$$

$$E(G_\xi) := \xi^{-1}(\ast).$$

$G_\xi$ is a random graph with distribution $\text{ER}(n, n^{-(\alpha+\delta)})$. 
Let
\[ \sqrt{2} := \text{inds}_{G^{\xi}}(H) \geq n^{|V(H)| - \alpha |E(H)| - \varepsilon}. \]

Observe
\[ \alpha + \delta < \text{threshold}(H) \quad \text{and} \quad \delta |E(H)| \leq \varepsilon / 2 \]

by
\[ \delta := \min \left\{ \frac{\varepsilon}{2|E(H)|}, \frac{\text{threshold}(H) - \alpha}{2} \right\}. \]
Proof (6)

\[
\Pr[-\sqrt{2}] = \Pr \left[ \text{inds}ub(G_{\xi}, H) < n |V(H)| - \alpha |E(H)| - \varepsilon \right]
\]

\[
= \Pr_{G \in \mathcal{ER}(n, n^{-\delta})} \left[ \text{inds}ub(G_{\xi}, H) < n |V(H)| - \alpha |E(H)| - \varepsilon \right]
\]

\[
\leq \Pr_{G \in \mathcal{ER}(n, n^{-\delta})} \left[ \text{inds}ub(G_{\xi}, H) < n |V(H)| - (\alpha + \delta) |E(H)| - \varepsilon / 2 \right]
\]

\[
= \exp \left( - n^{\Omega(1)} \right)
\]

by the lemma of Janson, Luczak, and Rucinski for \( \alpha + \delta < \text{threshold}(H) \) and \( \delta |E(H)| \leq \varepsilon / 2 \).
Thus we know

\[ \Pr \left[ \sqrt{1} \text{ and } \sqrt{2} \text{ both hold} \right] \geq 1 - O \left( \frac{1}{n^{(1-\beta)|V(H)|}} \right) - \exp \left( - n^{\Omega(1)} \right). \]

Now assume it’s the case, and we proceed to pick a random graph \( H' \) and a random graph-restriction \( \rho \) as follows.

1. Choose \( H' \) uniformly at random from among the induced subgraphs of \( G_\xi \) isomorphic to \( H \), for \( \sqrt{2} \to \text{indsub}(G_\xi, H) > 0 \).
2. Pick a random graph-restriction \( \rho : \binom{[n]}{2} \to \{0, 1, \ast\} \) as
   2.1 \( \rho^{-1}(\ast) = E(H') \) (with probability 1);
   2.2 \( \rho(e) = \xi(e) \) (with probability 1) for all \( e \in \xi^{-1}(\{0, 1\}) \);
   2.3 \( \Pr [\rho(e) = 1] = n^{-\alpha} \) independently for all \( e \in \xi^{-1}(\ast) \setminus E(H') \).
We observe:

(i) Conditioned on \( \sqrt{1} \) and \( \sqrt{2} \) (so that \( \rho \) is well-defined) \( \rho \) has distribution \( \mathcal{G}_{n}^{\alpha-\alpha}(H) \).

(ii) \( S(f | \rho) \subseteq S(f | \xi) \) and \( V(f | \rho) \subseteq V(f | \xi) \), since \( \rho \) refines \( \xi \), i.e., \( \xi(e) \in \{0, 1\} \rightarrow \rho(e) = \xi(e) \).

(iii) \( \sqrt{1} \) implies that \( V(f | \xi) \) contains at most \( (2c[|V(H)|]) \) subsets of size \( |V(H)| \).

(iv) \( \sqrt{2} \) implies that no matter what \( \xi \) is, there are at least \( n|V(H)|-\alpha|E(H)|-\varepsilon \) equally likely possibility for the random graph \( H' \).
Combining these observations

\[
\Pr \left[ \left| \mathbb{V}(f \upharpoonright \rho) \right| = \left| \mathbb{V}(H) \right| \right] \geq 1 \wedge \sqrt{2}
\]

\[
= \Pr \left[ \mathbb{V}(H') = \mathbb{V}(f \upharpoonright \rho) \right] \geq 1 \wedge \sqrt{2}
\]

\[
\leq \Pr \left[ \mathbb{V}(H') \subseteq \mathbb{V}(f \upharpoonright \xi) \right] \geq 1 \wedge \sqrt{2}
\]

(by (ii))

\[
\leq \frac{(2c \lceil n^\beta \rceil)}{n^{\left| V(H) \right| - \alpha \left| E(H) \right| - \varepsilon}} \left( \frac{n^\beta \left| V(H) \right| + o(1)}{n^{\left| V(H) \right| - \alpha \left| E(H) \right| - \varepsilon}} \right)
\]

(by (iii) and (iv))

\[
\leq \frac{n^\alpha \left| E(H) \right| + (\beta - 1) \left| V(H) \right| + \varepsilon + o(1)}{n^{\left| E(H) \right| + (\beta - 1) \left| V(H) \right| + \varepsilon + o(1)}}
\]
Finally we get

\[
\Pr \left[ |\nabla (f \upharpoonright \rho)| = |\nabla (H)| \right] \\
\leq \Pr \left[ |\nabla (f \upharpoonright \rho)| = |\nabla (H)| \bigg| \sqrt{1} \wedge \sqrt{2} \right] + \Pr[\neg \sqrt{1}] + \Pr[\neg \sqrt{2}] \\
\leq n^{\alpha |E(H)| + (\beta - 1)|V(H)| + \epsilon + o(1)} + O(1/ n^{(1-\beta)|V(H)|}) + \exp \left( - n^{\Omega(1)} \right) \\
= n^{\alpha |E(H)| + (\beta - 1)|V(H)| + \epsilon + o(1)}
\]
Recall

**Lemma**

Let $\mathcal{F} = (f_n)_{n \in \mathbb{N}}$ be an $\text{AC}^0$-computable sequence of functions

$$f_n : \{0, 1\}^{n \choose 2} \rightarrow \{0, 1\}^\lceil n^\beta \rceil$$

for some constant $\beta > 0$. Then for a random graph $G \in \text{ER}(n, n^{-\alpha})$ and a uniform random set $A \in \binom{[n]}{r}$ we have

$$\Pr\left[T_{f_n, G}^n(A) = A\right] \leq n^{\alpha \binom{r}{2} + (\beta - 1)r + o(1)}.$$
We can assume that

$$\alpha < 2/(r - 1) = \text{threshold}(K_r).$$

Let $f_n : \{0, 1\}^{\binom{n}{2}} \to \{0, 1\}^{\lfloor n^\beta \rfloor}$ be AC$^0$-computable, $G \in \text{ER}(n, n^{-\alpha})$, and a set $A \in \binom{[n]}{r}$ chosen uniformly at random.

Consider a graph-restriction $\rho_A^G : \binom{[n]}{2} \to \{0, 1, \star\}$ defined by

$$\rho_A^G(e) := \begin{cases} 
* & \text{if } e \in \binom{A}{2}, \\
1 & \text{if } e \in E(G) \setminus \binom{A}{2}, \\
0 & \text{otherwise.}
\end{cases}$$

We observe that $\rho_A^G$ has the distribution $\mathcal{GR}_n^{n^{-\alpha}}(K_r)$. 
It is easy to see

\[ T_{f,G}^f(A) \subseteq V(f \upharpoonright_{\rho_A^G}) \subseteq A. \]

Thus

\[ T_{f,G}^f(A) = A \implies V(f \upharpoonright_{\rho_A^G}) = r. \]

We obtain

\[
\Pr_{G,A}\left[T_{f,G}^f(A) = A\right] \leq \Pr_{G,A}\left[V(f \upharpoonright_{\rho_A^G}) = r\right] \\
= \Pr_{\rho_A^G \in \mathcal{B}_n^{\mathcal{L}^n_{\alpha}(K_r)}}\left[V(f \upharpoonright_{\rho_A^G}) = r\right] \\
\leq n^{\alpha|E(K_r)| + (\beta - 1)|V(K_r)| + o(1)} \\
= n^{\alpha\binom{r}{2} + (\beta - 1)r + o(1)}.
\]

□