Foundations of Quantum Programming

Lecture 5: Analysis of Quantum Programs

Mingsheng Ying

University of Technology Sydney, Australia
Outline

Analysis of Quantum Loops
  Quantum while-Loops with Unitary Bodies
  General Quantum while-Loops
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  Quantum while-Loops with Unitary Bodies
  General Quantum while-Loops
Quantum while-Loops with Unitary Bodies

\[ S \equiv \textbf{while } M[\bar{q}] = 1 \textbf{ do } \bar{q} := U[\bar{q}] \textbf{ od } \]

where:

» \( \bar{q} \) denotes quantum register \( q_1, \ldots, q_n \), its state Hilbert space:

\[ \mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_{q_i} \]
Quantum while-Loops with Unitary Bodies

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where:

- \( \overline{q} \) denotes quantum register \( q_1, \ldots, q_n \), its state Hilbert space:
  \[ \mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_{q_i} \]

- the loop body is unitary transformation \( \overline{q} := U[\overline{q}] \) in \( \mathcal{H} \).
Quantum while-Loops with Unitary Bodies

\[ S \equiv \textbf{while } M[\vec{q}] = 1 \textbf{ do } \vec{q} := U[\vec{q}] \textbf{ od } \]

where:

- \( \vec{q} \) denotes quantum register \( q_1, \ldots, q_n \), its state Hilbert space:
  \[ \mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_{q_i} \]

- the loop body is unitary transformation \( \vec{q} := U[\vec{q}] \) in \( \mathcal{H} \);

- the yes-no measurement \( M = \{ M_0, M_1 \} \) in the loop guard is projective: \( M_0 = P_{X^\perp}, M_1 = P_X \) with \( X \) being a subspace of \( \mathcal{H} \), \( X^\perp \) being the orthocomplement of \( X \).
Execution of Quantum Loops

- Initial step: Performs measurement $M$ on the input state $\rho$:

  $\rho(1)_{\text{out}} = P_{X}\rho P_{X\perp}$.

  The loop continues with probability $p(1)_{NT}(\rho) = 1 - p(1)_{T}(\rho) = \text{tr}(P_{X}\rho P_{X\perp})$.

  The program state after the measurement:

  $\rho(1)_{\text{mid}} = P_{X}\rho P_{X\perp} p(1)_{NT}(\rho)$.

  $\rho(1)_{\text{mid}}$ is fed to the unitary operation $U$:

  $\rho(2)_{\text{in}} = U\rho(1)_{\text{mid}} U^{\dagger}$ is returned.

  $\rho(2)_{\text{in}}$ will be used as the input state in the next step.
Execution of Quantum Loops

- **Initial step**: Performs measurement $M$ on the input state $\rho$:
  - The loop terminates with probability $p_T^{(1)}(\rho) = tr(P_{X^\perp} \rho)$.
    - The output at this step:
      \[
      \rho_{\text{out}}^{(1)} = \frac{P_{X^\perp} \rho P_{X^\perp}}{p_T^{(1)}(\rho)}.
      \]
Execution of Quantum Loops

- **Initial step:** Performs measurement $M$ on the input state $\rho$:
  - The loop terminates with probability $p_T^{(1)}(\rho) = tr(P_{X\perp}\rho)$. The output at this step:
    \[
    \rho_{out}^{(1)} = \frac{P_{X\perp}\rho P_{X\perp}}{p_T^{(1)}(\rho)}. 
    \]
  - The loop continues with probability
    \[
    p_{NT}^{(1)}(\rho) = 1 - p_T^{(1)}(\rho) = tr(P_X\rho). 
    \]
    The program state after the measurement:
    \[
    \rho_{mid}^{(1)} = \frac{P_X\rho P_X}{p_{NT}^{(1)}(\rho)}. 
    \]

$\rho_{out}^{(1)}$ is fed to the unitary operation $U$:
\[
\rho_{in}^{(2)} = U \rho_{out}^{(1)} U^\dagger 
\]
is returned. $\rho_{in}^{(2)}$ will be used as the input state in the next step.
Execution of Quantum Loops

- **Initial step**: Performs measurement $M$ on the input state $\rho$:
  - The loop terminates with probability $p_T^{(1)}(\rho) = tr(P_X \perp \rho)$. The output at this step:
    \[ \rho_{\text{out}}^{(1)} = \frac{P_X \perp \rho P_X \perp}{p_T^{(1)}(\rho)}. \]
  - The loop continues with probability $p_{NT}^{(1)}(\rho) = 1 - p_T^{(1)}(\rho) = tr(P_X \rho)$. The program state after the measurement:
    \[ \rho_{\text{mid}}^{(1)} = \frac{P_X \rho P_X}{p_{NT}^{(1)}(\rho)}. \]
  - $\rho_{\text{mid}}^{(1)}$ is fed to the unitary operation $U$:
    \[ \rho_{\text{in}}^{(2)} = U \rho_{\text{mid}}^{(1)} U^\dagger \]
    is returned. $\rho_{\text{in}}^{(2)}$ will be used as the input state in the next step.
>  **Induction step:** Suppose the loop has run $n$ steps, it did not terminate at the $n$th step: $p_{NT}^{(n)} > 0$. If $\rho_{in}^{(n+1)}$ is the program state at the end of the $n$th step, then in the $(n + 1)$th step:

The termination probability:

$$p_{NT}^{(n+1)} = \text{tr}(P_X \rho_{in}^{(n+1)})$$

The output at this step is

$$\rho_{out}^{(n+1)} = P_X \rho_{in}^{(n+1)}.$$
**Induction step:** Suppose the loop has run $n$ steps, it did not terminate at the $n$th step: $p_{NT}^{(n)} > 0$. If $\rho_{in}^{(n+1)}$ is the program state at the end of the $n$th step, then in the $(n + 1)$th step:

- The termination probability: $p_T^{(n+1)}(\rho) = tr(P_X \rho_{in}^{(n+1)})$. The output at this step is

$$
\rho_{out}^{(n+1)} = \frac{P_X \rho_{in}^{(n+1)} P_X}{p_T^{(n+1)}(\rho)}.
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Induction step: Suppose the loop has run \( n \) steps, it did not terminate at the \( n \)th step: \( p_{NT}^{(n)} > 0 \). If \( \rho_{in}^{(n+1)} \) is the program state at the end of the \( n \)th step, then in the \( (n+1) \)th step:

- The termination probability: \( p_{T}^{(n+1)}(\rho) = tr(P_X\rho_{in}^{(n+1)}) \). The output at this step is

\[
\rho_{out}^{(n+1)} = \frac{P_X\rho_{in}^{(n+1)}P_X}{p_{T}^{(n+1)}(\rho)}.
\]

- The loop continues to perform the unitary operation \( U \) on the post-measurement state

\[
\rho_{mid}^{(n+1)} = \frac{P_X\rho_{in}^{(n+1)}P_X}{p_{T}^{(n+1)}(\rho)}
\]

with probability \( p_{NT}^{(n+1)}(\rho) = 1 - p_{T}^{(n+1)}(\rho) = tr(P_X\rho_{in}^{(n+1)}) \). The state \( \rho_{in}^{(n+2)} = U\rho_{mid}^{(n+1)}U^\dagger \) will be returned. It will be the input of the \( (n+2) \)th step.
Termination

1. If probability $p_{NT}^{(n)}(\rho) = 0$ for some positive integer $n$, then the loop terminates from input $\rho$. 

Terminating

A quantum loop is terminating (resp. almost surely terminating) if it terminates (resp. almost surely terminates) from all input $\rho \in D(H)$. 


Termination

1. If probability $p_{NT}^{(n)}(\rho) = 0$ for some positive integer $n$, then the loop terminates from input $\rho$.

2. The nontermination probability of the loop from input $\rho$ is

$$p_{NT}(\rho) = \lim_{n \to \infty} p_{NT}^{(\leq n)}(\rho)$$

where

$$p_{NT}^{(\leq n)}(\rho) = \prod_{i=1}^{n} p_{NT}^{(i)}(\rho)$$

is the probability that the loop does not terminate after $n$ steps.

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$$p_{NT}^{(\leq n)}(\rho) = \prod_{i=1}^{n} p_{NT}^{(i)}(\rho)$$

is the probability that the loop does not terminate after $n$ steps.
3. The loop almost surely terminates from input $\rho$ whenever nontermination probability $p_{NT}(\rho) = 0$.

**Terminating**

A quantum loop is terminating (resp. almost surely terminating) if it terminates (resp. almost surely terminates) from all input $\rho \in D(\mathcal{H})$. 
Computed Function

- The function $\mathcal{F} : \mathcal{D}(\mathcal{H}) \rightarrow \mathcal{D}(\mathcal{H})$ computed by the loop:

$$
\mathcal{F}(\rho) = \sum_{n=1}^{\infty} p_{NT}^{(\leq n-1)}(\rho) \cdot p_{T}^{(n)}(\rho) \cdot \rho_{out}^{(n)}
$$

for each $\rho \in \mathcal{D}(\mathcal{H})$. 
Computed Function

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\mathcal{F}(\rho) = \sum_{n=1}^{\infty} p^{(\leq n-1)}_{NT}(\rho) \cdot p_{T}^{(n)}(\rho) \cdot \rho_{out}^{(n)}
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for each $\rho \in \mathcal{D}(\mathcal{H})$.

- For operator $A$ in Hilbert space $\mathcal{H}$, subspace $X$ of $\mathcal{H}$, the restriction of $A$ in $X$:

$$
A_X = P_X A P_X
$$
Computed Function

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A_X = P_X A P_X
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- 

$$
p_{NT}^{(\leq n)}(\rho) = tr(U_X^{n-1} \rho_X U_X^{+n-1})
$$
Computed Function

▶ The function $\mathcal{F} : \mathcal{D}(\mathcal{H}) \rightarrow \mathcal{D}(\mathcal{H})$ computed by the loop:

$$\mathcal{F}(\rho) = \sum_{n=1}^{\infty} p_{NT}^{(\leq n-1)}(\rho) \cdot p_T^{(n)}(\rho) \cdot \rho_{out}$$

for each $\rho \in \mathcal{D}(\mathcal{H})$.

▶ For operator $A$ in Hilbert space $\mathcal{H}$, subspace $X$ of $\mathcal{H}$, the restriction of $A$ in $X$:

$$A_X = P_X A P_X$$

▶

$$p_{NT}^{(\leq n)}(\rho) = tr(U_{n-1}^X \rho X U_X^{+n-1})$$

▶

$$\mathcal{F}(\rho) = P_{X \perp} \rho P_{X \perp} + P_{X \perp} U \left( \sum_{n=0}^{\infty} U_{X n}^X \rho X U_X^{+n} \right) U^{\dagger} P_{X \perp}$$
Termination Analysis

- Let $\rho = \sum_i p_i \rho_i$ with $p_i > 0$ for all $i$. Then the loop terminates from input $\rho$ if and only if it terminates from input $\rho_i$ for all $i$. 

A quantum loop is terminating if and only if it terminates from all pure input states.
Termination Analysis

- Let $\rho = \sum_i p_i \rho_i$ with $p_i > 0$ for all $i$. Then the loop terminates from input $\rho$ if and only if it terminates from input $\rho_i$ for all $i$.
- A quantum loop is terminating if and only if it terminates from all pure input states.
Let \( \{ |m_1\rangle, \ldots, |m_l\rangle \} \) be an orthonormal basis of \( \mathcal{H} \) such that

\[
\sum_{i=1}^{k} |m_i\rangle\langle m_i| = P_X, \quad \sum_{i=k+1}^{l} |m_i\rangle\langle m_i| = P_{X^\perp}
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Let \(\{ |m_1\rangle, \ldots, |m_l\rangle \} \) be an orthonormal basis of \( \mathcal{H} \) such that

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Write \( |\psi\rangle_X \) for (the vector representation of) projection \( P_X |\psi\rangle \).
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Let \( \{ |m_1\rangle, \ldots, |m_l\rangle \} \) be an orthonormal basis of \( \mathcal{H} \) such that

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The following statements are equivalent:

1. The loop terminates from input \( \rho \in \mathcal{D}(\mathcal{H}) \);
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Write \( |\psi\rangle_X \) for (the vector representation of) projection \( P_X |\psi\rangle \).

The following statements are equivalent:
1. The loop terminates from input \( \rho \in \mathcal{D}(\mathcal{H}) \);
2. \( U_X^n \rho_X U_X^{\dagger n} = 0_{k \times k} \) for some nonnegative integer \( n \), where \( 0_{k \times k} \) is the \((k \times k)\)-zero matrix.
Let \( \{|m_1\rangle, \ldots, |m_l\rangle\} \) be an orthonormal basis of \( \mathcal{H} \) such that
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The loop terminates from pure input state \( |\psi\rangle \) if and only if \( U_X^n |\psi\rangle_X = 0 \) for some nonnegative integer \( n \), where \( 0 \) is the \( k \)-dimensional zero vector.
From Quantum Loop to Classical Loop

- The condition $U^n_X |\psi\rangle_X = 0$ is a termination condition for the loop:

  $$\text{while } v \neq 0 \text{ do } v := U_X v \text{ od}$$

This loop must be understood as a classical computation in the field of complex numbers.
From Quantum Loop to Classical Loop

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- Let $S$ be a nonsingular $(k \times k)$-complex matrix. The following statements are equivalent:
From Quantum Loop to Classical Loop

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- Let $S$ be a nonsingular $(k \times k)$-complex matrix. The following statements are equivalent:
  
  1. The above classical loop (with $v \in C^k$) terminates from input $v_0 \in C^k$. 

From Quantum Loop to Classical Loop

- The condition $U^n_X|\psi\rangle_X = 0$ is a termination condition for the loop:

  while $v \neq 0$ do $v := U_X v$ od

  This loop must be understood as a classical computation in the field of complex numbers.

- Let $S$ be a nonsingular $(k \times k)$-complex matrix. The following statements are equivalent:
  1. The above classical loop (with $v \in \mathbb{C}^k$) terminates from input $v_0 \in \mathbb{C}^k$.
  2. The classical loop:

     while $v \neq 0$ do $v := (SU_XS^{-1})v$ od

     (with $v \in \mathbb{C}^k$) terminates from input $Sv_0$. 
Jordan Normal Form Theorem

For any \((k \times k)\)-complex matrix \(A\), there is a nonsingular \((k \times k)\)-complex matrix \(S\) such that

\[
A = S J(A) S^{-1}
\]

where

\[
J(A) = \bigoplus_{i=1}^{l} J_{k_i}(\lambda_i)
\]

\[
= \text{diag}(J_{k_1}(\lambda_1), J_{k_2}(\lambda_2), \ldots, J_{k_l}(\lambda_l))
\]

\[
= \begin{pmatrix}
J_{k_1}(\lambda_1) & & \\
& J_{k_2}(\lambda_2) & \\
& & \ddots \\
& & & J_{k_l}(\lambda_l)
\end{pmatrix}
\]

is the Jordan normal form of \(A\),
Jordan Normal Form Theorem (Continued)

\[ \sum_{i=1}^{l} k_i = k, \]

\[ J_{k_i}(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \ddots & \ddots \\ & & & \ddots & 1 \\ & & & & \lambda_i \end{pmatrix} \]

is a \((k_i \times k_i)\)-Jordan block for each \(1 \leq i \leq l\).
Technical Lemma

Let $J_r(\lambda)$ be a $(r \times r)$-Jordan block, $\mathbf{v}$ an $r$-dimensional complex vector. Then

$$J_r(\lambda)^n \mathbf{v} = 0$$

for some nonnegative integer $n$ if and only if $\lambda = 0$ or $\mathbf{v} = \mathbf{0}$.
Theorem

- The Jordan decomposition of $U_X$: $U_X = SJ(U_X)S^{-1}$, where

$$J(U_X) = \bigoplus_{i=1}^{l} J_{k_i}(\lambda_i) = \text{diag}(J_{k_1}(\lambda_1), J_{k_2}(\lambda_2), \ldots, J_{k_l}(\lambda_l)).$$
Theorem

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- Let $S^{-1}|\psi\rangle_X$ be divided into $l$ sub-vectors $v_1, v_2, \ldots, v_l$ such that the length of $v_i$ is $k_i$.

Corollary

The quantum loop is terminating if and only if $U_X$ has only zero eigenvalues.
Theorem

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$$J(U_X) = \bigoplus_{i=1}^{l} J_{k_i}(\lambda_i) = \text{diag}(J_{k_1}(\lambda_1), J_{k_2}(\lambda_2), \ldots, J_{k_l}(\lambda_l)).$$

- Let $S^{-1}|\psi\rangle_X$ be divided into $l$ sub-vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_l$ such that the length of $\mathbf{v}_i$ is $k_i$.

- Then: the quantum loop terminates from input $|\psi\rangle$ if and only if for each $1 \leq i \leq l$, $\lambda_i = 0$ or $\mathbf{v}_i = \mathbf{0}$.

Corollary

The quantum loop is terminating if and only if $U_X$ has only zero eigenvalues.
Almost sure termination

Let $\rho = \sum_i p_i \rho_i$ with $p_i > 0$ for all $i$. Then the quantum loop almost surely terminates from input $\rho$ if and only if it almost surely terminates from input $\rho_i$ for all $i$. 

A quantum loop is almost surely terminating if and only if it almost surely terminates from all pure input states. 

The quantum loop almost surely terminates from pure input state $|\psi\rangle$ if and only if $\lim_{n \to \infty} ||U^n X|\psi\rangle|| = 0$. 

The quantum loop is almost surely terminating if and only if all the eigenvalues of $U_X$ have norms less than 1.
Almost sure termination

- Let $\rho = \sum_i p_i \rho_i$ with $p_i > 0$ for all $i$. Then the quantum loop almost surely terminates from input $\rho$ if and only if it almost surely terminates from input $\rho_i$ for all $i$.
- A quantum loop is almost surely terminating if and only if it almost surely terminates from all pure input states.
Almost sure termination

- Let $\rho = \sum_i p_i \rho_i$ with $p_i > 0$ for all $i$. Then the quantum loop almost surely terminates from input $\rho$ if and only if it almost surely terminates from input $\rho_i$ for all $i$.

- A quantum loop is almost surely terminating if and only if it almost surely terminates from all pure input states.

- The quantum loop almost surely terminates from pure input state $|\psi\rangle$ if and only if

$$\lim_{n \to \infty} ||U^n_x |\psi\rangle|| = 0.$$
Almost sure termination

Let $\rho = \sum_i p_i \rho_i$ with $p_i > 0$ for all $i$. Then the quantum loop almost surely terminates from input $\rho$ if and only if it almost surely terminates from input $\rho_i$ for all $i$.

A quantum loop is almost surely terminating if and only if it almost surely terminates from all pure input states.

The quantum loop almost surely terminates from pure input state $|\psi\rangle$ if and only if

$$\lim_{n \to \infty} \|U^n_X|\psi\rangle\| = 0.$$ 

The quantum loop almost surely terminates from input $|\psi\rangle$ if and only if for each $1 \leq i \leq l$, $|\lambda_i| < 1$ or $v_i = 0$. 

Almost sure termination

Let $\rho = \sum_i p_i \rho_i$ with $p_i > 0$ for all $i$. Then the quantum loop almost surely terminates from input $\rho$ if and only if it almost surely terminates from input $\rho_i$ for all $i$.

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The quantum loop almost surely terminates from pure input state $|\psi\rangle$ if and only if

$$\lim_{n \to \infty} ||U^n_X |\psi\rangle|| = 0.$$

The quantum loop almost surely terminates from input $|\psi\rangle$ if and only if for each $1 \leq i \leq l$, $|\lambda_i| < 1$ or $v_i = 0$.

The quantum loop is almost surely terminating if and only if all the eigenvalues of $U_X$ have norms less than 1.
General Quantum while-Loops

\[
\text{while } M[\overline{q}] = 1 \text{ do } S \text{ od}
\]

where:

- \( M = \{M_0, M_1\} \) is a yes-no measurement;

\[
\text{while } M[\overline{q}] = 1 \text{ do } \overline{q} := \mathcal{E}[\overline{q}] \text{ od}.
\]
General Quantum while-Loops

\[
\textbf{while } M[\bar{q}] = 1 \textbf{ do } S \textbf{ od}
\]

where:

- \( M = \{M_0, M_1\} \) is a yes-no measurement;
- \( \bar{q} \) is a quantum register;

\[
\textbf{while } M[\bar{q}] = 1 \textbf{ do } \bar{q} := E[\bar{q}] \textbf{ od}.
\]

Notation
For \( i = 0, 1 \), define quantum operation \( E_i \):

\[
E_i(\sigma) = M_i \sigma M_i^\dagger
\]
General Quantum while-Loops

while $M[\bar{q}] = 1$ do $S$ od

where:

- $M = \{M_0, M_1\}$ is a yes-no measurement;
- $\bar{q}$ is a quantum register;
- the loop body $S$ is a general quantum program.

while $M[\bar{q}] = 1$ do $\bar{q} := \mathcal{E}[\bar{q}]$ od.

Notation

For $i = 0, 1$, define quantum operation $\mathcal{E}_i$:

$$\mathcal{E}_i(\sigma) = M_i \sigma M_i^\dagger$$
Execution of Loops

Initial step: Perform the termination measurement \{M_0, M_1\} on the input state \(\rho\).

- The probability that the program terminates (the measurement outcome is 0):
  \[ p^{(1)}_T(\rho) = \text{tr}[\mathcal{E}_0(\rho)]. \]

The program state after termination:

\[ \rho^{(1)}_{out} = \mathcal{E}_0(\rho) / p^{(1)}_T(\rho). \]

Encode probability \(p^{(1)}_T(\rho)\) and density operator \(\rho^{(1)}_{out}\) into a partial density operator

\[ p^{(1)}_T(\rho)\rho^{(1)}_{out} = \mathcal{E}_0(\rho). \]

So, \(\mathcal{E}_0(\rho)\) is the partial output state at the first step.
Execution of Loops (Continued)

- The probability that the program does not terminate (the measurement outcome is 1):

\[ p_{NT}^{(1)}(\rho) = tr[\mathcal{E}_1(\rho)] \]

The program state after the outcome 1 is obtained:

\[ \rho_{mid}^{(1)} = \mathcal{E}_1(\rho) / p_{NT}^{(1)}(\rho). \]

It is transformed by the loop body \( \mathcal{E} \) to

\[ \rho_{in}^{(2)} = (\mathcal{E} \circ \mathcal{E}_1)(\rho) / p_{NT}^{(1)}(\rho), \]

upon which the second step will be executed.

Combine \( p_{NT}^{(1)} \) and \( \rho_{in}^{(2)} \) into a partial density operator

\[ p_{NT}^{(1)}(\rho)\rho_{in}^{(2)} = (\mathcal{E} \circ \mathcal{E}_1)(\rho). \]
Execution of Loops (Continued)

**Induction step:** Write \( p_{NT}^{(\leq n)} = \prod_{i=1}^{n} p_{NT}^{(i)} \) for the probability that the program does not terminate within \( n \) steps, where \( p_{NT}^{(i)} \) is the probability that the program does not terminate at the \( i \)th step for every \( 1 \leq i \leq n \).

The program state after the \( n \)th measurement with outcome 1:

\[
\rho_{mid}^{(n)} = \frac{[\mathcal{E}_1 \circ (\mathcal{E} \circ \mathcal{E}_1)^{n-1}] (\rho)}{p_{NT}^{(\leq n)}}
\]

It is transformed by the loop body \( \mathcal{E} \) into

\[
\rho_{in}^{(n+1)} = \frac{(\mathcal{E} \circ \mathcal{E}_1)^n (\rho)}{p_{NT}^{(\leq n)}}.
\]

Combine \( p_{NT}^{(\leq n)} \) and \( \rho_{in}^{(n+1)} \) into a partial density operator

\[
p_{NT}^{(\leq n)} (\rho) \rho_{in}^{(n+1)} = (\mathcal{E} \circ \mathcal{E}_1)^n (\rho).
\]
Execution of Loops (Continued)

- The \((n + 1)\)st step is executed upon \(\rho_{in}^{(n+1)}\).
Execution of Loops (Continued)

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    \]
  - Combining \(q_T^{(n+1)}(\rho)\) and \(\rho_{out}^{(n+1)}\) yields the partial output state of the program at the \((n + 1)\)st step:
    \[
    q_T^{(n+1)}(\rho) \rho_{out}^{(n+1)} = [E_0 \circ (E \circ E_1)^n] (\rho).
    \]
Execution of Loops (Continued)

- The probability that the program does not terminate within $(n + 1)$ steps:

$$p_{NT}^{(\leq n+1)}(\rho) = tr([\mathcal{E}_1 \circ (\mathcal{E} \circ \mathcal{E}_1)^n](\rho)).$$
Execution of Loops (Continued)

- The probability that the program does not terminate within \((n + 1)\) steps:

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Termination

1. The quantum loop terminates from input state \(\rho\) if probability \(p_{NT}^{(n)}(\rho) = 0\) for some positive integer \(n\).
Execution of Loops (Continued)

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Termination

1. The quantum loop terminates from input state \(\rho\) if probability \(p_{NT}^{(n)}(\rho) = 0\) for some positive integer \(n\).
2. The loop almost surely terminates from input state \(\rho\) if nontermination probability

\[
p_{NT}(\rho) = \lim_{n \to \infty} p_{NT}^{(\leq n)}(\rho) = 0
\]

where \(p_{NT}^{(\leq n)}\) is the probability that the program does not terminate within \(n\) steps.
**Terminating**

The quantum loop is terminating (resp. almost surely terminating) if it terminates (resp. almost surely terminates) from any input $\rho$. 

**Computed Function**

The function $F : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H})$ computed by the quantum loop:

$$F(\rho) = \infty \sum_{n=1}^{\infty} q(n) T(\rho) \rho(n)_{\text{out}}$$

for each $\rho \in \mathcal{D}(\mathcal{H})$, where $q(n) T = p(\leq n - 1) NT p(n) T$ is the probability that the program does not terminate within $n - 1$ steps but it terminates at the $n$th step.
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**Computed Function**

The function $\mathcal{F} : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H})$ computed by the quantum loop:

$$\mathcal{F}(\rho) = \sum_{n=1}^{\infty} q_T^{(n)}(\rho) \rho_{\text{out}}^{(n)} = \sum_{n=0}^{\infty} \left[ \mathcal{E}_0 \circ (\mathcal{E} \circ \mathcal{E}_1)^n \right] (\rho)$$

for each $\rho \in \mathcal{D}(\mathcal{H})$, where

$$q_T^{(n)} = p_T^{(\leq n-1)} p_T^{(n)}$$

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Recursive Characterisation of Computed Function

The quantum operation $\mathcal{F}$ computed by a loop satisfies the recursive equation:

$$\mathcal{F}(\rho) = \mathcal{E}_0(\rho) + \mathcal{F}[(\mathcal{E} \circ \mathcal{E}_1)(\rho)].$$
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Matrix Representation of Quantum Operations

Suppose quantum operation $\mathcal{E}$ in a $d$-dimensional Hilbert space $\mathcal{H}$ has the Kraus operator-sum representation:

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger.$$

Then the matrix representation of $\mathcal{E}$ is the $d^2 \times d^2$ matrix:

$$M = \sum_i E_i \otimes E_i^*,$$

where $A^*$ stands for the conjugate of matrix $A$. 
Lemma
Write $|\Phi\rangle = \sum_j |jj\rangle$ for the (unnormalized) maximally entangled state in $\mathcal{H} \otimes \mathcal{H}$, where $\{|j\rangle\}$ is an orthonormal basis of $\mathcal{H}$. Let $M$ be the matrix representation of quantum operation $\mathcal{E}$. Then for any $d \times d$ matrix $A$:

$$(\mathcal{E}(A) \otimes I)|\Phi\rangle = M(A \otimes I)|\Phi\rangle.$$
Notations

Let the quantum operation $\mathcal{E}$ in the loop body has the operator-sum representation:

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger.$$

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Let $\mathcal{E}_i$ ($i = 0, 1$) be the quantum operations defined by the measurement operations $M_0, M_1$ in the loop guard: $\mathcal{E}_i = M_i \circ M_i^\dagger$.

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- Write $\mathcal{G}$ for the composition of $\mathcal{E}$ and $\mathcal{E}_1$: $\mathcal{G} = \mathcal{E} \circ \mathcal{E}_1$.

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The matrix representations of $\mathcal{E}_0$ and $\mathcal{G}$ are:

$$N_0 = M_0 \otimes M_0^*,$$

$$R = \sum_i (E_i M_1) \otimes (E_i M_1)^*.$$
Lemma

- Suppose that the Jordan decomposition of $R$ is

$$R = SJ(R)S^{-1}$$

where $S$ is a nonsingular matrix, and $J(R)$ is the Jordan normal form of $R$:

$$J(R) = \bigoplus_{i=1}^{l} J_{k_i}(\lambda_i) = \text{diag}(J_{k_1}(\lambda_1), J_{k_2}(\lambda_2), \cdots, J_{k_l}(\lambda_l)).$$

Then:

1. $|\lambda_s| \leq 1$ for all $1 \leq s \leq l$.
2. If $|\lambda_s| = 1$ then the $s$th Jordan block is 1-dimensional; that is, $k_s = 1$. 

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Theorem: Terminating and Almost Sure Terminating
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1. If $R^k|\Phi\rangle = 0$ for some integer $k \geq 0$, then quantum loop is terminating. Conversely, if loop is terminating, then $R^k|\Phi\rangle = 0$ for all integer $k \geq k_0$, where $k_0$ is the maximal size of Jordan blocks of $R$ corresponding to eigenvalue 0.
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2. Quantum loop is almost surely terminating if and only if $|\Phi\rangle$ is orthogonal to all eigenvectors of $R^\dagger$ corresponding to eigenvalues $\lambda$ with $|\lambda| = 1.$
Expectation of Observables at the Outputs

- The expectation $tr(P\mathcal{F}(\rho))$ of observable $P$ in the output state $\mathcal{F}(\rho)$. 
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where $R$ is the matrix representation of $\mathcal{G} = \mathcal{E} \circ \mathcal{E}_1$.
- This series may not converge when some eigenvalues of $R$ has module 1.
- Idea to overcome this objection: modify the Jordan normal form $J(R)$ of $R$ by vanishing the Jordan blocks corresponding to those eigenvalues with module 1: $N = SJ(N)S^{-1}$

$$J(N) = \text{diag}(J'_1, J'_2, \cdots, J'_3),$$

$$J'_s = \begin{cases} 
0 & \text{if } |\lambda_s| = 1, \\
J_{k_s}(\lambda_s) & \text{otherwise.}
\end{cases}$$
Lemma
For any integer $n \geq 0$:

$$N_0 R^n = N_0 N^n,$$

where $N_0 = M_0 \otimes M_0^*$ is the matrix representation of $E_0$. 

Theorem
The expectation of observable $P$ in the output state $F(\rho)$ of quantum loop with input state $\rho$:

$$\text{tr}(P F(\rho)) = \langle \Phi | (P \otimes I) N_0 (I \otimes I - N) - 1 \rangle (\rho \otimes I) | \Phi \rangle.$$
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Average Running Time

- The average running time loop with input state $\rho$:

$$\sum_{n=1}^{\infty} np_T^{(n)}$$

where for each $n \geq 1$,

$$p_T^{(n)} = tr \left[ \left( \mathcal{E}_0 \circ (\mathcal{E} \circ \mathcal{E}_1)^{n-1} \right) (\rho) \right] = tr \left[ \left( \mathcal{E}_0 \circ \mathcal{G}^{n-1} \right) (\rho) \right]$$

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**Theorem**

The average running time of quantum loop with input state $\rho$:

$$\langle \Phi | N_0 (I \otimes I - N)^{-2} (\rho \otimes I) | \Phi \rangle.$$
Example: Quantum Walk on a Circle

- Let $\mathcal{H}_d$ be the direction space — a 2-dimensional Hilbert space with orthonormal basis state $|L\rangle$ and $|R\rangle$, indicating directions Left and Right.
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- The state space of the quantum walk: \( \mathcal{H} = \mathcal{H}_d \otimes \mathcal{H}_p \).
- The initial state: \(|L\rangle|0\rangle\).
- This walk has an absorbing boundary at position 1.
Example: Quantum Walk on a Circle, Continued

Each step of the walk consists of:

1. Measure the position of the system to see whether the current position is 1. If the outcome is “yes”, then the walk terminates; otherwise, it continues. This measurement models the absorbing boundary:

\[ M = \{M_{yes} = I_d \otimes |1\rangle\langle 1|, M_{no} = I - M_{yes}\}. \]
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   \[ H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \]

   is applied in the direction space \( \mathcal{H}_d \).
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3. A shift operator

   \[ S = \sum_{i=0}^{n-1} |L\rangle\langle L| \otimes |i \ominus 1\rangle\langle i| + \sum_{i=0}^{n-1} |R\rangle\langle R| \otimes |i \oplus 1\rangle\langle i| \]

   is performed in the space \( \mathcal{H} \).
Example: Quantum Walk on a Circle, Continued

- Quantum while-loop:

  ```
  while $M[d, p] = yes$ do $d, p := W[d, p]$ od
  ```

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A MATLAB program shows that average running time is $n$ for $n < 30$.

Question: The average running time is $n$ for all $n \geq 30$?
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