# Oracle Construction of Non-R.E. Degrees

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- A more basic problem is whether there exist any degrees between $\mathbf{0}$ and $\mathbf{0}'$.

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- A more basic problem is whether there exist any degrees between $\mathbf{0}$ and $\mathbf{0}'$.
  - Positive answer!
  - Kleene and Post, 1954, "The upper semi-lattice of degrees of recursive unsolvability"
  - By constructing Turing incomparable sets $A, B \leq_T \varnothing'$ using diagonalisation method

- We have an infinite sequence $\{R_e\}_{e\in\omega}$ of conditions called requirements to satisfy.
- The desired sets, or their characteristic functions are constructed stage by stage.
- At each stage of the construction we will meet a single requirement, and this requirement will be satisfied from then on.

- $C$-recursive oracle construction
    - In this chapter the construction in each stage will be nonrecursive, requiring some oracle $C$, such as $\varnothing'$.
- Therefore the sets constructed in this chapter are not r.e.

- $C$-recursive oracle construction
  - In this chapter the construction in each stage will be nonrecursive, requiring some oracle $C$, such as $\varnothing'$.
- Therefore the sets constructed in this chapter are not r.e.
- We will extend the constructed function at each stage.
  - finite extension construction: $\mathrm{dom}\ f_s$ is finite for all $s$
  - infinite extension construction: otherwise

# Overview

1 A Pair of Incomparable Degrees Below $\mathbf{0}'$

- finite extension $\varnothing'$-oracle construction

3 Inverting the Jump

- finite extension $B$-recursive construction

2 Avoiding Cones of Degrees

- finite extension $B'$-recursive construction

4 Upper and Lower Bounds for Degrees

- infinite extension $((A^{[<s]})' \oplus A_s)$-recursive construction

# A Pair of Incomparable Degrees Below $\mathbf{0}'$

**Theorem (1.2)**

*There exist degrees* $\mathbf{a}, \mathbf{b} \leq \mathbf{0}'$ *such that* $\mathbf{a}|\mathbf{b}$, *i.e.* $\mathbf{a}$ *is incomparable with* $\mathbf{b}$, $\mathbf{a} \not\leq_T \mathbf{b}$ *and* $\mathbf{b} \not\leq_T \mathbf{a}$.

**Corollary**

*The above* $\mathbf{a}, \mathbf{b}$ *satisfy that* $\mathbf{0} < \mathbf{a}, \mathbf{b} < \mathbf{0}'$.

# Finite Extension $\varnothing'$-oracle Construction

- Construct $f_s$ and $g_s$ at each stage $s$, let $c_A = \bigcup_s f_s$ and $c_B = \bigcup_s g_s$.
- $f_s$ and $g_s$ are finite binary strings viewed as partial functions.
- $f_s \subset f_{s+1}$ and $g_s \subset f_{s+1}$ hence $c_A$ and $c_B$ are total functions.

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- Requirements:

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- As usual, at stage 0 we initiate with $f_0 = g_0 = \varnothing$.

- At stage $s+1=2e+1$, we satisfy $R_e\colon A \neq \phi_e^B$:
  - Let $n = |f_s| = \mu x(x \notin \mathrm{dom}\ f_s)$
  - Using a $\emptyset'$-oracle test whether

$$(\exists t)(\exists \sigma)(g_s \subset \sigma \wedge \phi_{e,t}^{\sigma}(n)\downarrow) \tag{1}$$

  (Note that $g_s \subset \sigma$ is recursive on $\sigma$)
    - True: Choose the least $\langle \sigma, t\rangle$ satisfying the matrix of (1). Set $g_{s+1} = \sigma$ and $f_{s+1} = f_s{}^\frown(1 - \phi_{e,t}^{\sigma}(n))$
    - False: $f_{s+1} = f_s{}^\frown 0$ and $g_{s+1} = g_s{}^\frown 0$

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    - False: $f_{s+1} = f_s\widehat{\ }0$ and $g_{s+1} = g_s\widehat{\ }0$
- At stage $s+1 = 2e+2$, we satisfy $S_e\colon B \neq \phi_e^A$: similarly with the roles of $f_s$ and $g_s$ interchanged.

- At each stage $s$,
  - $|f_s| \geq s$ and $|g_s| \geq s$, hence $c_A$ and $c_B$ are total;
  - in either cases,

$$f_{s+1}(n) \not\simeq \phi_e^{g_{s+1}}(n)$$

since $f_{s+1} \subset c_A$ and $g_{s+1} \subset c_B$,

$$c_A(n) \not\simeq \phi_e^{c_B}(n).$$

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□

- Remark: indeed we don't have to do anything when the test fails, since there are infinitely many stages $s$ with $f_s \subset f_{s+1}$.

# Generalized Results

**Theorem (1.3)**

*For any degree* $\mathbf{c}$, *there are degrees* $\mathbf{a}, \mathbf{b}$ *such that* $\mathbf{c} \leq \mathbf{a}, \mathbf{b} \leq \mathbf{c}'$ *and* $\mathbf{a} | \mathbf{b}$.

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**Definition (1.4)**

A countable sequence of sets $\{A_i\}_{i \in \omega}$ is recursively independent if for each $i$, $A_i \not\leq_T \oplus\{A_j : j \neq i\}$.

**Theorem (1.5)**

*There exists a recursively independent sequence of sets* $\{A_i\}_{i \in \omega}$ *each recursive in* $\varnothing'$*.*

- Any Turing jump is above $\mathbf{0}'$.
- The Turing jump map has range contained in $\{\mathbf{b} \colon \mathbf{b} \geq \mathbf{0}'\}$.
- The following theorem shows that this map is onto.

**Theorem (3.1)**

*For every degree $\mathbf{b} \geq \mathbf{0}'$, there is a degree $\mathbf{a}$ such that $\mathbf{a}' = \mathbf{a} \cup \mathbf{0}' = \mathbf{b}$.*

- Fix $B \in \mathbf{b} \geq \mathbf{0}'$, we must construct $A$ with
$$A' \equiv_T A \oplus \varnothing' \equiv_T B$$
.
- We will construct $c_A = \bigcup_s f_s$ using a finite extension $B$-recursive construction.

# Finite Extension $B$-recursive Construction

- At stage 0, set $f_0 = \varnothing$.
- At stage $s + 1 = 2e + 1$, (we decide whether $e \in A'$) we satisfy the following requirement:

$$R_e : (\exists \sigma \subset c_A)(\phi_e^\sigma(e)\downarrow \vee (\forall \tau \supseteq \sigma, \phi_e^\tau(e)\uparrow)).$$

Using a $\varnothing'$-oracle test whether

$$(\exists t)(\exists \sigma)(f_s \subset \sigma \wedge \phi_{e,t}^\sigma(e)\downarrow) \tag{2}$$

  - True: let $f_{s+1} =$ the least such $\sigma$.
  - False: set $f_{s+1} = f_s$.

# Finite Extension $B$-recursive Construction

- At stage 0, set $f_0 = \varnothing$.
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Using a $\varnothing'$-oracle test whether

$$(\exists t)(\exists \sigma)(f_s \subset \sigma \wedge \phi_{e,t}^\sigma(e)\downarrow) \tag{2}$$

  - True: let $f_{s+1} =$ the least such $\sigma$.
  - False: set $f_{s+1} = f_s$.
- At stage $s+1 = 2e+2$, (we code $B(e)$ into $A$) Let $f_{s+1} = f_s{}^\frown B(e)$.

# Correctness

- $|f_{2e}| \geq e$, hence $c_A$ is total.
- The construction is $B$-recursive:
  - at odd stage we use a $\varnothing'$-oracle,
  - at even stage we use a $B$-oracle.

- $|f_{2e}| \geq e$, hence $c_A$ is total.
- The construction is $B$-recursive:
  - at odd stage we use a $\varnothing'$-oracle,
  - at even stage we use a $B$-oracle.
- We just have to prove the following conditions are met:
  - $A' \leq_T B$:
    - To decide whether $e \in A'$, $B$-recursively compute $f_{2e}$, then $B$-recursively test whether (2) holds.
  - $B \leq_T A \oplus \varnothing'$:
    - To compute $B(e)$, just compute $f_s$ till stage $2e+2$. $f_s$ is $(A \oplus \varnothing')$-recursive for all $s$.

$\square$

**Corollary (3.3)**

*For every $n \geq 1$ and every degree $\mathbf{c}$,*

$$F_n(\mathbf{c})\colon (\forall \mathbf{b})(\mathbf{b} \geq \mathbf{c}^{(n)} \Rightarrow (\exists \mathbf{a})(\mathbf{a} \geq \mathbf{c} \wedge \mathbf{a}^{(n)} = \mathbf{a} \cup \mathbf{c}^{(n)} = \mathbf{b}))$$

**Theorem (3.4)**

*For all degree* $\mathbf{b} \geq \mathbf{0}'$, *there exists degrees* $\mathbf{a}_0, \mathbf{a}_1$ *such that* $\mathbf{a}_0 | \mathbf{a}_1$ *and*

$$\mathbf{a}_0' = \mathbf{a}_0 \cup \mathbf{0}' = \mathbf{b} = \mathbf{a}_1 \cup \mathbf{0}' = \mathbf{a}_1'.$$

# More Results

**Theorem (3.4)**

*For all degree* $\mathbf{b} \geq \mathbf{0}'$, *there exists degrees* $\mathbf{a}_0, \mathbf{a}_1$ *such that* $\mathbf{a}_0 | \mathbf{a}_1$ *and*

$$\mathbf{a}_0' = \mathbf{a}_0 \cup \mathbf{0}' = \mathbf{b} = \mathbf{a}_1 \cup \mathbf{0}' = \mathbf{a}_1'.$$

**Corollary (3.5)**

*There is a degree* $\mathbf{a}$ *such that*

$$\mathbf{0} < \mathbf{a} < \mathbf{0}' \wedge \mathbf{a}' = \mathbf{0}'$$

Turing jump map is not 1 : 1.

**Theorem (2.1)**

*For every degree* $\mathbf{b} > \mathbf{0}$ *there exists a degree* $\mathbf{a} < \mathbf{b}'$ *such that* $\mathbf{a} | \mathbf{b}$.

- $\mathbf{a}$ must avoid the lower cone of degrees $\{\mathbf{d} \colon \mathbf{d} \leq \mathbf{b}\}$ and the upper cone $\{\mathbf{d} \colon \mathbf{d} \geq \mathbf{b}\}$.

**Theorem (2.1)**

*For every degree $\mathbf{b} > \mathbf{0}$ there exists a degree $\mathbf{a} < \mathbf{b}'$ such that $\mathbf{a}|\mathbf{b}$.*

- $\mathbf{a}$ must avoid the lower cone of degrees $\{\mathbf{d} \colon \mathbf{d} \leq \mathbf{b}\}$ and the upper cone $\{\mathbf{d} \colon \mathbf{d} \geq \mathbf{b}\}$.
- We have the same requirements $\{R_e\}_{e\in\omega}$ and $\{S_e\}_{e\in\omega}$ to satisfy as in the first construction.
- The lower cone is easier to avoid than what we've done in the first construction, while the upper cone needs more tricky requirements.

- At stage 0, set $f_0 = \varnothing$.
- At stage $s+1 = 2e+1$, we satisfy $R_e : A \neq \phi_e^B$: let $n = |f_s|$, using a $B'$-oracle test whether $\phi_e^B(n)\downarrow$:
  - True: $f_{s+1} = f_s\widehat{\ }(1 - \phi_e^B(n))$
  - False: $f_{s+1} = f_s\widehat{\ }0$

- At stage 0, set $f_0 = \varnothing$.
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  - True: $f_{s+1} = f_s{}^\frown(1 - \phi_e^B(n))$
  - False: $f_{s+1} = f_s{}^\frown 0$
- At stage $s + 1 = 2e + 2$, we satisfy $S_e : B \neq \phi_e^A$: using a $\varnothing'$-oracle test whether

$$(\exists\sigma,\tau)(\exists x,y,z)(\exists t)(f_s \subset \sigma,\tau \wedge \phi_{e,t}^{\sigma}(x)\downarrow = y \neq z = \phi_{e,t}^{\tau}(x)\downarrow). \tag{3}$$

  - True: then one of $y, z$ must differ from $B(x)$, choose $f_{s+1}$ to be the first $\sigma \supset f_s$ such that $\phi_e^{\sigma}(x)\downarrow \neq B(x)$
  - False: let $f_{s+1} = f_s{}^\frown 0$

- The construction is $B'$-recursive.
- Even if test (3) fails, requirement $S_e$: $B \neq \phi_e^A$ is satisfied:

$$(\forall f \supset f_s)(\phi_e^f = g \text{ is total } \Rightarrow g \text{ is recursive}).$$

  - To recursively compute $g(x)$, find the least $\sigma \supset f_s$ with $\phi_e^\sigma(x)\downarrow$, then $\phi_e^{c_A}(x)$ must be the same value.

Hence for such $f$, $\phi_e^f \neq B$ since $B >_T \phi$.

□

**Theorem (2.2)**

*Let $\{\mathbf{b}_n\}_{n\in\omega}$ be any countable sequence of nonrecursive degrees, then there exist pairwise incomparable degrees $\{\mathbf{a}_n\}_{n\in\omega}$ such that $\mathbf{a}_n|\mathbf{b}_m$ for all $n, m \in \omega$.*

# Upper and Lower Bounds for Degrees

- Any finite set of degrees has a least upper bound.
- We will show this is not true for greatest lower bound.

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**Definition**

A sequence of degrees $\{\mathbf{a}_n\}_{n\in\omega}$ is ascending if $\mathbf{a}_n < \mathbf{a}_{n+1}$ for all $n$.

**Theorem (4.2)**

*For any ascending sequence $\{\mathbf{a}_n\}_{n\in\omega}$ of degrees there exists upper bounds $\mathbf{b}$ and $\mathbf{c}$ (called exact pair) such that*

$$(\forall \mathbf{d})((\mathbf{d} \leq \mathbf{b} \wedge \mathbf{d} \leq \mathbf{c}) \Rightarrow (\exists n)(\mathbf{d} \leq \mathbf{a}_n)).$$

**Theorem (4.2)**

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Its corollaries seem to be more natural in description.

**Corollary (4.4)**

*There are degrees $\mathbf{b}$ and $\mathbf{c}$ with no greatest lower bound.*

**Corollary (4.3)**

*An ascending sequence has no least upper bound.*

## Definition (4.5)

For any set $A \subseteq \omega$ define the *y-section* of $A$ as

$$A^{[y]} = \{\langle x, y\rangle : \langle x, y\rangle \in A\},$$

and define

$$A^{[<y]} = \{\langle x, z\rangle : \langle x, z\rangle \in A \wedge z < y\}.$$

## Definition (4.6)

A set $B \subset A$ is a *thick* subset of $A$ if for every $y$ the following thickness requirement is met:

$$T_y : B^{[y]} =^* A^{[y]},$$

where $X =^* Y$ denotes that the size of their symmetric difference $|(X - Y) \cup (Y - X)|$ is finite.

**Definition (4.7)**

Two partial functions $\phi$ and $\psi$ are compatible (written compat($\phi$,$\psi$)) if $\phi(x) = \psi(x)$ for all $x$ such that $\phi(x)\downarrow$ and $\psi(x)\downarrow$.

# Infinite Extension $((A^{[<s]})' \oplus A_s)$-recursive Construction

Fix a sequence of sets $\{A_i\}_{i\in\omega}$ with $A_i \in \mathbf{a}_i$, Let

$$A = \{\langle x, i\rangle | x \in A_i\}.$$

We will use an infinite extension construction to construct $f_s$ and $g_s$, namely $B$ and $C$.

# Infinite Extension $((A^{[<s]})' \oplus A_s)$-recursive Construction

At stage $s+1$:

Step 1 for $\langle e, i \rangle = s$, we satisfy the following requirement:

$$R_{\langle e,i \rangle} : \phi_e^B = \phi_i^C = h \text{ is total } \Rightarrow (\exists y)(h \leq_T A_y).$$

We test whether

$$(\exists \sigma, \tau)(\exists x)(\exists t)(compat(f_s, \sigma) \wedge compat(g_s, \tau) \wedge \phi_{e,t}^{\sigma}(x) \downarrow \neq \phi_{i,t}^{\tau}(x) \downarrow) \quad (4)$$

- True: let $\sigma, \tau$ be the first such strings, let $\hat{f} = f_s \cup \sigma$, $\hat{g} = g_s \cup \tau$.
- False: let $\hat{f} = f_s$, $\hat{g} = g_s$.

# Infinite Extension $((A^{[<s]})' \oplus A_s)$-recursive Construction

At stage $s+1$:

Step 2 We satisfy the thickness requirement $T_s^B$ and $T_s^C$ for $B$ and $C$:

$$T_s^B : \ : B^{[s]} =^* A^{[s]}$$

$$T_s^C : \ : C^{[s]} =^* A^{[s]}$$

For all $x \in \omega^{[s]} - \mathrm{dom}\ \widehat{f}$ define $f_{s+1}(x) = A(x)$, and define $f_{s+1}(x) = \widehat{f}(x)$ for all $x \in \mathrm{dom}\ \widehat{f}$. Similarly for $g_{s+1}$.

# Correctness

- We can prove inductively the following claims, hence $A_s \equiv_T B^{[s]} \leq_T B$, and likewise for $C$:
  - on $\omega^{[<s]}$, $f_s$ and $g_s$ are defined
  - $B^{[s]} =^* C^{[s]} =^* A^{[s]}$
  - $\mathrm{dom}\ f_s - \omega^{[<s]} =^* \varnothing =^* \mathrm{dom}\ g_s - \omega^{[<s]}$
- Step 1 requires an $(A^{[<s]})'$ oracle
  - $compat(f_s, \sigma)$ is an $A^{[<s]}$-recursive relation on $\sigma$, since $f_s \equiv_T A^{[<s]} \equiv_T g_s$
- Even if (4) fails, if $\phi_e^B = \phi_i^C = h$ is total, then $h \leq_T A^{[<s]} \leq_T A_s$ for $s = \langle e, i \rangle$:
  - to $A^{[<s]}$-recursively compute $h(x)$, find the least $\sigma$ such that $compat(f_s, \sigma)$ and $\phi_e^\sigma(x)\downarrow$, then $\phi_e^B(x)$ must be the same value.

□

# Summary

1 A Pair of Incomparable Degrees Below $\mathbf{0}'$

- construct $\mathbf{a}, \mathbf{b} \leq_T \varnothing'$ such that $\mathbf{a}|\mathbf{b}$
- finite extension $\varnothing'$-oracle construction

3 Inverting the Jump

- given $\mathbf{b} \geq 0'$, construct $\mathbf{a}$ with $\mathbf{a}' = \mathbf{a} \cup \mathbf{0}' = \mathbf{b}$
- finite extension $B$-recursive construction

2 Avoiding Cones of Degrees

- given $\mathbf{b} > 0$, construct $\mathbf{a} < \mathbf{b}'$ with $\mathbf{a}|\mathbf{b}$
- finite extension $B'$-recursive construction

4 Upper and Lower Bounds for Degrees

- given ascending sequence $\{\mathbf{a}_n\}_{n\in\omega}$, construct exact pair $\mathbf{b}$ and $\mathbf{c}$
- infinite extension $((A^{[<s]})' \oplus A_s)$-recursive construction

Thank you!