# Finite Injury Priority Method

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# Post's Problem

**Post's Problem**: is there is a r.e. degree $\mathbf{a}$ such that $\mathbf{0} < \mathbf{a} < \mathbf{0}'$.

It was solved by Friedberg[1957] and Muchnik[1956] independently.

The method used to solve the problem was named finite injury priority method.

# Finite Injury Priority Method

Recursively enumerate a set $A = \bigcup_s A_s$ to meet certain requirements $\{R_n\}_{n\in\omega}$.

1. If $n < m$, then $R_n$ is given priority over $R_m$.
2. Actions take for $R_m$ at some stage $s$ may at a later stage $t > s$ be undone when action is taken for for $R_n$, $n < m$. (Injury)
3. Finite Injury Property. Each requirement is injured at most finitely often.

# Overview

1. Low Simple Set

   Construct a simple set $A$ s.t. $A' \equiv_T \emptyset'$

2. The Original Friedberg-Muchnik Theorem

   Construct two r.e. set $A$ and $B$ such that $A|_T B$

3. Sacks Splitting Theorems

   Every nonrecursive r.e set can be split as a disjoint union two incomparable r.e. subsets.

# Notation

- $\{e\}_s^A(x) = y$ if $x, y, e < s$, $s > 0$, $\phi_e^A(x) = y$ in $< s$ steps and only numbers $z < s$ are used the computation.
- The use function $u(A; e, x, s)$ is $1+$ the maximum number used the computation if $\{e\}_s^A(y) \downarrow$ and $0$ otherwise.

## Lemma

*Let $\{A_s\}_{s\in\omega}$ be an enumeration such that $A = \bigcup_s A_s$, then*

- *If $\{e\}^A(x) = y$, then $\exists s\ \forall t \geq s\ \{e\}_t^{A_t}(x) = y$.*
- *$\{e\}_s^{A_s}(x) = y$ and let $r = u(A_s, e, x, s)$. If $A_s \restriction r = A \restriction r$ then $\{e\}^A(x) = y$.*

# Low Simple Set

### Theorem

*There is a simple set $A$ such that $A' \equiv_T \emptyset'$.*

**Remark**. The jump operator is not 1:1.

### Corollary (Friedberg-Muchnik)

*There is a nonrecursive incomplete r.e. degree $\mathbf{a}$ (i.e., $\mathbf{0} < \mathbf{a} < \mathbf{0}'$).*

# Requirements

It's sufficient to recursively enumerate a coinfinite r.e. set $A = \bigcup_s A_s$ to meet for all $e$ the requirements:

**(simplicity)** $P_e$: $W_e$ is infinite $\Longrightarrow W_e \cap A \neq \emptyset$.

**(lowness)** $N_e$: $(\exists^\infty s)\ [\{e\}_e^{A_s}(e)\downarrow] \Longrightarrow \{e\}^A(e)\downarrow$.

**Fact**. $\{N_e\}_{e\in\omega}$ implies $A' \leq_T \emptyset'$

$$g(e,s) = \begin{cases} 1 & \text{if } \{e\}_s^{A_s}(e)\downarrow; \\ 0 & \text{otherwise.} \end{cases}$$

$\{N_e\}_{e\in\omega}$ implies $\hat{g}(e) = \lim_s g(e,s)$ exists for all $e$. But $\hat{g} \leq_T \emptyset'$, and hence $A' \leq_T \emptyset'$.

# Strategy to Meet $N_e$

$$N_e : (\exists^\infty s)\ [\{e\}_e^{A_s}(e)\downarrow] \Longrightarrow \{e\}^A(e)\downarrow$$

Given $A_s$ define for all $e$

(restraint function) $\qquad r(e,s) = u(A_s, e, e, s).$

**Observation**. If $\{e\}_s^{A_s}(e)\downarrow$ and $N_e$ succeeds in preventing any $x \leq r(e,s)$ from entering $A$, then $\{e\}^A(e)\downarrow$.

**Strategy**. Restrain with priority $N_e$ any elements $x \leq r(e,s)$ from entering $A_{s+1}$.

Such elements can only enter $A$ for the sake of some $P_i$ with stronger priority.

$$N_0 > P_0 > N_1 > P_1 > N_2 > P_2 > \dots$$

# Construction

Stage $s = 0$. Let $A_0 = \emptyset$.
Stage $s + 1$. Given $A_s$ we have $r(e, s)$ for all $e$. Choose the least $i \leq s$ such that

$$W_{i,s} \cap A_s = \emptyset; \tag{1}$$

and

$$(\exists x)[x \in W_{i,s} \wedge x > 2i \wedge (\forall e \leq i)[r(e, x) < x]] \tag{2}$$

- If $i$ exists, chooses the least $x$ satisfying (2), let $A_{s+1} = A_s \cup \{x\}$ and say that $P_i$ receives attention. Hence $W_i \cap A_{s+1} \neq \emptyset$ and $P_i$ is satisfied for all stage $> s + 1$.
- If $i$ dose not exists, let $A_{s+1} = A_s$.

Let $A = \bigcup_s A_s$.

# Finite Injury Property

$x$ injury $N_e$ at stage $s+1$ if $x \in A_{s+1} - A_s$ and $x \leq r(e,s)$.
Defined the injury set for $N_e$ as

$$I_e = \{x : (\exists s)[x \in A_{s+1} - A_s \wedge \ x \leq r(e,s)]\}$$

**Lemma (FIP)**

$(\forall e)$ $[I_e$ *is finite* $]$.

**Lemma**

*For every $e$ requirement $N_e$ is meet and $r(e) = \lim_s r(e, s)$ exists.*

**Proof.**

Fix $e$. By FIP, choose $s_e$ such that $N_e$ is not injured at any stage $s > s_e$. If $\{e\}_s^{A_s}(e)$ converges for $s > s_e$ then by induction on $t \geq s$, $r(e,t) = r(e,s)$ and $\{e\}_t^{A_t}(e) = \{e\}_s^{A_s}(e)$ for all $t \geq s$. □

**Lemma**

*For every $i$, requirement $P_i$ is meet.*

# The Original Friedberg-Muchnik Theorem

Recursively enumerate $A$ and $B$ to meet for all $e$ the requirements:

$$R_{2e}: \quad A \neq \{e\}^B,$$
$$R_{2e+1}: \quad B \neq \{e\}^A.$$

**Theorem (Friedberg 1957, Muchnik 1956)**

*There exsit r.e. set $A$ and $B$ such that $A|_T B$, and hence $\emptyset <_T A, B <_T \emptyset'$.*

# Strategy to Meet a Requirement

Attach to $R_{2e}$ a witness $x$ not yet enumerate in $A$ and look for a stage $s+1$ such that

$$e_s^{B_s}(x)\downarrow = 0.$$

If no such stage exists then we do nothing.

If $s+1$ exists, we say $R_{2e}$ receive attention at stage $s+1$.

- Enumerate $x$ into $A_{s+1}$;
- Rest the wall $r(2e, s+1) = u(B_s; e, x, s)$;
- Restrain with priority $R_{2e}$ any numbers $y \leq r(2e, s+1)$ from later entering $B$.

# Construction of $A$ and $B$

Stage $s = 0$.
Set $A_0 = B_0 = \emptyset$, $x_e^0 = \langle 0, e \rangle$ and $r(e, 0) = -1$ for all $e$.

Stage $s + 1$. Requirement $R_{2e}$ requires attention if

$$\{e\}_s^{B_s}(x_{2e}^s) \downarrow = 0 \text{ and } r(2e, s) = -1 \qquad (3)$$

and $R_{2e+1}$ require attention if

$$\{e\}_s^{A_s}(x_{2e+1}^s) \downarrow = 0 \text{ and } r(2e + 1, s) = -1 \qquad (4)$$

# Construction of $A$ and $B$ (cont'd)

Choose the least $i \leq s$ such that $R_i$ requires attention and we say $R_i$ receive attention by doing the follows. Suppose that $i = 2e$.

- Enumerate $x^s_{2e}$ into $A_{s+1}$ and set $x^{s+1}_{2e} = x^s_{2e}$;
- Reset the wall $r(2e, s+1) = u(B_s, e, x^s_{2e}, s)$;
- For $j < 2e$, set $r(j, s+1) = r(j, s)$ and $x^{s+1}_j = x^s_j$;
- For $j > 2e$, set $r(j, s+1) = -1$ ($R_j$ may require attention in the future) and let $x^{s+1}_j$ to be

$$\text{min}\{y \in \omega^j : \quad y \notin A_{s+1} \cup B_{s+1} \wedge y > x^s_j \\ \wedge\, y > \max\{r(k, s+1) : k \leq 2e\}\}$$

If $i$ fails to exist then do nothing.

**Remark**. $R_i$ can be injured at most $2^i - 1$ times.

# Correctness

## Lemma

*For every $i$, $R_i$ receives attention at most finitely often and is eventually satisfied.*

## Proof.

Fix $i$ and assume by induction that the Lemma holds for all $j < i$. Choose $s$ minimal so that no $R_j$, $j < i$ received attention at a stage $t \geq s$.

1. $\forall t \geq s$, $x_i^t = x_i^s = x_i$, and $x_i \notin A_s \cup B_s$.
2. If $R_i$ receive attention at some stage $t \geq s$, then by assumption $R_i$ will not injured thereafter, and hence $R_i$ is satisfied.
3. If $R_i$ never receive attention after stage $s$, then $A(x_i) = 0$ and it is not the case that $\{e\}^D(x_i) \downarrow= 0$. ($D = B$ if $i$ is even and $D = A$ if $i$ is odd.)

□

# Sacks Theorem

Given a nonrecursive r.e. set $C$, recursively enumerate a coinfinite set $A = \bigcup_s A_s$ that meets the requirements

$$N_e: \quad C \neq \{e\}^A;$$
$$P_e: \quad W_e \text{ is finite } \Longrightarrow W_e \cap A \neq \emptyset.$$

## Theorem (Sacks)

*For every nonrecursive r.e. set $C$ there is a simple set $A$ such that $C \not\leq_T A$ ( and hence $\emptyset <_T A <_T \emptyset'$).*

# Strategy to Meet $N_e$

Sacks's method for constructing $A$ to meet $N_e$ of the form $C \neq \{e\}^A$ is to preserve agreement between $C_s$ and $\{e\}_s^{A_s}$ rather than disagreement as in the Friedberg strategy.

Sufficient preservation of agreement will guarantee that if $C = \{e\}^A$ then $C$ is recursive.

# Sacks's Construction

Stage $s = 0$. Set $A_0 = \emptyset$.
Stage $s + 1$. Given $A_s$, defined for all $e$
**length function**:

$$l(e, s) = \max\{x : (\forall y < x)[\{e\}_s^{A_s}(y) \downarrow= C_s(y)]\}$$

**restraint function**:

$$r(e, s) = \max\{u(A_s; e, x, s) : x \leq l(e, s)\}.$$

$P_i$ requires attention at stage $s + 1$ if $i \leq s$, $W_{i,s} \cap A_s = \emptyset$, and

$$(\exists x)[x \in W_{i,s} \wedge x > 2i \wedge (\forall e \leq i)[r(e, s) < x]]$$

We enumerate in $A_{s+1}$ the least such $x$ for each $P_i$ requires attention at $s + 1$.

# Finite Injury Property

**Injury Set**:

$$I_e = \{x : (\exists s)[x \in A_{s+1} - A_s \wedge x \leq r(e, s)]\}$$

## Lemma (FIP)

*$I_e$ is finite for all $e$. (Indeed $|I_e| < e$.)*

Lemma

$(\forall e)[C \neq \{e\}^A]$.

Proof.

Assume that $C = \{e\}^A$. Then $\lim_s l(e,s){=}\infty$. By FIP we can choose $s'$ such that $N_e$ is never injured after stage $s'$. We shall recursively compute $C$ contrary to hypothesis.

To compute $C(p)$ for some $p \in \omega$ find the least $s > s'$ such that $l(e,s) > p$. It follows by induction on $t \geq s$ that

$$(\forall t \geq s)[l(e,t) > p \wedge r(e,t) > \max\{u(A_s; e, x, s) : x \leq p\}]$$

and hence $C(p) = \{e\}^A P = \{e\}_s^{A_s}(p)$. □

### Lemma

$(\forall e)[\lim_s r(e,s)$ *exists and is finite* $]$.

### Proof.

Choose $p = (\mu x)[C(x) \neq \{e\}^A(x)]$. Choose $s'$ sufficient large such that for all $s \geq s'$

1. $N_e$ is not injured at stage $s$;
2. $(\forall x \leq p)[\{e\}_s^{A_s}(x) \downarrow = \{e\}^A(x)]$; and
3. $(\forall x \leq p)[C_s(x) = C(x)]$.

If $\{e\}_t^{A_t}(p) \downarrow$ for some $t \geq s'$, then $\lim_s r(e,s) = r(e,t)$, otherwise $\lim_s r(e,s) = r(e,s')$. □

### Lemma

$(\forall e)[W_e$ *is infinite* $\implies W_e \cap A \neq \emptyset]$.

## Theorem (Sacks 1963)

*Let $B$ and $C$ be r.e. sets such that $C$ is nonrecursive. Then there exist r.e. sets $A_0$ and $A_1$ such that*

1. $A_0 \cup A_1 = B$ *and* $A_0 \cap A_1 = \emptyset$;
2. $C \not\leq_T A_i$ *for* $i = 0, 1$; *and*
3. $A_i \leq_T \emptyset'$ *for* $i = 0, 1$.

## Corollary

*Let $C$ be a nonrecursive r.e. set. Then there exist low r.e. sets $A_0$ and $A_1$ such that $A_0|_T A_1$, $C = A_0 \cup A_1$ and $A_0 \cap A_1 = \emptyset$.*

# Requirements

Let $\{B_s\}_{s\in\omega}$ and $\{C_s\}_{s\in\omega}$ be recursive enumeration of $B$ and $C$ such that $B_0 = \emptyset$ and $|B_{s+1} - B_s| = 1$ for all $s$.

We give recursive enumerations $\{A_{i,s}\}_{s\in\omega}$, $i = 0, 1$ satisfying the requirements

$$P : x \in B_{s+1} - B_s \Longrightarrow [x \in A_{0,s+1} \vee x \in A_{i,s+1}]$$

$$N_{\langle e,i\rangle} : C \neq \{e\}^{A_i} \quad (i = 0, 1)$$

# Sacks's Construction

Stage $s = 0$. Define $A_{i,0} = \emptyset$, $i = 0, 1$.
Stage $s + 1$. Given $i, s$ define the recursive function $l^i(e, s)$ and $r^i(e, s)$ as follows

$$l^i(e,s) = \max\{x : (\forall y < x)[\{e\}_s^{A_s^i}(y) \downarrow = C_s(y)]\}$$

$$r^i(e,s) = \max\{u(A_s^i; e, x, s) : x \leq l(e,s)\}.$$

Let $x \in B_{s+1} - B_s$. Choose the $\langle e', i' \rangle$ to be the least $\langle e, i \rangle$ such that $x \leq r^i(e, s)$, and enumerate $x$ in $A_{1-i',s+1}$.

If $\langle e', i' \rangle$ fails to exist, enumerate $x$ into $A_0$.

# Correctness

**Lemma**

*For all $i = 0, 1$ and all $e$*

- $I_e^i$ *is finite.*
- $C \neq \{e\}^{A_i}$; *and*
- $r^i(e) = \lim_s r^i(e, s)$ *exists and is finite.*

**Lemma**

$A_i \leq_T \emptyset'$ *for* $i = 0, 1$.

**Proof.**

Define a recursive function $g$ as follows

$$g(e)^{A_i}(y) = \begin{cases} C(0) & \text{if } y = 0 \text{ and } \{e\}^{A_i}(e)\downarrow, \\ \uparrow & \text{otherwise.} \end{cases}$$

$$e \in A_i' \iff \{e\}^{A_i}(e)\downarrow \iff \{g(e)\}^{A_i}(0) = C(0) \iff \lim_s l^i(g(e), s) > 0$$

□

# Thanks.

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