Searching for Natural Unsolvable Problems

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Cantor space

Cantor space: A topology over 2^ω so that the collection of basic open sets is $\{\{x \mid\succ \sigma\} \mid \sigma \in 2^{<\omega}\}$.
Measure over Cantor space: $\mu([\sigma]) = 2^{-|\sigma|}$, where $\sigma \in 2^{<\omega}$ and $[\sigma] = \{x \mid x \succ \sigma\}$.

We call an element of 2^ω as a \`\`real". And every real is a characteristic function of a set of natural numbers.

Compactness

Theorem

The Cantor space is compact.

Proof.

Fix a set $U \subset 2^{<\omega}$ so that $2^{\omega} \subseteq \bigcup_{\sigma \in U}[\sigma]$. Suppose that U has no subset covering 2^{ω}.
Then using the fact, by an induction, we may construct a real x so that $x \notin \bigcup_{\sigma \in U}[\sigma]$, a contradiction. □

Baire Space

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Exercise

Baire space is not compact.

The set not above $0'$ is comeager.

Theorem

If x is not recursive, then the set $\{y \mid y \geq_T x\}$ is comeager.

Proof.

Fix any recursive functional Φ.
Let $U_\Phi = \{\sigma \mid \exists n(\Phi^\sigma(n) \downarrow \neq x(n) \vee \forall \tau \succeq \sigma(\Phi^\tau(n) \uparrow))\}$.
If U_Φ is not dense, then there is some ν so that $\forall n \forall \sigma \succ \nu(\exists \tau \succeq \sigma(\Phi^\tau(n) \downarrow) \wedge (\Phi^\sigma(n) \downarrow \rightarrow \Phi^\sigma(n) = x(n)))$. Then x must be recursive.
Then $\bigcap_\Phi U_\Phi = \{y \mid y \not\geq_T x\}$ is a comeager set. □

The set not above $0'$ is conull.

Theorem

If x is not recursive, then the set $\{y \mid y \geq_T x\}$ is null.

Proof.

Fix any recursive functional Φ.
Let $A_\Phi = \{y \mid \Phi^y \neq x\}$ is a Borel set and so measurable.
Suppose that A_Φ has positive measure, then by Lesbegue density, there must be some σ so that $A_\Phi \cap [\sigma] = \{y \succ \sigma \mid \Phi^y = x\}$ has measure greater than $\frac{3}{4} \cdot 2^{-|\sigma|}$. Then it can be proved that x must be recursive.
Then $\{y \mid y \geq_T x\} = \bigcup_\Phi A_\Phi$ is null. □

Finding unsolvable problems.

So in the both Baire category and measure theory senses, almost no reals compute halting problem. In other words, those problems not more difficult than the halting problem are everywhere.

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But wait a minute, can you give me a "concrete example"?

Exercises

Exercise

1. The set $\{x \mid x \oplus \emptyset' \equiv_T x'\}$ is comeager.
2. The set $\{x \mid x \oplus \emptyset' \equiv_T x'\}$ is conull.

Posner–Robinson Theorem (1)

Theorem

If x is not recursive, then there is a real g so that $x \oplus g \geq_T g'$.

Proof.

We may assume that x is neither r.e. nor co–r.e. (Replace x with $x \oplus (\omega \setminus x)$ if necessary).
We will construct a real g so that
$\forall e \exists n(\Phi_e^{g \upharpoonright n}(e) \downarrow \vee \forall \sigma \succeq g \upharpoonright n(\Phi_e^{\sigma}(e) \uparrow))$.
Then $g' \equiv_T g \oplus \emptyset'$. □

Posner–Robinson Theorem (2)

Proof.

For each e and σ, the fact

$$\exists n((x(n) = 1 \wedge \forall \tau \succ \sigma^\frown 0^n 1(\Phi_e^\tau(e) \uparrow)) \vee (x(n) = 0 \wedge \exists \tau \succ \sigma^\frown 0^n 1 \Phi_e^\tau(e) \downarrow))$$

must hold.

Using the fact to construct g.

The construction can be decoded by $x \oplus g$. So $x \oplus g \geq_T g'$. □

Exercise

Exercise

1. If x is not recursive, then there is some g so that $x \oplus g \equiv_T g'$.
2. If x is not recursive, then for any real $y \geq_T x'$, there is some g so that $x \oplus g \equiv_T y$.

The philosophy of Posner–Robinson Theorem

"Every unsolvable problem is a halting problem."

Degree functions

A function $F : 2^\omega \to 2^\omega$ is degree invariant on a cone (of reals) $C = \{x \mid x \geq_T z\}$ for some z if any $x, y \geq_T z$ in C of the same Turing degree satisfy $F(x) \equiv_T F(y)$.
F is increasing on a cone C if $F(x) \geq_T x$ for $x \in C$.
If F and G are degree invariant on a cone, write $F \leq_M G$ if $F(x) \leq_T G(x)$ for all $x \in C$.

Martin's conjecture

Conjecture

1. Every degree invariant function that is not increasing on a cone is a constant on a cone;
2. $\leq_M$ prewellorders the degree invariant functions which are increasing on a cone. Furthermore, if the $\leq_M$-rank of F is α, then F′ has $\leq_M$-rank $\alpha + 1$, where $F'(x) = (F(x))'$, the Turing jump of F(x).

Exercise

Exercise

Martin's conjecture does not hold under ZFC.

Game theory

Let $A \subseteq \omega^\omega$. An infinite game G_A with perfect information has two players labelled I and II. The game is played by choosing a natural numbers alternately between the two players and ends in ω−many steps. Each game generates a real $x = (n_0, m_0, \dots, n_i, m_i, \dots) \in \omega^\omega$ where n_i and m_i are the numbers played by I and II respectively at step i. If $x \in A$, then I wins the game. Otherwise, II wins.

Winning strategies

Definition

Given a set $A \subseteq \omega^\omega$, I has a winning strategy $\hat{\sigma} : \omega^{<\omega} \to \omega$ for the game G_A if for every $g \in \omega^\omega$, the real $\hat{\sigma} * g = (n_0, g(0), \ldots, n_i, g(n_i), \ldots) \in A$ where $n_0 = \hat{\sigma}(\emptyset)$ and $n_i = \hat{\sigma}((n_0, g(0), \ldots, n_{i-1}, g(i-1)))$ for $i > 0$. The string $(n_0, g(0), \ldots, n_i)$ is called a play at step i by I following $\hat{\sigma}$. II has a winning strategy $\hat{\tau} : \omega^{<\omega} \to \omega$ for the game G_A if for every $f \in \omega^\omega$, the real $f * \hat{\tau} = (f(0), m_0, \ldots, f(i), m_i, \ldots) \notin A$ where $m_0 = \hat{\tau}(f(0))$ and $m_i = \hat{\tau}((f(0), m_0, , \cdots, f(i)))$ for $i > 0$. The string $(f(0), m_0, \ldots, m_i)$ is called a play by II at step i following $\hat{\tau}$.

Determinacy

G_A is determined if either of the two players has a winning strategy.

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The Axiom of Determinacy (AD) states that for any $A \subseteq \omega^\omega$, either I or II has a winning strategy for G_A.

Determinacy of closed sets

Theorem

If A is closed, then G_A is determined.

Proof.

Suppose that II has no winning strategy. We construct a winning strategy for I. □

On Borel sets

Martin proved the following result.

Theorem

If A is a Borel set, then G_A is determined.

Exercise

Exercise

1. AD does not hold under ZFC.
2. There is some A so that A is determined but $\omega^\omega \setminus A$ is not.

On cones

Theorem

Assume that ZF + AD. For any set A of Turing degrees, either A or its complement contains a cone .

Proof.

Suppose I has a winning strategy $\hat{\sigma}$ for G_A. Then $\{x \mid x \geq_T \hat{\sigma}\} \subseteq A$. Otherwise, the complement of A contains a cone. □

A measure on the set of Turing degrees

Define $\lambda : \mathscr{P}(\mathcal{D}) \to \{0,1\}$ so that for any $A \subseteq \mathcal{D}$,

$$\lambda(A) = \begin{cases} 1 & \text{If A contains a cone,} \\ 0 & \text{Otherwise.} \end{cases}$$

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$$\lambda(A) = \begin{cases} 1 & \text{If A contains a cone,} \\ 0 & \text{Otherwise.} \end{cases}$$

Theorem

Assume that $ZF + AD + DC$, λ is a countable additive measure on $\mathscr{P}(\mathcal{D})$.

Martin's conjecture revisit

Conjecture

Assume that ZF + AD + DC, Martin's conjecture holds.

So Martin's conjecture says that the collection of degrees functions are well ordered module a null set of Turing degrees.

On increasing functions

A degree invariant function F is increasing if $x \geq_T y$ implies $F(x) \geq_T F(y)$.

Theorem

Assume that $ZF + DC + AD$. If F is a degree invariant increasing function so that $F(x) >_T x$ on some cone, then $F(x) \geq_T x'$ on some cone.

Proof.

By Posner–Robinson Theorem, for any x, there is some $y >_T x$ so that $F(x) \oplus y \geq y'$. Then $F(y) \geq_T F(x) \oplus y \geq y'$.
Then by AD, $F(x) \geq_T x'$ on some cone. □

Uniformly degree invariant functions

Definition

- If $e = \langle e_0, e_1 \rangle$, we say that $x \equiv_T y$ via e if $x = \Phi_{e_0}^{y}$ and $y = \Phi_{e_1}^{x}$;
- A function $F : 2^\omega \to 2^\omega$ is uniformly degree invariant on a cone $C = \{x \mid x \geq_T z\}$ with base z if there is a function $t : \omega \to \omega$ such that for any $x, y \geq_T z$ with $x \equiv_T y$ via e, we have $F(x) \equiv_T F(y)$ via $t(e)$.

Slaman–Steel's results

Lachlan initiated the study of uniformly degree invariant functions. Then Slaman and Steel proved the following remarkable result.

Theorem

Assume that $\mathrm{ZF} + \mathrm{AD} + \mathrm{DC}$, Martin's conjecture holds for uniformly degree invariant functions.

An example

Theorem

Assume ZF + AD + DC. Then $\leq_M$ is a linear ordering on the collection of functions which are increasing and uniformly degree invariant.

Proof.

Define a set $A \subseteq \omega^\omega$ so that $x_0 \oplus x_1 \in A$ if and only if either $x_1^+ \notin 2^\omega$ or

- $x_0^+ \in 2^\omega$;
- $\Phi_{e_0}^{x_0^+} = x_1^+$ where $x_0(0) = \langle e_0, n_0 \rangle$, and
- Either $\Phi_{e_1}^{x_1^+} \neq x_0^+$ or $n_0 \in F(x_0^+) \Leftrightarrow n_1 \notin G(x_1^+)$ where $x_1(0) = \langle e_1, n_1 \rangle$.

□

The reverse

Let PD be the determinacy for projective sets.

Theorem

Assume that ZF + DC, if Martin′s conjecture holds for uniformly degree invariant functions, then PD is consistent.

Martin′s conjecture v.s. information theory

Though we have a conull set of unsolvable problems not stronger than those halting problem, they are actually "random". In other words, they exist everywhere but there is no way to find it.

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Though we have a conull set of unsolvable problems not stronger than those halting problem, they are actually "random". In other words, they exist everywhere but there is no way to find it.

Martin's conjecture rules out \`\`the junk information".

Finish