Advanced Algorithms (III)

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Review
Partial $k$-Trees
**Definition**

Let $k \in \mathbb{N}$. Then the set of $k$-trees is defined as follows.

(K1) A complete graph $K_{k+1}$ is a $k$-tree.

(K2) Let $G$ be a graph and $v \in V$ such that

- $N^G[v]$ is isomorphic to $K_{k+1}$, where $N^G[v]$ is the induced subgraph of $G$ on

$$N^G[v] := \{ u \in V(G) \mid \{u, v\} \in E(G) \} \cup \{v\}$$

- $G[V(G) \setminus \{v\}]$ is a $k$-tree.

Then $G$ is a $k$-tree.

**Definition**

A graph is a **partial** $k$-tree if it is a subgraph of a $k$-tree.
Theorem
A graph $G$ is a partial $k$-tree if and only if $\text{tw}(G) \leq k$.

Lemma
Let $G$ be a subgraph of $H$, i.e., $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$. Then $\text{tw}(G) \leq \text{tw}(H)$.

Theorem
1. Every graph of treewidth $\leq k$ is a partial $k$-tree.
2. Every $k$-tree has a tree decomposition of width $\leq k$. 
Algebraic Construction
Definition
Let $k \in \mathbb{N}$. A $k$-terminal graph $(G, v_1, \ldots, v_k)$ is a pair of a graph and a tuple of its $k$ pairwise distinct vertices, called terminals.

Ordinary graphs are obtained as 0-terminal graphs.
The algebra $\mathbf{B}$

**Definition**

For every $k \in \mathbb{N}$, we use $\mathbf{B}_k$ to denote a set of terms defined recursively as follows.

1. $0$ is a term in $\mathbf{B}_0$.
2. $e^2$ is a term in $\mathbf{B}_2$.
3. If $s_1$ and $s_2$ are two $\mathbf{B}_k$-terms, then $s_1 \oplus_k s_2$ is also a $\mathbf{B}_k$-term.
4. Let $i < k$. If $s$ is a $\mathbf{B}_k$-term, then $\sigma_i^k(s)$ is also a $\mathbf{B}_k$-term.
5. Let $i \leq k$. If $s$ is a $\mathbf{B}_{k-1}$-term, then $\ell_i^k(s)$ is a $\mathbf{B}_k$-term.
6. If $s$ is a $\mathbf{B}_k$-term, then $r_k(s)$ is a $\mathbf{B}_{k-1}$-term.

Let

$$\mathbf{B} := \bigcup_{k \geq 0} \mathbf{B}_k.$$
The mapping $\psi$

**Definition**

$\psi$ maps every $B_k$-term to a $k$-terminal graph in the following way:

1. $\psi(e^2)$ is an edge with two terminals.
2. $\psi(0)$ is the empty graph.
3. $\psi(s_1 \oplus_k s_2)$ is a parallel composition, i.e., fuse each $i$-th terminal in $\psi(s_1)$ and $\psi(s_2)$ for every $i \in [k]$.
4. $\psi(\sigma^i_k(s))$ is a permutation, i.e., permute the $i$-th terminal and the $i+1$-th terminal in $\psi(s)$.
5. $\psi(\ell^i_k(s))$ is a lifting, i.e., insert a new isolated terminal (as a new vertex) to $\psi(s)$ at the $i$-th position in $k-1$ terminals.
6. $r_k(s)$ removes the last terminal from $\psi(s)$. 
**Definition**

Let $k \geq 0$ and $s$ be a $B$-term. We say that $s$ is a hereditary $B_{\leq k}$-term, if every subterm of $s$ (including $s$ itself) is a $B_{k'}$-term for some $k' \leq k$.

**Theorem (Ogawa, 2004)**

Let $k \geq 0$. Then a graph $G$ is isomorphic to the graph $\psi(s)$ for a hereditary $B_{\leq k+1}$-term (neglecting terminals) if and only if $tw(G) \leq k$. 

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**Hereditary $B_{\leq k+1}$-terms and treewidth $\leq k$**
Matchings in Bipartite Graphs
Two edges $e \neq f$ in a graph $G$ are adjacent if they have a vertex in common. Otherwise, they are independent.

**Definition**
A set $M$ of pairwise independent edges in a graph $G = (V, E)$ is a matching. $M$ is a matching of $U \subseteq V$ if every vertex in $U$ is incident with an edge in $M$. The vertices in $U$ are then called matched (by $M$); vertices not incident with any edge of $M$ are unmatched.
Definition
A matching $M$ in a graph $G = (V, E)$ is perfect if every vertex is matched by $M$. Or equivalently, $|M| = |V|/2$. 
Augmenting paths

We fix a bipartite graph $G = (V, E)$ with bipartition $A|B$. That is, $V = A \cup B$, $A \cap B = \emptyset$, and every edge in $G$ has one vertex in $A$ and one in $B$.

Let $M$ be a matching in $G$. A path in $G$ which starts in $A$ at an unmatched vertex and then contains, alternately, edges from $E \setminus M$ and from $M$, is an alternating path with respect to $M$.

An alternating path that ends in an unmatched vertex of $B$ is an augmenting path.

Lemma

Let $M$ be a matching in a graph $G$. If there is an augmenting path with respect to $M$, then we have a matching $M'$ in $G$ with $|M'| > |M|$. 
**Definition**

Let $G = (V, E)$ be a graph. Then a set $S \subseteq V$ a *vertex cover* of $E$ if every edge of $G$ is incident with a vertex in $S$. 
König’s Theorem

Theorem (König, 1931)

The maximum cardinality of a matching in $G$ is equal to the minimum cardinality of a vertex cover of its edges.

Proof.
Let $M$ be a matching of maximum cardinality.

Define a set $S$ in the following way: For every $\{a, b\} \in M$ with $a \in A$, if there is an alternating path ending in $b$, then $b \in S$; otherwise $a \in S$. 

$\square$
Hall’s Theorem

Theorem (Hall, 1935)

Let $G = (V, E)$ be a bipartite graph with partition $A \mid B$. Then $G$ contains a matching of $A$ if and only if

$$|N^G(S)| \geq |S|$$

for all $S \subseteq A$, where

$$N^G(S) := \{ v \in B \mid \text{for some } u \in S \text{ we have } \{u, v\} \in E \}. $$
Induction on $|A|$.

Trivial for $|A| = 1$.

If $|N(S)| \geq |S| + 1$ for every non-empty set $S \subsetneq A$, we pick an edge $\{a, b\} \in E$ and apply induction hypothesis on the graph $G' := G - \{a, b\}$.

Assume $A$ has a nonempty proper subset $A'$ with $|B'| = |A'|$ for $B' := N(A)$. We apply the induction hypothesis on $G[A' \cup B']$ and $G - (A' \cup B')$. \qed
Let $M$ be a matching that leaves a vertex in $A$ unmatched. We will construct an augmenting path with respect to $M$.

Let $a_0, b_1, a_1, b_2, a_2, \ldots$ be a maximal sequence of distinct vertices $a_i \in A$ and $b_i \in B$ satisfying the following conditions for all $i \geq 1$:

1. $a_0$ is unmatched;
2. $b_i$ is adjacent to some vertex $a_{f(i)} \in \{a_0, \ldots, a_{i-1}\}$;
3. $\{a_i, b_i\} \in M$.

The sequence will end in some vertex $b_k \in B$.

Consider

$$P := b_k a_{f(k)} b_{f(k)} a_{f^2(k)} b_{f^2(k)} a_{f^3(k)} \ldots, a_{f^r(k)}$$

with $f^r(k) = 0$ is an alternating path.

It is easy to see that $P$ is an augmenting path. \qed
Graph Isomorphism Problems
Graph isomorphism

**Definition**

Let $\mathcal{G}$ and $\mathcal{H}$ be two graphs. A function $f : V(\mathcal{G}) \rightarrow V(\mathcal{H})$ is an isomorphism if

1. (GI1) $f$ is a bijection;
2. (GI2) for every $u, v \in V(\mathcal{G})$ we have $\{u, v\} \in E(\mathcal{G})$ if and only if $\{f(u), f(v)\} \in E(\mathcal{H})$.

If such an $f$ exists, then $\mathcal{G}$ and $\mathcal{H}$ are isomorphic.
Graph Isomorphism (GI) problem

<table>
<thead>
<tr>
<th>GI</th>
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<tbody>
<tr>
<td><strong>Input:</strong> Two graphs ( G ) and ( H ).</td>
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<tr>
<td><strong>Problem:</strong> Decides whether ( G ) and ( H ) are isomorphic.</td>
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**Remark**

1. GI is in \( \text{NP} \).
2. GI is not \( \text{NP} \)-complete, unless Polynomial Hierarchy collapses (which most people do not believe).
3. We don’t know whether GI is \( \text{P} \)-hard.
4. Some people believe GI is in \( \text{P} \), but we don’t even have a quantum polynomial time algorithm.
Graph isomorphism problems and treewidth
Theorem (Bodlaender, 1990)

Let $k \in \mathbb{N}$. Then there is a polynomial time algorithm which decides GI on graphs $G$ with $\text{tw}(G) \leq k$. 

I will present an algorithm deciding the problem

<table>
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<tr>
<th>Input:</th>
<th>Two graphs $G$ and $H$ and a smooth tree decomposition of $G$ of width $k$.</th>
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<tbody>
<tr>
<td>Problem:</td>
<td>Decides whether $G$ and $H$ are isomorphic.</td>
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in time

$$O\left((|V(G)| + |V(H)|)^O(k)\right).$$
Let $\mathcal{C}_G$ be the set of connected components of $G$ and $\mathcal{C}_H$ the set of connected components of $H$.

Then $G$ and $H$ are isomorphic if and only if there is a bijection $h : \mathcal{C}_G \rightarrow \mathcal{C}_H$ such that $G[C]$ and $H[h(C)]$ are isomorphic for every $C \in \mathcal{C}_G$.

This is equivalent to that there is a perfect matching in the following bipartite graph.

1. The left part is $\mathcal{C}_G$ and the right part $\mathcal{C}_H$.
2. There is an edge between a $C \in \mathcal{C}_G$ and a $C' \in \mathcal{C}_H$ if $G[C]$ and $H[C']$ are isomorphic.
Let $S \subseteq V(\mathcal{G})$ and

$$\mathcal{G}\backslash S := \{C \mid C \text{ a connected component of } \mathcal{G}\backslash S\}.$$ 

Then $\mathcal{G}$ and $\mathcal{H}$ are isomorphic if and only if there is a set $S' \subseteq V(\mathcal{H})$, a function $h : \mathcal{G}\backslash S \to \mathcal{H}\backslash S'$ and functions $f_C : S \cup C \to S' \cup h(C)$ for all $C \in \mathcal{G}\backslash S$ such that

1. $|S| = |S'|$;
2. $h$ is a bijection;
3. $f_C$ is an isomorphism between $\mathcal{G}[S \cup C]$ and $\mathcal{H}[S' \cup h(C)]$ for every $C \in \mathcal{G}\backslash S$, and $f_C(S) = S'$;
4. $f_{C_1} \upharpoonright S = f_{C_2} \upharpoonright S$ for every $C_1, C_2 \in \mathcal{G}\backslash S$. 

Let \((\mathcal{T}, (B_t)_{t \in V(\mathcal{T})})\) be a smooth tree decomposition of width \(k\) for the graph \(G\). Again we choose an arbitrary root \(r\) in \(\mathcal{T}\).

For every \(t \in V(\mathcal{T})\) we define

\[\mathcal{C}_t := \{ C \mid C = \emptyset \text{ or } C \text{ a connected component of } G_{\leq t} \setminus B_t \}.\]
Lemma
Every nonempty $C \in \mathcal{C}_t$, i.e., a connected component in $G_{\leq t} \setminus B_t$, is a connected component of $G \setminus B_t$.

Proof.
Clearly there is a connected component $C'$ in $G \setminus B_t$ with $C \subseteq C'$.

Assume that $C' \setminus C \neq \emptyset$. Then there is an edge $\{u, v\} \in E(G)$ with $u \in V(G_{\leq t}) \setminus B_t$ and $v \in V(G) \setminus V(G_{\leq t})$.

But then, $\{u, v\}$ is not contained in any bag of the tree decomposition.
Lemma

Let $t_1$ be a child of $t$. Then for every nonempty $C_1 \in \mathcal{C}_{t_1}$ there is a unique $C \in \mathcal{C}_t$ with $C_1 \subseteq C$, and $C_1 \cap C' = \emptyset$ for all other $C' \in \mathcal{C}_t$.

Proof.

Let $C_1$ be a connected component of $\mathcal{G}_{\leq t_1} \setminus B_{t_1}$.

Observe that

$$\mathcal{G}_{\leq t_1} \setminus B_{t_1} \subseteq \mathcal{G}_{\leq t} \setminus B_t,$$

so $C_1$ is connected in $\mathcal{G}_{\leq t} \setminus B_t$, and the result follows.
Lemma

Let $t$ be a node in $T$ with children $t_1, \ldots, t_n$. And let $C \in \mathcal{C}_t$ be nonempty. Then, there is a unique $i \in [m]$ such that

$$C \subseteq \bigcup \mathcal{C}_{t_i} \cup \{v\} \quad \text{where} \quad \{v\} = B_{t_i} \setminus B_t.$$ 

Intuitively, $C$ is shattered, i.e., broken into several smaller connected components, by the bag of exactly one child of $t$. 
Lemma

Let $t_1, t_2$ be two distinct children of $t$. For every $i \in [2]$, let $v_i$ be the vertex in $G$ with $\{v_i\} = B_{t_i} \setminus B_t$; and $C_i \in \mathcal{C}_{t_i}$. Then for every $C \in \mathcal{C}_t$:

$$(C_1 \cup \{v_1\}) \cap C = \emptyset \quad \text{or} \quad (C_2 \cup \{v_2\}) \cap C = \emptyset.$$
Proof.
It is easy to see
\[(C_1 \cup \{v_1\}) \cap (C_2 \cup \{v_2\}) = \emptyset.\]
Assume \((C_1 \cup \{v_1\}) \cap C \neq \emptyset \neq (C_2 \cup \{v_2\}) \cap C\). Then there is a path \(P\) from \(C_1 \cup \{v_1\}\) to \(C_2 \cup \{v_2\}\) in \(C\). Without loss of generality, we can assume that all vertices on \(P\) are in
\[(C_1 \cup \{v_1\}) \cup (C_2 \cup \{v_2\}).\]
Then there is an edge between \(C_1 \cup \{v_1\}\) and \(C_2 \cup \{v_2\}\), which cannot be contained in any bag of the tree decomposition. \(\square\)
Let $\mathcal{H}$ be a second graph for which we want to decide whether $\mathcal{G}$ and $\mathcal{H}$ are isomorphic.

We define (the set of pairs of separators and connected components)

$$\mathcal{SC}(\mathcal{H}) := \{(S, C) \mid S \subseteq V(\mathcal{H}) \text{ with } |S| = k + 1$$

$$\text{and } (C = \emptyset \text{ or } C \text{ a connected component of } \mathcal{H} \setminus S)\}$$
Definition
Let \( t \in V(T) \), \( S_1 := B_t \), and \( C_1 \in \mathcal{C}_t \). Moreover, let \( (S_2, C_2) \in \mathcal{IE}(H) \). We say \((S_1, C_1)\) and \((S_2, C_2)\) are \(f\)-isomorphic for a function \( f : S_1 \to S_2 \), denoted by \((S_1, C_1) \equiv^f (S_2, C_2)\), if there is a function \( F : S_1 \cup C_1 \to S_2 \cup C_2 \) such that

1. \( F \upharpoonright S_1 = f \);
2. for every \( u, v \in S_1 \cup C_1 \) we have \( \{u, v\} \in E(G) \) if and only if \( \{F(u), F(v)\} \in E(H) \).

That is, \( F \) is an isomorphism between \( G[S_1 \cup C_1] \) and \( H[S_2 \cup C_2] \) which extends \( f \).
Our goal is to compute for each $t \in V(T)$ the set

$\mathcal{F}_t := \{(f, B_t, C_1, S_2, C_2) \mid (B_t, C_1) \equiv^f (S_2, C_2) \}

where $C_1 \in \mathcal{C}_t$ and $(S_2, C_2) \in \mathcal{H}(\mathcal{H})$. 

using dynamic programming.
Leaves

Let \( t \) be a leaf of \( \mathcal{T} \).
Then \( \mathcal{C}_t = \{ \emptyset \} \). Hence,
\[
\mathcal{F}_t := \{ (f, B_t, \emptyset, S_2, \emptyset) \mid (B_t, \emptyset) \equiv^f (S, \emptyset) \\
\text{ where } S_2 \subseteq V(\mathcal{H}) \text{ with } |S_2| = k + 1 \}.
\]
This can be computed in time
\[
(k + 1)! \cdot |V(\mathcal{H})|^{O(k)}.
\]
Non-leaves (1)

Let $t$ be a node in $T$ with children $t_1, \ldots, t_m$ for some $m \geq 1$.

Now let $C_1 \in C_t$ be nonempty. By Lemma 3, there is a unique $i \in [m]$ such that

$$C_1 \subseteq \bigcup C_{t_i} \cup \{v\} \quad \text{where} \quad \{v\} = B_{t_i} \setminus B_t.$$

For every $(S_2, C_2) \in \mathcal{L}(\mathcal{H})$ and every $f : B_t \to S_2$ with $(B_t, \emptyset) \equiv^f (S_2, \emptyset)$ we want to check whether $(B_t, C_1) \equiv^f (S_2, C_2)$. 
(\(B_t, C_1\)) \equiv^f (S_2, C_2) \text{ if and only if some for } v' \in V(\mathcal{H}) \setminus S_2, u \in S_2, \text{ and }

- S'_2 := S_2 \cup \{v'\} \setminus \{u\},
- C^*_1 := \{C^* \mid C^* \text{ a connected component of } G \setminus B_{t_i} \text{ with } C^* \subseteq C_1\} \text{ and } \quad C^*_2 := \{C^* \mid C^* \text{ a connected component of } \mathcal{H} \setminus S'_2 \text{ with } C^* \subseteq C_2\},
- f' : B_{t_i} \rightarrow S'_2 \text{ defined by }

\[
f'(w) = \begin{cases} 
  v' & \text{if } w = v \\
  f(w) & \text{otherwise},
\end{cases}
\]

we have

(N1) every connected component of \(\mathcal{H} \setminus S'_2\) is either contained in or disjoint with \(C_2\);
(N2) \(C_2 \subseteq \bigcup C^*_2 \cup \{v'\}\);
(N3) there is a bijection \(h : C^*_1 \rightarrow C^*_2\) such that for every \(C^* \in C^*_1\)

\[
(\mathcal{B}_{t_i}, C^*) \equiv^f (S'_2, h(C^*)).
\]
(N1) and (N2) can be checked in polynomial time.

To verify (N3) we create a bipartite graph $\mathcal{B}$:

1. the left part is $\mathcal{C}_1^*$ and the right part $\mathcal{C}_2^*$;
2. there is an edge between $C_1^* \in \mathcal{C}_1^*$ and $C_2^* \in \mathcal{C}_2^*$ if $(B_{t_i}, C_1^*) \equiv f'(S_2', C_2^*)$.

Then (N3) holds if and only if there is a perfect matching in $\mathcal{B}$, which can be decided in polynomial time.
The final step

$G$ and $H$ are isomorphic if and only if for some $S_2 \subseteq V(H)$ with $|S_2| = k + 1$ and $f : B_r \rightarrow S_2$ there is a perfect matching in the following bipartite graph.

1. The left part is $C_r$ and the right part $C^* := \{ C_2 \mid (S_2, C_2) \in \mathcal{L}(H) \}$.
2. There is an edge between a $C_1 \in C_r$ and a $C_2 \in C^*$ if $(B_r, C_1) \equiv^f (S_2, C^*)$. 