Finite Injury Priority Method

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Post’s Problem: is there is a r.e. degree $a$ such that $0 < a < 0'$. It was solved by Friedberg[1957] and Muchnik[1956] independently. The method used to solve the problem was named finite injury priority method.
Recursively enumerate a set \( A = \bigcup_s A_s \) to meet certain requirements \( \{R_n\}_{n \in \omega} \).

1. If \( n < m \), then \( R_n \) is given priority over \( R_m \).
2. Actions take for \( R_m \) at some stage \( s \) may at a later stage \( t > s \) be undone when action is taken for \( R_n \), \( n < m \). (Injury)
3. Finite Injury Property. Each requirement is injured at most finitely often.
1. **Low Simple Set**
   Construct a simple set $A$ s.t. $A' \equiv_T \emptyset'$

2. **The Original Friedberg-Muchnik Theorem**
   Construct two r.e. set $A$ and $B$ such that $A|_TB$

3. **Sacks Splitting Theorems**
   Every nonrecursive r.e set can be split as a disjoint union two incomparable r.e. subsets.
Notation

- $\{e\}^A_s(x) = y$ if $x, y, e < s$, $s > 0$, $\phi^A_e(x) = y$ in $< s$ steps and only numbers $z < s$ are used the computation.
- The use function $u(A; e, x, s)$ is $1+$ the maximum number used the computation if $\{e\}^A_s(y) \downarrow$ and 0 otherwise.

Lemma

Let $\{A_s\}_{s \in \omega}$ be an enumeration such that $A = \bigcup_s A_s$, then

- If $\{e\}^A(x) = y$, then $\exists s \forall t \geq s \{e\}^{A_t}_t(x) = y$.
- $\{e\}^{A_s}_s(x) = y$ and let $r = u(A_s, e, x, s)$. If $A_s \upharpoonright r = A \upharpoonright r$ then $\{e\}^A(x) = y$. 
Theorem
There is a simple set $A$ such that $A' \equiv_T \emptyset'$.

Remark. The jump operator is not 1:1.

Corollary (Friedberg-Muchnik)
There is a nonrecursive incomplete r.e. degree $a$ (i.e., $0 < a < 0'$).
It’s sufficient to recursively enumerate a coinfinite r.e. set $A = \bigcup_s A_s$ to meet for all $e$ the requirements:

**(simplicity)** $P_e$: $W_e$ is infinite $\implies W_e \cap A \neq \emptyset$.

**(lowness)** $N_e$: $(\exists \infty s) \{e\}^A_s(e) \downarrow \implies \{e\}^A(e) \downarrow$.

**Fact.** $\{N_e\}_{e \in \omega}$ implies $A' \leq_T \emptyset'$

$$g(e, s) = \begin{cases} 1 & \text{if } \{e\}^A_s(e) \downarrow; \\ 0 & \text{otherwise.} \end{cases}$$

$\{N_e\}_{e \in \omega}$ implies $\hat{g}(e) = \lim_s g(e, s)$ exists for all $e$. But $\hat{g} \leq_T \emptyset'$, and hence $A' \leq_T \emptyset'$.
Strategy to Meet $N_e$

$$N_e : (\exists \infty s) \left[ \{e\}^{A_s}(e) \downarrow \right] \implies \{e\}^{A}(e) \downarrow$$

Given $A_s$ define for all $e$

(restraint function) \hspace{1cm} r(e, s) = u(A_s, e, e, s).

**Observation.** If $\{e\}^{A_s}(e) \downarrow$ and $N_e$ succeeds in preventing any $x \leq r(e, s)$ from entering $A$, then $\{e\}^{A}(e) \downarrow$.

**Strategy.** Restrain with priority $N_e$ any elements $x \leq r(e, s)$ from entering $A_{s+1}$.

Such elements can only enter $A$ for the sake of some $P_i$ with stronger priority.

$$N_0 > P_0 > N_1 > P_1 > N_2 > P_2 > \ldots$$
Stage $s = 0$. Let $A_0 = \emptyset$.

Stage $s + 1$. Given $A_s$ we have $r(e, s)$ for all $e$. Choose the least $i \leq s$ such that

$$W_{i,s} \cap A_s = \emptyset;$$  \hspace{1cm} (1)

and

$$(\exists x)[x \in W_{i,s} \land x > 2i \land (\forall e \leq i)[r(e, x) < x]]$$  \hspace{1cm} (2)

- If $i$ exists, chooses the least $x$ satisfying (2), let $A_{s+1} = A_s \cup \{x\}$ and say that $P_i$ receives attention. Hence $W_i \cap A_{s+1} \neq \emptyset$ and $P_i$ is satisfied for all stage $> s + 1$.

- If $i$ dose not exists, let $A_{s+1} = A_s$.

Let $A = \bigcup_s A_s$. 
Finite Injury Property

$x$ injury $N_e$ at stage $s + 1$ if $x \in A_{s+1} - A_s$ and $x \leq r(e, s)$.

Defined the injury set for $N_e$ as

$$I_e = \{x : (\exists s)[x \in A_{s+1} - A_s \land x \leq r(e, s)]\}$$

**Lemma (FIP)**

$(\forall e)$ [$I_e$ is finite].
Lemma

For every \( e \) requirement \( N_e \) is meet and \( r(e) = \lim_s r(e, s) \) exists.

Proof.
Fix \( e \). By FIP, choose \( s_e \) such that \( N_e \) is not injured at any stage \( s > s_e \). If \( \{e\}^A_s(e) \) converges for \( s > s_e \) then by induction on \( t \geq s \), \( r(e, t) = r(e, s) \) and \( \{e\}_t^A(e) = \{e\}_s^A(e) \) for all \( t \geq s \).

Lemma

For every \( i \), requirement \( P_i \) is meet.
Recursively enumerate $A$ and $B$ to meet for all $e$ the requirements:

\[
R_{2e} : \quad A \neq \{e\}^B,
\]
\[
R_{2e+1} : \quad B \neq \{e\}^A.
\]

**Theorem (Friedberg 1957, Muchnik 1956)**

*There exist r.e. set $A$ and $B$ such that $A|_T B$, and hence $\emptyset <_T A, B <_T \emptyset'$.*
Attach to $R_{2e}$ a witness $x$ not yet enumerate in $A$ and look for a stage $s + 1$ such that

$$e^B_s(x) \downarrow = 0.$$ 

If no such stage exists then we do nothing.

If $s + 1$ exists, we say $R_{2e}$ receive attention at stage $s + 1$.

- Enumerate $x$ into $A_{s+1}$;
- Rest the wall $r(2e, s + 1) = u(B_s; e, x, s)$;
- Restrain with priority $R_{2e}$ any numbers $y \leq r(2e, s + 1)$ from later entering $B$. 
Stage $s = 0$.
Set $A_0 = B_0 = \emptyset$, $x_e^0 = \langle 0, e \rangle$ and $r(e, 0) = -1$ for all $e$.

Stage $s + 1$. Requirement $R_{2e}$ requires attention if

\[ \{e\}^B_s (x_{2e}^s) \downarrow = 0 \text{ and } r(2e, s) = -1 \]  \hspace{1cm} (3)

and $R_{2e+1}$ require attention if

\[ \{e\}^A_s (x_{2e+1}^s) \downarrow = 0 \text{ and } r(2e + 1, s) = -1 \]  \hspace{1cm} (4)
Choose the least $i \leq s$ such that $R_i$ requires attention and we say $R_i$ receive attention by doing the follows. Suppose that $i = 2e$.

- Enumerate $x_{2e}^s$ into $A_{s+1}$ and set $x_{2e}^{s+1} = x_{2e}^s$;
- Reset the wall $r(2e, s + 1) = u(B_s, e, x_{2e}^s, s)$;
- For $j < 2e$, set $r(j, s + 1) = r(j, s)$ and $x_j^{s+1} = x_j^s$;
- For $j > 2e$, set $r(j, s + 1) = -1$ ($R_j$ may require attention in the future) and let $x_j^{s+1}$ to be

$$\min\{y \in \omega^j : y \not\in A_{s+1} \cup B_{s+1} \land y > x_j^s \land y > \max\{r(k, s + 1) : k \leq 2e\}\}$$

If $i$ fails to exist then do nothing.

**Remark.** $R_i$ can be injured at most $2^i - 1$ times.
**Lemma**

*For every* \( i \), \( R_i \) *receives attention at most finitely often and is eventually satisfied.*

**Proof.**

Fix \( i \) and assume by induction that the Lemma holds for all \( j < i \). Choose \( s \) minimal so that no \( R_j \), \( j < i \) received attention at a stage \( t \geq s \).

1. \( \forall t \geq s, \ x^t_i = x^s_i = x_i, \) and \( x_i \notin A_s \cup B_s \).
2. If \( R_i \) receive attention at some stage \( t \geq s \), then by assumption \( R_i \) will not injured thereafter, and hence \( R_i \) is satisfied.
3. If \( R_i \) never receive attention after stage \( s \), then \( A(x_i) = 0 \) and it is not the case that \( \{e\}^D(x_i) \downarrow = 0. \) (\( D = B \) if \( i \) is even and \( D = A \) if \( i \) is odd.)
Sacks Theorem

Given a nonrecursive r.e. set $C$, recursively enumerate a coinfinite set $A = \bigcup_s A_s$ that meets the requirements

$$Ne : \quad C \neq \{e\}^A;$$
$$Pe : \quad W_e \text{ is finite} \implies W_e \cap A \neq \emptyset.$$

**Theorem (Sacks)**

*For every nonrecursive r.e. set $C$ there is a simple set $A$ such that $C \not\leq_T A$ (and hence $\emptyset <_T A <_T \emptyset'$).*
Sacks’s method for constructing $A$ to meet $N_e$ of the form $C \neq \{e\}^A$ is to preserve agreement between $C_s$ and $\{e\}^A_s$ rather than disagreement as in the Friedberg strategy.

Sufficient preservation of agreement will guarantee that if $C = \{e\}^A$ then $C$ is recursive.
Stage $s = 0$. Set $A_0 = \emptyset$.

Stage $s + 1$. Given $A_s$, defined for all $e$

**length function:**

\[ l(e, s) = \max\{x : (\forall y < x)[\{e\}_{s}^{A_s}(y) \downarrow = C_s(y)]\} \]

**restraint function:**

\[ r(e, s) = \max\{u(A_s; e, x, s) : x \leq l(e, s)\}. \]

$P_i$ requires attention at stage $s + 1$ if $i \leq s$, $W_{i,s} \cap A_s = \emptyset$, and

\[ (\exists x)[x \in W_{i,s} \land x > 2i \land (\forall e \leq i)[r(e, s) < x]] \]

We enumerate in $A_{s+1}$ the least such $x$ for each $P_i$ requires attention at $s + 1$. 
Injury Set:

\[ I_e = \{ x : (\exists s)[x \in A_{s+1} - A_s \land x \leq r(e, s)] \} \]

Lemma (FIP)

\( I_e \) is finite for all \( e \). (Indeed \( |I_e| < e \).)
Lemma
$(\forall e)[C \neq \{e\}^A]$.

Proof.
Assume that $C = \{e\}^A$. Then $\lim_s l(e, s) = \infty$. By FIP we can choose $s'$ such that $N_e$ is never injured after stage $s'$. We shall recursively compute $C$ contrary to hypothesis.

To compute $C(p)$ for some $p \in \omega$ find the least $s > s'$ such that $l(e, s) > p$. It follows by induction on $t \geq s$ that

$$(\forall t \geq s)[l(e, t) > p \land r(e, t) > \max\{u(A_s; e, x, s) : x \leq p\}]$$

and hence $C(p) = \{e\}^A P = \{e\}_{A_s}^s (p)$. \qed
Correctness (cont’d)

**Lemma**

\((\forall e)[\lim_s r(e, s) \text{ exists and is finite}]\).

**Proof.**

Choose \(p = (\mu x)[C'(x) \neq \{e\}^A(x)]\). Choose \(s'\) sufficient large such that for all \(s \geq s'\)

1. \(N_e\) is not injured at stage \(s\);
2. \((\forall x \leq p)[\{e\}^A_s(x) \downarrow = \{e\}^A(x)]; \text{ and}\)
3. \((\forall x \leq p)[C_s(x) = C'(x)]\).

If \(\{e\}^A_t(p) \downarrow\) for some \(t \geq s'\), then \(\lim_s r(e, s) = r(e, t)\), otherwise \(\lim_s r(e, s) = r(e, s')\).

**Lemma**

\((\forall e)[W_e \text{ is infinite} \implies W_e \cap A \neq \emptyset]\).
Sacks Splitting Theorem

Theorem (Sacks 1963)

Let $B$ and $C$ be r.e. sets such that $C$ is nonrecursive. Then there exist r.e. sets $A_0$ and $A_1$ such that

1. $A_0 \cup A_1 = B$ and $A_0 \cap A_1 = \emptyset$;
2. $C \not\leq_T A_i$ for $i = 0, 1$; and
3. $A_i \leq_T \emptyset'$ for $i = 0, 1$.

Corollary

Let $C$ be a nonrecursive r.e. set. Then there exist low r.e. sets $A_0$ and $A_1$ such that $A_0|_T A_1$, $C = A_0 \cup A_1$ and $A_0 \cap A_1 = \emptyset$. 
Let $\{B_s\}_{s \in \omega}$ and $\{C_s\}_{s \in \omega}$ be recursive enumeration of $B$ and $C$ such that $B_0 = \emptyset$ and $|B_{s+1} - B_s| = 1$ for all $s$.

We give recursive enumerations $\{A_{i,s}\}_{s \in \omega}$, $i = 0, 1$ satisfying the requirements

\[
P : x \in B_{s+1} - B_s \implies [x \in A_{0,s+1} \lor x \in A_{i,s+1}]
\]

\[
N_{\langle e, i \rangle} : C \neq \{e\}^{A_i} \quad (i = 0, 1)
\]
Stage $s = 0$. Define $A_{i,0} = \emptyset$, $i = 0, 1$.

Stage $s + 1$. Given $i, s$ define the recursive function $l_i^i(e, s)$ and $r_i^i(e, s)$ as follows

$$l_i^i(e, s) = \max\{x : (\forall y < x)[\{e\}_{s}^{A_i^i s}(y) \downarrow = C_s(y)]\}$$

$$r_i^i(e, s) = \max\{u(A_i^i s; e, x, s) : x \leq l(e, s)\}.$$ 

Let $x \in B_{s+1} - B_s$. Choose the $\langle e', i' \rangle$ to be the least $\langle e, i \rangle$ such that $x \leq r_i^i(e, s)$, and enumerate $x$ in $A_{1-i',s+1}$.

If $\langle e', i' \rangle$ fails to exist, enumerate $x$ into $A_0$. 

Sacks’s Construction
Correctness

Lemma
For all $i = 0, 1$ and all $e$

- $I_e^i$ is finite.
- $C \neq \{e\}^{A_i}$; and
- $r^i(e) = \lim_s r^i(e, s)$ exists and is finite.

Lemma
$A_i \leq_T \emptyset'$ for $i = 0, 1$.

Proof.
Define a recursive function $g$ as follows

$$g(e)^{A_i}(y) = \begin{cases} C(0) & \text{if } y = 0 \text{ and } \{e\}^{A_i}(e) \downarrow, \\ \uparrow & \text{otherwise.} \end{cases}$$

$e \in A'_i \iff \{e\}^{A_i}(e) \downarrow \iff \{g(e)\}^{A_i}(0) = C(0) \iff \lim_s l^i(g(e), s) > 0$
Thanks.

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