

Slicewise definability in first-order logic with bounded quantifier rank

Yijia Chen
School of Computer Science
Fudan University
yijiachen@fudan.edu.cn

Jörg Flum
Mathematisches Institut
Universität Freiburg
joerg.flum@math.uni-freiburg.de

Xuangui Huang
Department of Computer Science
Shanghai Jiao Tong University
stslxg@gmail.com

Abstract

For every $q \in \mathbb{N}$ let FO_q denote the class of sentences of first-order logic FO of quantifier rank at most q . If a graph property can be defined in FO_q , then it can be decided in time $O(n^q)$. Thus, minimizing q has favorable algorithmic consequences. Many graph properties amount to the existence of a certain set of vertices of size k . Usually this can only be expressed by a sentence of quantifier rank at least k . We use the color-coding method to demonstrate that some (hyper)graph problems can be defined in FO_q where q is independent of k . This property of a graph problem is equivalent to the question of whether the corresponding parameterized problem is in the class para-AC^0 .

It is crucial for our results that the FO-sentences have access to built-in addition and multiplication. It is known that then FO corresponds to the circuit complexity class uniform AC^0 . We explore the connection between the quantifier rank of FO-sentences and the depth of AC^0 -circuits, and prove that $\text{FO}_q \subsetneq \text{FO}_{q+1}$ for structures with built-in addition and multiplication.

Keywords. first-order logic, quantifier rank, parameterized AC^0 , circuit depth.

1. Introduction

Let φ be a sentence of first-order logic FO. The *quantifier rank* of φ , denoted by $\text{qr}(\varphi)$, is the maximum nested depth of quantifiers in φ . If φ defines a graph property \mathbf{K} , that is,

$$\mathbf{K} = \{ \mathcal{G} \mid \mathcal{G} \text{ a graph and } \mathcal{G} \text{ has the property } \varphi \},$$

then a straightforward algorithm can decide whether an input graph \mathcal{G} belongs to \mathbf{K} in time $O(|\mathcal{G}|^{\text{qr}(\varphi)})$. Therefore, minimizing the quantifier rank of φ would lead to better algorithms for deciding the graph property \mathbf{K} . Many graph properties amount to the existence of a certain set of vertices of size k , where k is a fixed constant. A well-known example is the *k -vertex-cover problem* of deciding whether a given graph \mathcal{G} contains a set C of k vertices such that every edge in \mathcal{G} has one end in C . The set C is then called a *k -vertex-cover* of \mathcal{G} . Clearly, the existence of a k -vertex-cover can be expressed by the

following sentence of FO

$$\psi_k := \exists x_1 \cdots \exists x_k \left(\bigwedge_{1 \leq i < j \leq k} x_i \neq x_j \wedge \forall u \forall v (Euv \rightarrow \bigvee_{i=1}^k (u = x_i \vee v = x_i)) \right).$$

In other words, a graph \mathcal{G} has a k -vertex-cover if and only if \mathcal{G} satisfies ψ_k . Observe that $\text{qr}(\psi_k) = k + 2$, hence the naive algorithm derived from ψ_k would have running time $O(|\mathcal{G}|^{k+2})$. Clearly it is far worse than the existing linear time algorithms for the k -vertex-cover problem. An immediate question is whether the k -vertex-cover problem can be defined by a sentence φ_k with $\text{qr}(\varphi_k) < k + 2$. As the first main result of this paper we show that this is indeed possible for a φ_k with $\text{qr}(\varphi_k) \leq 17$. Note that this holds for every k even though we need different φ_k 's for different k 's. The k -vertex-cover problem is the k th slice of the parameterized vertex cover problem

p -VERTEX-COVER

Input: A graph \mathcal{G} .
Parameter: k .
Question: Does \mathcal{G} have a vertex cover of size k ?

For $q \in \mathbb{N}$ we denote by FO_q the class of FO-sentences of quantifier rank at most q . Our result can be phrased in terms of the *slicewise definability* [9] of p -VERTEX-COVER:

Theorem 1.1. p -VERTEX-COVER is slicewise definable in FO_{17} .

The vertex cover problem is a special case of the *hitting set problem* on hypergraphs of bounded hyperedge size. For every $d \in \mathbb{N}$ a d -hypergraph is a hypergraph with hyperedges of size at most d . Then, the *parameterized d -hitting set problem* p - d -HITTING-SET asks whether an input d -hypergraph \mathcal{G} contains a set of k vertices that intersects with every hyperedge in \mathcal{G} . Thus p -VERTEX-COVER is basically the parameterized 2-hitting set problem. Extending Theorem 1.1 we prove that p - d -HITTING-SET is slicewise definable in FO_q , where $q = O(d^2)$. The problem p - d -HITTING-SET can be Fagin-defined [8] by an FO-formula with a second-order variable which does not occur in the scope of an existential quantifier or negation symbol. We show that all problems Fagin-definable in this form are slicewise definable in some FO_q .

What is the complexity of the class of parameterized problems that are slicewise definable in FO with bounded quantifier rank? We prove that it coincides with para-FO [6], the class of problems FO-definable after a precomputation on the parameter. Thus we obtain a descriptive characterization of the class para-FO, or equivalently of the parameterized circuit complexity class para- AC^0 [7, 3, 6].

The equivalence between para-FO and para- AC^0 is an easy consequence of the equivalence between FO and the classical circuit complexity class uniform AC^0 [4]. This equivalence crucially relies on the assumption that the input graphs (or more generally, the input structures) are equipped with built-in addition and multiplication. In fact, the main technical tool for proving Theorem 1.1 and the subsequent results, the *color-coding* method [1], makes essential use of arithmetic. Without addition and multiplication, it is not difficult to show that p -VERTEX-COVER cannot be slicewise defined in FO_q for any $q \in \mathbb{N}$. Thus Theorem 1.1 exhibits the power of addition and multiplication, although on the face of it, the vertex cover problem has nothing to do with arithmetic operations.

In finite model theory there is consensus that inexpressibility results for FO and for fragments of FO are very hard to obtain in the presence of addition and multiplication. To get such a result we exploit the equivalence between FO and uniform AC^0 , more precisely, we analyze the connection between the quantifier rank of a sentence φ and the depth of the corresponding AC^0 circuits. Together with a theorem [11, 14] on a version of Sipser functions we show that the hierarchy $(\text{FO}_q)_{q \in \mathbb{N}}$ is strict:

Theorem 1.2. *Let $q \in \mathbb{N}$. Then there is a parameterized problem slicewise definable in FO_{q+1} but not in FO_q .*

Organization of the paper. In Section 2 we prove Theorem 1.1, and then extend it to the hitting set problem in Section 3. We give a natural class of Fagin-definable problems that are slicewise definable in FO with bounded quantifier rank in Section 4. We prove the hierarchy theorem, i.e., Theorem 1.2, in Section 6. In the final section we conclude with some open problems.

Some logic preliminaries. A *vocabulary* τ is a finite set of relation symbols. Each relation symbol has an *arity*. A *structure* \mathcal{A} of vocabulary τ , or τ -*structure*, consists of a nonempty set A called the *universe* of \mathcal{A} , and of an interpretation $R^{\mathcal{A}} \subseteq A^r$ of each r -ary relation symbol $R \in \tau$. In this paper all structures have a finite universe. Occasionally we allow the use of constants: For a vocabulary τ we consider $\tau \cup \{c_1, \dots, c_s\}$ -structures \mathcal{A} . Then $c_1^{\mathcal{A}}, \dots, c_s^{\mathcal{A}}$, the interpretations of the constants c_1, \dots, c_s , are elements of \mathcal{A} . However the letters τ, τ', \dots will always denote *relational* vocabularies (without constants). If τ contains a binary relation symbol $<$ and in the structure \mathcal{A} the relation $<^{\mathcal{A}}$ is an order of the universe, then \mathcal{A} is an *ordered structure*.

Let τ be a vocabulary and C a set of constant. Formulas φ of first-order logic of vocabulary $\tau \cup C$ are built up from atomic formulas $t_1 = t_2$ and $Rt_1 \dots t_r$ where t_1, t_2, \dots, t_r are either variables or constants in C , and where $R \in \tau$ is of arity r , using the Boolean connectives and existential and universal quantification. A formula φ is a *sentence* if it has no free variables. The quantifier rank of φ is defined inductively as:

$$\text{qr}(\varphi) := \begin{cases} 0 & \text{if } \varphi \text{ is atomic} \\ \text{qr}(\psi) & \text{if } \varphi = \neg\psi \\ \max\{\text{qr}(\psi_1), \text{qr}(\psi_2)\} & \text{if } \varphi = \psi_1 \wedge \psi_2 \text{ or } \varphi = \psi_1 \vee \psi_2 \\ 1 + \text{qr}(\psi) & \text{if } \varphi = \exists x\psi \text{ or } \varphi = \forall x\psi. \end{cases}$$

2. Slicewise-definability in FO_q and the vertex cover problem

In this section we prove Theorem 1.1, i.e., p -VERTEX-COVER is slicewise definable in FO_{17} . Our main tool is Theorem 2.2. It shows how we can express that there are k elements having a first-order property by a number of quantifiers independent of k . We give further applications of this tool in this and the next section.

For $n \in \mathbb{N}$ let $[n] := \{0, 1, \dots, n-1\}$. Denote by $<^{[n]}$ the natural order on $[n]$. Clearly, if \mathcal{A} is any ordered structure, then $(A, <^{\mathcal{A}})$ is isomorphic to $([|A|], <^{[|A|]})$ and the isomorphism is unique. For ternary relation symbols $+$ and \times we consider the ternary relations $+^{[n]}$ and $\times^{[n]}$ on $[n]$ that are the relations of addition and multiplication of \mathbb{N} restricted to $[n]$. That is,

$$\begin{aligned} +^{[n]} &:= \{(a, b, c) \mid a, b, c \in [n] \text{ with } c = a + b\}, \\ \times^{[n]} &:= \{(a, b, c) \mid a, b, c \in [n] \text{ with } c = a \cdot b\}. \end{aligned}$$

Finally, for every $m \in \mathbb{N}$ let $C(m) := \{\bar{\ell} \mid \ell < m\}$ be a set of constants and set

$$\bar{\ell}^{[n]} := \ell, \text{ if } \ell < n \quad \text{and} \quad \bar{\ell}^{[n]} := n-1, \text{ if } \ell \geq n.$$

Assume a relational vocabulary τ contains $<$, $+$, and \times . A $\tau \cup C(m)$ -structure \mathcal{A} has *built-in* $<$, $+$, \times , $C(m)$ if its $\{<, +, \times, C(m)\}$ -reduct is isomorphic to $([n], <^{[n]}, +^{[n]}, \times^{[n]}, (\bar{\ell}^{[n]})_{\ell < m})$.

If $m = 0$, we briefly say that \mathcal{A} has built-in addition and multiplication. We denote by $\text{ARITHM}[\tau]$ the class of τ -structures with built-in addition and multiplication. If $\mathcal{A} \in \text{ARITHM}[\tau]$ and $m \in \mathbb{N}$, we denote by $\mathcal{A}_{C(m)}$ its unique expansion to a $\tau \cup C(m)$ -structure with built-in $<, +, \times, C(m)$.

In the proof of Theorem 2.2 we use the color-coding technique of Alon et al. [1] essentially in the form presented in [10, page 347]:

Lemma 2.1. *There is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, all $k \leq n$ and for every k -element subset X of $[n]$, there exists a prime $p < k^2 \cdot \log_2 n$ and a $q < p$ such that the function $h_{p,q} : [n] \rightarrow \{0, \dots, k^2 - 1\}$ given by $h_{p,q}(m) := (q \cdot m \bmod p) \bmod k^2$ is injective on X .*

As already mentioned the following result allows to express the existence of k elements satisfying a first-order property by a bounded number of quantifiers.

Theorem 2.2. *Let τ be a vocabulary containing $<, +, \times$. Then there is an algorithm that assigns to every $k \in \mathbb{N}$ and every $\text{FO}[\tau]$ -formula $\varphi(\bar{x}, y)$ an $\text{FO}[\tau \cup C(k^2 + 1)]$ -formula $\chi_\varphi^k(\bar{x})$ such that for every $\mathcal{A} \in \text{ARITHM}[\tau]$ with $k^2 \leq |A|/\log |A|$ and $|A| \geq n_0$ and $\bar{u} \in A$,*

$$\mathcal{A}_{C(k^2)} \models \chi_\varphi^k(\bar{u}) \iff \text{there are pairwise distinct } v_0, \dots, v_{k-1} \in A \text{ with } \mathcal{A} \models \varphi(\bar{u}, v_i) \text{ for every } i \in [k]. \quad (1)$$

Furthermore, $\text{qr}(\chi_\varphi^k(\bar{x})) = \max\{12, \text{qr}(\varphi(\bar{x}, y)) + 3\}$.

Note that the conditions “ $k^2 \leq |A|/\log |A|$ and $|A| \geq n_0$ ” on $|A|$ are fulfilled if $|A| \geq \max\{2^{k^2}, n_0\}$, so we have a lower bound of $|A|$ in terms of k (here n_0 is a natural number according to Lemma 2.1).

Proof: Let \mathcal{A} be as above, set $n := |A|$, and w.l.o.g. assume that $A := [n]$. In order to make formulas more readable, we introduce some abbreviations. Clearly, $x = (y \bmod z)$ is an abbreviation for

$$\exists u(y = u \times z + x \wedge x < z),$$

more precisely, as $+$ and \times are relation symbols, an abbreviation for

$$\exists u \exists u'(u' = u \times z \wedge y = u' + x \wedge x < z).$$

Now let

$$\chi_\varphi^k(\bar{x}) := \exists p \exists q \left(\bigvee_{0 \leq i_1 < \dots < i_{k-1} < k^2} \bigwedge_{j \in [k]} \exists y (“h_{p,q}(y) = i_j” \wedge \varphi(\bar{x}, y)) \right),$$

where

$$“h_{p,q}(y) = i_j” := (q \times (u \bmod p) \bmod p) \bmod k^2 = \bar{i}_j.$$

We replaced $(q \times u \bmod p)$ by $(q \times (u \bmod p) \bmod p)$, since $q \times u$ might exceed $|A|$. To count the quantifier rank note that “ $h_{p,q}(y) = i_j$ ” means

$$\exists v \exists v' \exists \alpha \left(v' = v \times k^2 \wedge \alpha = v' + \bar{i}_j \wedge \bar{i}_j < k^2 \right),$$

where the intended meaning of α is $(q \times (u \bmod p) \bmod p)$. So α is the unique element satisfying

$$\exists w \exists w' \exists \beta (w' = w \times p \wedge \beta = w' + \alpha \wedge \alpha < p).$$

Here the intended meaning of β is $q \times (u \bmod p)$. Thus β is the unique element satisfying

$$\exists \gamma (\beta = q \times \gamma \wedge \text{“}\gamma = u \bmod p\text{”}).$$

So we can replace “ $\gamma = u \bmod p$ ” by

$$\exists z \exists z' (z' = z \times p \wedge u = z' + \gamma \wedge \gamma < p).$$

Thus, $\text{qr}(\text{“}h_{p,q}(y) = i_j\text{”}) = 9$ and hence, $\text{qr}(\chi_\varphi^k(\bar{x})) = \max\{12, \text{qr}(\varphi(\bar{x}, y)) + 3\}$. \square

We use the previous result to show that two parameterized problems are slicewise definable in FO_q for some q , one is an easy application, the other the more intricate p -VERTEX-COVER. First we give the precise definitions of parameterized problem in our context and of slicewise definability.

Definition 2.3. A parameterized problem is a subclass Q of $\text{ARITHM}[\tau] \times \mathbb{N}$ for some vocabulary τ , where for each $k \in \mathbb{N}$ the class $Q_k := \{\mathcal{A} \in \text{ARITHM}[\tau] \mid (\mathcal{A}, k) \in Q\}$ is closed under isomorphism. The class Q_k is the k th slice of Q .

Every pair $(\mathcal{A}, k) \in \text{ARITHM}[\tau] \times \mathbb{N}$ is an *instance* of Q , \mathcal{A} its *input* and k its *parameter*.

Definition 2.4. Q is *slicewise definable* in FO with bounded quantifier rank, briefly $Q \in \text{XFO}_{\text{qr}}$, if there is a $q \in \mathbb{N}$ and computable functions $h : \mathbb{N} \rightarrow \mathbb{N}$ and $f : \mathbb{N} \rightarrow \text{FO}_q[\tau \cup C(h(k))]$ such that for all $(\mathcal{A}, k) \in \text{ARITHM}[\tau] \times \mathbb{N}$,

$$(\mathcal{A}, k) \in Q \iff \mathcal{A}_{C(h(k))} \models f(k).$$

That is, if $m_k := h(k)$ and $f(k) := \varphi_k$, then

$$(\mathcal{A}, k) \in Q \iff \mathcal{A}_{C(m_k)} \models \varphi_k.$$

We then say that Q is *slicewise definable* in FO_q and write $Q \in \text{XFO}_q$.

Using the constants in $C(m)$ we can characterize arithmetical structures with less than m elements by a quantifier free sentence, more precisely:

Lemma 2.5. Assume that $\mathcal{A} \in \text{ARITHM}[\tau]$ and that $|A| < m$. Then there is a quantifier free $\text{FO}[\tau \cup C(m)]$ -sentence $\varphi_{\mathcal{A}_{C(m)}}$ (that is, $\varphi_{\mathcal{A}_{C(m)}} \in \text{FO}_0[\tau \cup C(m)]$) such that for all structures $\mathcal{B} \in \text{ARITHM}[\tau]$ we have

$$\mathcal{B}_{C(m)} \models \varphi_{\mathcal{A}_{C(m)}} \iff \mathcal{A} \cong \mathcal{B}.$$

Using this lemma we get the following simple but useful observation.

Proposition 2.6. Let $Q \in \text{ARITHM}[\tau] \times \mathbb{N}$ be a decidable parameterized problem and $q \in \mathbb{N}$. Assume that Q is eventually slicewise definable in FO_q , that is, there are computable functions $k \mapsto m_k$ with $m_k \in \mathbb{N}$ and $k \mapsto \varphi_k$ with $\varphi_k \in \text{FO}_q[\tau \cup C(m_k)]$ and a computable and increasing function $g : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $(\mathcal{A}, k) \in \text{ARITHM}[\tau] \times \mathbb{N}$ with $|A| \geq g(k)$,

$$(\mathcal{A}, k) \in Q \iff \mathcal{A}_{C(m_k)} \models \varphi_k.$$

Then Q is slicewise definable in FO_q .

Proof: Assume Q is eventually slicewise definable in FO_q and let m_k, φ_k , and g be as above. The sentence ψ_k defining the k th slice of Q essentially says

(the structure has at least $g(k)$ elements and satisfies φ_k) or
 (the structure has less than $g(k)$ elements and is in Q).

To express this we use the set $C(m'_k)$ of constants where $m'_k := \max\{g(k), m_k\}$. In structures with built-in $<, +, \times$ and $C(m'_k)$ the sentence $\overline{g(k) - 1} \neq \overline{g(k) - 2}$ says that the universe has $\geq g(k)$ elements. So we can set (compare Lemma 2.5)

$$\psi_k := \left(\overline{g(k) - 1} \neq \overline{g(k) - 2} \wedge \varphi_k \right) \vee \bigvee_{(\mathcal{A}, k) \in Q, |A| < g(k)} \varphi_{\mathcal{A}_{C(g(k))}}.$$

Hence, the quantifier rank of each ψ_k coincides with the quantifier rank of φ_k . As Q is decidable, the mapping $k \mapsto \psi_k$ is computable. \square

We now turn to our first application of Theorem 2.2.

Theorem 2.7. *The parameterized problem*

p-deg-INDEPENDENT-SET

Input: A graph \mathcal{G} .

Parameter: $k \in \mathbb{N}$.

Question: Is $k \geq \text{deg}(\mathcal{G})$ and does \mathcal{G} have an independent set of $k - \text{deg}(\mathcal{G})$ elements?

is slicewise definable in FO_{13} .

Let $\tau_{\text{GRAPH}} := \{E, <, +, \times\}$ with binary E . More formally, by *p-deg-INDEPENDENT-SET* we mean in our context the class

$$\left\{ (\mathcal{G}, k) \in \text{ARITHM}[\tau_{\text{GRAPH}}] \times \mathbb{N} \mid k \geq \text{deg}(\mathcal{G}) \text{ and } \right. \\ \left. \text{(the } \{E\}\text{-reduct of } \mathcal{G} \text{ has an independent set of size } k - \text{deg}(\mathcal{G})) \right\}^1$$

Proof: An easy induction on $\ell := k - \text{deg}(\mathcal{G})$ shows that every graph \mathcal{G} with at least $(\text{deg}(\mathcal{G}) + 1) \cdot \ell$ vertices has an independent set of size ℓ . Hence, for $(\mathcal{G}, k) \in \text{ARITHM}[\tau]$, where the graph \mathcal{G} has at least $(k + 1) \cdot k$ vertices, we have

$$(\mathcal{G}, k) \in \text{p-deg-INDEPENDENT-SET} \iff k \geq \text{deg}(\mathcal{G}). \quad (2)$$

We use this fact to prove that *p-deg-INDEPENDENT-SET* is eventually slicewise definable in FO_{13} , which yields our claim by Proposition 2.6.

Let $d \in \mathbb{N}$ and $\varphi(u, y) := Euy$. Then, by Theorem 2.2, we have for every graph \mathcal{G} with at least $h(k)$ vertices for some computable $h : \mathbb{N} \rightarrow \mathbb{N}$ and every vertex u of \mathcal{G} ,

$$\mathcal{G} \models \chi_{\varphi}^d(u) \iff \text{the degree of } u \text{ in } \mathcal{G} \text{ is } \geq d.$$

¹In the following we will present parameterized graph problems in the more liberal form as given by the box above.

So the degree of \mathcal{G} is the unique d such that

$$\mathcal{G} \models \exists u \chi_\varphi^d(u) \wedge \neg \exists u \chi_\varphi^{d+1}(u).$$

Thus, for $k \in \mathbb{N}$ and every graph $\mathcal{G} \in \text{ARITHM}[\tau_{\text{GRAPH}}]$ with at least $\max\{h(k), (k+1) \cdot k\}$ vertices, by (2),

$$(\mathcal{G}, k) \in p\text{-deg-INDEPENDENT-SET} \iff \mathcal{G} \models \bigvee_{d \leq k} \left(\exists u \chi_\varphi^d(u) \wedge \neg \exists u \chi_\varphi^{d+1}(u) \right).$$

As $\text{qr}(\varphi) = 0$, Theorem 2.2 and the previous equivalence show that $p\text{-deg-INDEPENDENT-SET}$ is eventually in XFO_{13} (and hence in XFO_{13} by Proposition 2.6). \square

Now we are ready to show the slicewise definability of $p\text{-VERTEX-COVER}$ in FO_{17} .

Proof of Theorem 1.1.: Recall the main ingredient of Buss' kernelization for an instance (\mathcal{G}, k) of the vertex cover problem.

- (1) If a vertex v has degree $\geq k+1$ in \mathcal{G} , then v must be in every vertex cover of size k . We remove all v of degree $\geq k+1$ in \mathcal{G} , say ℓ many, and decrease k to $k' := k - \ell$.
- (2) Remove all isolated vertices.
- (3) Let \mathcal{G}' be the resulting induced graph. If $k' < 0$ or \mathcal{G}' has $> k' \cdot (k+1)$ vertices, then (\mathcal{G}', k') , and hence also (\mathcal{G}, k) , is a NO instance of $p\text{-VERTEX-COVER}$.

Again let $\varphi(x, y) := Exy$. Then, by Theorem 2.2, for every instance (\mathcal{G}, k) of $p\text{-VERTEX-COVER}$, where the vertex set G of \mathcal{G} is sufficiently large compared with k and every vertex $v \in G$,

$$\mathcal{G} \models \chi_\varphi^{k+1}(v) \iff v \text{ has degree } \geq k+1.$$

Therefore, applying again Theorem 2.2 we get for $\ell \in \mathbb{N}$,

$$\mathcal{G} \models \left(\chi_{\chi_\varphi^{k+1}}^\ell \wedge \neg \chi_{\chi_\varphi^{k+1}}^{\ell+1} \right) \iff \mathcal{G} \text{ has exactly } \ell \text{ vertices of degree } \geq k+1.$$

For every vertex v of \mathcal{G} we have

$$\mathcal{G} \models \text{uni}(v) \iff v \text{ is a vertex of } \mathcal{G}',$$

where

$$\text{uni}(x) := \left(\neg \chi_\varphi^{k+1}(x) \wedge \neg \forall y (Exy \rightarrow \chi_\varphi^{k+1}(y)) \right).$$

Then,

$$\begin{aligned} (\mathcal{G}, k) \in p\text{-VERTEX-COVER} & \\ \iff \text{for some } \ell \text{ with } 0 \leq \ell \leq k, \mathcal{G} \text{ has exactly } \ell \text{ vertices of degree } \geq k+1 \text{ and} & \\ \text{there is a } j \leq (k-\ell) \cdot (k+1) \text{ such that } \mathcal{G}' \text{ has } j \text{ vertices and} & \\ (\mathcal{G}', k-\ell) \text{ is a YES instance of } p\text{-VERTEX-COVER} & \\ \iff \mathcal{G} \models \bigvee_{0 \leq \ell \leq k} \left(\chi_{\chi_\varphi^{k+1}}^\ell \wedge \neg \chi_{\chi_\varphi^{k+1}}^{\ell+1} \wedge \bigvee_{0 \leq j \leq (k-\ell) \cdot (k+1)} (\chi_{\text{uni}}^j \wedge \neg \chi_{\text{uni}}^{j+1} \wedge \rho_j) \right). & \quad (3) \end{aligned}$$

Here the formula ρ_j , a formula expressing (in \mathcal{G} with a \mathcal{G}' with exactly j vertices) that \mathcal{G}' has a vertex cover of size $k - \ell$, still has to be defined. We do that by saying that \mathcal{G}' (with built-in arithmetic) is isomorphic to one of the graphs with j vertices (and with built-in arithmetics) that have vertex covers of size $k - \ell$. For this we have to be able to define an order of \mathcal{G}' by a formula of quantifier rank bounded by a constant number independent of k . Again this is done with the color-coding method: We find p and q , and $0 \leq i_0 < \dots < i_{j-1} < j^2$ with

$$h_{p,q}(G') = \{i_0, \dots, i_{j-1}\}.$$

Then, we can speak of the first, the second, . . . , vertex in \mathcal{G}' . Therefore ρ_j has the form

$$\exists p \exists q \left(\bigvee_{0 \leq i_0 < \dots < i_{j-1} < j^2} \left(\bigwedge_{s \in [j]} \exists y (\text{"}h_{p,q}(y) = i_s\text{"} \wedge \text{uni}(y)) \wedge \bigvee_{\substack{(\mathcal{H}, k-\ell) \in p\text{-VERTEX-COVER} \\ H=[j]}} \rho'_{\mathcal{H}} \right) \right),$$

where

$$\begin{aligned} \rho'_{\mathcal{H}} := & \bigwedge_{\substack{s,t \in [j] \\ E^{\mathcal{H}}_{st}}} \exists y \exists z (\text{uni}(y) \wedge \text{uni}(z) \wedge \text{"}h_{p,q}(y) = i_s\text{"} \wedge \text{"}h_{p,q}(z) = i_t\text{"} \wedge Eyz) \\ & \wedge \bigwedge_{\substack{s,t \in [j] \\ \neg E^{\mathcal{H}}_{st}}} \exists y \exists z (\text{uni}(y) \wedge \text{uni}(z) \wedge \text{"}h_{p,q}(y) = i_s\text{"} \wedge \text{"}h_{p,q}(z) = i_t\text{"} \wedge \neg Eyz). \end{aligned}$$

As $\text{qr}(\chi_{\varphi}^{k+1}) \leq 12$, we have $\text{qr}(\text{uni}(x)) \leq 13$. Thus, $\text{qr}(\chi_{\text{uni}}^j) \leq 16$ and $\text{qr}(\rho_j) \leq 17$. As the remaining formulas in (3) have at most quantifier rank 16, we get $p\text{-VERTEX-COVER} \in \text{XFO}_{17}$. \square

3. The hitting set problems with bounded hyperedge size

We consider the parameterized problem

<p><i>p</i>-d-HITTING-SET</p> <p><i>Input:</i> A hypergraph \mathcal{G} with edges of size at most d.</p> <p><i>Parameter:</i> $k \in \mathbb{N}$.</p> <p><i>Question:</i> Does \mathcal{G} have a hitting set of size k?</p>
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A hypergraph \mathcal{G} is a pair (V, E) , where V is a set, the set of vertices of \mathcal{G} , and every element of E is a hyperedge, that is, a nonempty subset of V . A hitting set in \mathcal{G} is a set H that intersects each hyperedge (that is, $H \cap e \neq \emptyset$ for all $e \in E$).

We view a hypergraph $\mathcal{G} := (V, E)$ as an $\{E_0, \varepsilon\}$ -structure $(V \cup E, E, \varepsilon^{\mathcal{G}})$, where E_0 is a unary relation symbol and ε is a binary relation symbol and

$$E_0^{\mathcal{G}} := E \quad \text{and} \quad \varepsilon^{\mathcal{G}} := \{(v, e) \mid v \in V, e \in E \text{ and } v \in e\}.$$

The goal of this section is to show:

Theorem 3.1. *Let $d \geq 1$. Then p -d-HITTING-SET is slicewise definable in FO with bounded quantifier rank; more precisely, p -d-HITTING-SET $\in \text{XFO}_q$ with $q = O(d^2)$.*

The following lemma can be viewed as a generalization of part of Buss' kernelization algorithm for p -VERTEX-COVER to p -d-HITTING-SET. The case for p -3-HITTING-SET was first shown in [12].

Lemma 3.2. Let (\mathcal{G}, k) with $\mathcal{G} = (V, E)$ be an instance of p - d -HITTING-SET. Let $1 < \ell \leq d$ and assume that every ℓ -set (i.e., set with exactly ℓ elements) of vertices has at most $k^{d-\ell}$ extensions in E .

If $v_1, \dots, v_{\ell-1}$ are pairwise distinct vertices such that there is a hitting set H of size $\leq k$ that contains none of these vertices, then $\{v_1, \dots, v_{\ell-1}\}$ has at most $k^{d-(\ell-1)}$ extensions in E .

Proof: Every hyperedge that extends $\{v_1, \dots, v_{\ell-1}\}$ must contain a vertex u of the hitting set H . By the assumptions, u is distinct from the v_i 's and therefore, the set $\{v_1, \dots, v_{\ell-1}, u\}$ has at most $k^{d-\ell}$ extensions in E . As $|H| \leq k$, we see that there are at most $k \cdot k^{d-\ell}$ ($= k^{d-(\ell-1)}$) extensions in E . \square

Let (\mathcal{G}, k) and $1 < \ell \leq d$ satisfy the hypotheses of the lemma, that is, (\mathcal{G}, k) with $\mathcal{G} = (V, E)$ is an instance of p - d -HITTING-SET and every ℓ -set has at most $k^{d-\ell}$ extensions in E . For every pairwise distinct vertices $v_1, \dots, v_{\ell-1}$ such that $\{v_1, \dots, v_{\ell-1}\}$ has more than $k^{d-(\ell-1)}$ extensions in E , we delete from E all hyperedges extending $\{v_1, \dots, v_{\ell-1}\}$ and add the hyperedge $\{v_1, \dots, v_{\ell-1}\}$. Let $\mathcal{G}^\ell = (V, E^\ell)$ be the the resulting hypergraph. Then:

- (a) For every pairwise distinct vertices $v_1, \dots, v_{\ell-1}$ there are at most $k^{d-(\ell-1)}$ hyperedges in E^ℓ extending $\{v_1, \dots, v_{\ell-1}\}$.
- (b) If H is a subset of V and $|H| \leq k$, then

$$H \text{ is a hitting set of } \mathcal{G} \iff H \text{ is a hitting set of } \mathcal{G}^\ell,$$

in particular,

$$(\mathcal{G}, k) \in p\text{-}d\text{-HITTING-SET} \iff (\mathcal{G}^\ell, k) \in p\text{-}d\text{-HITTING-SET}.$$

Let (\mathcal{G}, k) be an instance of p - d -HITTING-SET. For $\ell := d$ the hypothesis of Lemma 3.2 is fulfilled: Every d -set of vertices has at most one extension in E , namely at most, itself. Hence, applying the above procedure for $\ell = d$ we get the hypergraph \mathcal{G}^d , which satisfies the hypotheses of Lemma 3.2 for $\ell := d - 1$. So we get, again by the above procedure the hypergraph $(\mathcal{G}^d)^{\ell-1}$, which we denote by $\mathcal{G}^{\ell, \ell-1}$. Following this way, we finally obtain the hypergraph $\mathcal{G}^{\ell, \ell-1, \dots, 2}$, which we denote by \mathcal{G}' . Note that $\mathcal{G}' = (V, E')$ for some E' . From (a) and (b) we get (a') and (b').

(a') For every vertex v there are at most k^{d-1} hyperedges in E' containing v .

(b') If H is a subset of V and $|H| \leq k$, then

$$H \text{ is a hitting set of } \mathcal{G} \iff H \text{ is a hitting set of } \mathcal{G}',$$

Moreover,

(c') If $(\mathcal{G}, k) \in p$ - d -HITTING-SET, then $|E'| \leq k^d$ and $|V'| \leq d \cdot k^d$, where

$$V' := \{v \in V \mid \text{there is an } e \in E' \text{ with } v \text{ in } e\}$$

is the set of non-isolated vertices of \mathcal{G}' .

In fact, let H be a hitting set with $|H| = k$ of \mathcal{G} and hence, by (b') of \mathcal{G}' . As every hyperedge must contain a vertex of H , we get $|E'| \leq k^d$ from (a'). As every hyperedge $e \in E'$ contains at most d vertices, we have $|V'| \leq d \cdot k^d$.

We fix k and look at the k th slice of p - d -HITTING-SET. In the proof of Theorem 3.1 we will see that for hypergraphs \mathcal{G} sufficiently large compared with k we can FO-define \mathcal{G}' in \mathcal{G} . By (b') and (c'), we know that $(\mathcal{G}, k) \in p$ - d -HITTING-SET implies $|E'| \leq k^d$. By Theorem 2.2, we can express $|E'| \leq k^d$ in first-order logic with a bounded number of quantifiers if we add built-in addition and multiplication. Essentially this shows that p - d -HITTING-SET is eventually slicewise definable in FO with bounded quantifier rank and thus, p - d -HITTING-SET \in XFO_{qr} (by Proposition 2.6). This idea underlies the following proof of Theorem 3.1.

Proof of Theorem 3.1: To simplify the presentation we restrict ourselves to the case $d = 3$.

Let $\mathcal{G}_0 = (V_0 \cup E_0, E_0, \varepsilon)$ be a hypergraph with hyperedges of size at most three. Assume that $V_0 := \{1, \dots, n\}$.

To present the application of the color-coding method in a readable fashion we pass to a further structure \mathcal{H} . Let σ be the vocabulary $\{\text{Zero}, E, \text{First}, \text{Second}, \text{Third}, <\}$, where Zero and E are unary relation symbols and all others symbols are binary. Let \mathcal{H} be the σ -structure with

- $H = V^3$, the set of ordered triples of elements of $V := V_0 \cup \{0\} = \{0, 1, \dots, n\}$ (for technical reasons, in V we add 0 to V_0),
- $<^{\mathcal{H}}$ is the lexicographic order on V^3 ,
- $\text{Zero}^{\mathcal{H}} = \{(0, 0, 0)\}$,
- $\text{First}^{\mathcal{H}} = \left\{ ((u, v, w), (0, 0, u)) \mid (u, v, w) \in V^3 \right\}$,
- $\text{Second}^{\mathcal{H}} = \left\{ ((u, v, w), (0, 0, v)) \mid (u, v, w) \in V^3 \right\}$,
- $\text{Third}^{\mathcal{H}} = \left\{ ((u, v, w), (0, 0, w)) \mid (u, v, w) \in V^3 \right\}$,
- $E^{\mathcal{H}} = \left\{ (0, 0, w) \mid \{w\} \in E_0 \right\} \cup \left\{ (0, v, w) \mid \{v, w\} \in E^{\mathcal{G}}, v < w \right\} \cup \left\{ (u, v, w) \mid \{u, v, w\} \in E^{\mathcal{G}}, u < v < w \right\}$.

It is easy to show that \mathcal{H} can be defined by an FO-interpretation in \mathcal{G} .

We can express in \mathcal{H} that x is an i -set (for $i = 1, 2, 3$) by an FO-formula $\varphi_{i\text{-set}}$, where, say for $i = 2$,

$$\varphi_{2\text{-set}}(x) := \exists x_1 \exists x_2 \exists x_3 \left(\text{First } x x_1 \wedge \text{Zero } x_1 \wedge \text{Second } x x_2 \wedge \text{Third } x x_3 \wedge x_1 < x_2 < x_3 \right).$$

Similarly, there is an FO-formula $\varphi_{x \subseteq y}$ expressing that “ x and y are sets and that $x \subseteq y$.”

Fix $k \in \mathbb{N}$ and assume the vertex set V^3 of the hypergraph \mathcal{H} is sufficiently large compared with k . Furthermore, add built-in addition and multiplication to \mathcal{H} . Then we can FO-define in \mathcal{H} the hypergraph corresponding to the hypergraph \mathcal{H}^3 in the terminology introduced after Lemma 3.2. In the transition to \mathcal{H}^3 for every 2-set x , which has more than k extensions that are hyperedges, we have to delete all these hyperedges and then add the hyperedge x . Note that for the formula

$$\varphi(x, y) := \left(\varphi_{x \subseteq y}(x, y) \wedge E y \right)$$

the FO-formula $\chi_{\varphi}^{k+1}(x)$ expresses that “ x has more than k extensions that are hyperedges” (see Theorem 2.2). Thus, the new hyperedge relation (that is, the hyperedge relation corresponding to the

hypergraph \mathcal{H}^3 is given by

$$\begin{aligned} \varphi_{E^3}(x) := & \left((\varphi_{1\text{-set}}(x) \wedge Ex) \vee (\varphi_{2\text{-set}}(x) \wedge \chi_\varphi^{k+1}(x)) \right. \\ & \left. \vee (Ex \wedge \neg \exists y (\varphi_{2\text{-set}}(y) \wedge \varphi_{x \subseteq y}(y, x) \wedge \neg \chi_{\varphi(y,z)}^{k+1}(y))) \right). \end{aligned}$$

Similarly we can define the hyperedge relation corresponding to the hypergraph $\mathcal{H}^{3,2} = (V^3, E')$. By (b') and (c') on page 9, we know that

$$\begin{aligned} (\mathcal{H}, k) \in p\text{-3-HITTING-SET} & \iff (\mathcal{H}^{3,2}, k) \in p\text{-3-HITTING-SET}, \\ \text{and if } (\mathcal{H}, k) \in p\text{-}d\text{-HITTING-SET, then } |E'| & \leq k^3 \text{ and } |V'| \leq 3 \cdot k^3, \text{ where} \\ V' := \{v \in V^3 \mid \text{there is an } e \in E' \text{ with } v \in e\}. & \end{aligned}$$

So the k th slice of p - d -HITTING-SET can eventually be defined by a sentence expressing

$$|E'| \leq k^3 \text{ and } ((V', E'), k) \text{ is a YES instance of } p\text{-}d\text{-HITTING-SET}.$$

Again such a formula is obtained using Theorem 2.2 as in the proof of Theorem 1.1.

For the structures \mathcal{G} and \mathcal{H} without built-in addition and multiplication, we already saw that the second one can be obtained from the first one by an FO-interpretation. We need this result for the structures with built-in addition and multiplication, too. This follows from Proposition 3.3. Moreover, it is not hard to see that the final FO-sentence we obtain has quantifier rank $q = O(d^2)$. \square

A part of an FO-interpretation I is an FO-formula $\varphi_{uni}^I(x_1, \dots, x_s)$ defining the universe of the defined structure, that is: if I is an interpretation of σ -structures in a class \mathbf{K} of τ -structures, then for every structure $\mathcal{A} \in \mathbf{K}$ the set

$$(\varphi_{uni}^I)^{\mathcal{A}} := \{(a_1, \dots, a_s) \in A^s \mid \mathcal{A} \models \varphi(a_1, \dots, a_s)\}$$

is the universe of the σ -structure $I(\mathcal{A})$ defined by I in \mathcal{A} .

Assume that σ does not contain the relation symbols $<, +, \times$, but that the structures in \mathbf{K} are structures with built-in addition and multiplication, i.e., $\mathbf{K} \subseteq \text{ARITHM}[\tau]$. In general, we can not extend the interpretation I to an interpretation J such that

$$J(\mathcal{A}) = \left(I(\mathcal{A}), <^{J(\mathcal{A})}, +^{J(\mathcal{A})}, \times^{J(\mathcal{A})} \right)$$

has built-in addition and multiplication (that is, so that $J(\mathcal{A})$ is $I(\mathcal{A})$ together with an order and the corresponding addition and multiplication).

For example, for $\tau = \{P, <, +, \times\}$ with unary P let \mathbf{K} be the class of τ -structures \mathcal{A} with $P^{\mathcal{A}} \neq \emptyset$. Let σ be the empty vocabulary and consider the interpretation I yielding in \mathcal{A} the σ -structure with universe $P^{\mathcal{A}}$ (take $\varphi_{uni}^I(x) := Px$). If we could extend I to an interpretation J such that $J(\mathcal{A}) := (P^{\mathcal{A}}, <^{\mathcal{A}}, +^{\mathcal{A}}, \times^{\mathcal{A}})$ has built-in addition and multiplication, then we could express in $J(\mathcal{A})$, and thus in \mathcal{A} , that “ $P^{\mathcal{A}}$ is even,” i.e., the parity problem, which is well known to be impossible.

The next result shows that the situation is different if for $\varphi_{uni}^I(x_1, \dots, x_s)$ we have $(\varphi_{uni}^I)^{\mathcal{A}} = A^s$.

Proposition 3.3. *Let τ contain $<, +, \times$ and assume that none of these symbols is in the vocabulary σ . Let $\mathbf{K} \subseteq \text{ARITHM}[\tau]$ and let I be an interpretation of σ -structures in the structures in \mathbf{K} with $\varphi_{uni}^I = \varphi_{uni}^I(x_1, \dots, x_s)$. If for all $\mathcal{A} \in \mathbf{K}$,*

$$(\varphi_{uni}^I)^{\mathcal{A}} = A^s,$$

then the interpretation I can be extended to an interpretation of $\sigma \cup \{<, +, \times\}$ such that $J(\mathcal{A}) = (I(\mathcal{A}), <^{J(\mathcal{A})}, +^{J(\mathcal{A})}, \times^{J(\mathcal{A})})$ has built-in addition and multiplication for all $\mathcal{A} \in \mathbf{K}$.

Proof: Let $\mathcal{A} \in \mathbf{K}$ and assume $A = [n]$ and $<, +, \times$ have their natural interpretations. We define the extension $J(\mathcal{A})$ of $I(\mathcal{A})$ (the construction will be independent of \mathcal{A}). Of course, the lexicographic order of $[n]^s$ (the universe of $I(\mathcal{A})$) is FO-definable in \mathcal{A} . So we define J such that $<^{J(\mathcal{A})}$ is the lexicographic order. Then $(a_1, \dots, a_s) \in [n]^s$ is the element at the position

$$a_1 \cdot n^{s-1} + \dots + a_{s-1} \cdot n + a_s$$

in $<^{J(\mathcal{A})}$.

For $a, b \in A$ with $a + b \geq n$ and $0 \leq i < s$, we have

$$a \cdot n^i + b \cdot n^i = n^{i+1} + (a + b - n) \cdot n^i = n^{i+1} + (a - (n - b)) \cdot n^i$$

and $n - b, a - (n - b) \in A$. Thus, the built-in addition (with respect to the lexicographic order) can be FO-defined using $+^{\mathcal{A}}$ by formalizing the addition of base n numbers with at most s digits.

The FO-definition of the multiplication is not so easy. Note that

$$\sum_{i=1}^s a_i \cdot n^{s-i} \cdot \sum_{j=1}^s b_j \cdot n^{s-j} = \sum_{k=0}^{2s-2} \left(\sum_{i+j=2s-k} a_i b_j \right) \cdot n^k.$$

From this equation, we see that once we know how to FO-define in \mathcal{A} the product $(0, \dots, 0, a) \times (0, \dots, 0, b)$ with $a, b \in A$, we can FO-define $(a_1, \dots, a_s) \times (b_1, \dots, b_s)$ for arbitrary tuples in $[n]^s$. Of course, thereby taking into account whether this product is $< n^s$. As $a \cdot b < n^2$ for $a, b \in [n]$, we can restrict ourselves to the case $s = 2$, that is, we have to FO-define $(0, a) \times (0, b)$ with help of FO-definition of the addition. We assume $n > 2$ (and leave the case $n = 2$ to the reader, the case $n = 1$ being trivial).

For this purpose we consider the smallest element $e \in [n]$ such that $e^2 \geq n$ (exceeds $n - 1$) and the largest element $\ell \in [n]$ such that $\ell^2 \leq n - 1$. By $n > 2$, we have

$$e = \ell + 1 \quad \text{and} \quad \ell + \ell \leq n - 1 \tag{4}$$

and both, e and ℓ , are FO-definable in \mathcal{A} .

We first FO-define $(0, e) \times (0, e)$. This will allow us to FO-define $(0, a) \times (0, b)$, essentially by writing a and b in base e notation.

By (4), $e^2 - \ell^2 = e + \ell$. Hence, $e^2 = \ell + \ell + 1 + \ell^2 = n + \ell + \ell - ((n - 1) - \ell^2)$. Thus,

$$(0, e) \times (0, e) = (1, t) \quad \text{with} \quad t = \ell + \ell - ((n - 1) - \ell^2). \tag{5}$$

Note that $(n - 1) - \ell^2 \in A$, thus by (4), $t \in A$. Therefore we can FO-define $(0, e) \times (0, e)$.

With the following two claims we will obtain the full result.

Claim 1. For $d \leq \ell$ we can FO-define $d \cdot e$, $d \cdot t$, and $d \cdot e^2$.

Proof of Claim 1: $d \cdot e$: We have $d \cdot e = d \cdot (\ell + 1) = d \cdot \ell + d$. As $d \cdot \ell \leq \ell^2 \in A$, the claim follows.

$d \cdot t$: By (5), $t \leq 2 \cdot \ell$. Therefore there is $t' \leq \ell$ and $q \in \{0, 1\}$ with $t = t' + t' + q$. Hence, $d \cdot t = d \cdot t' + d \cdot t' + d \cdot q$. As $d \cdot t' \in A$ and $d \cdot q \in \{0, q\}$, the claim follows.

$d \cdot e^2$: We know that $(0, e) \times (0, e) = (1, t)$ and $(0, d) \times (1, t) = (0, d) \times (1, 0) + (0, d) \times (0, t)$. Clearly, $(0, d) \times (1, 0) = (d, 0)$. Furthermore, we know how to FO-define $d \cdot t$ by the previous step. Therefore, the claim follows. \dashv

The following result extends Claim 1.

Claim 2. For $d \leq n - 1$ we can FO-define $d \cdot e$, $d \cdot t$, and $d \cdot e^2$.

Proof of Claim 2: We write d in the form $d = d_1 \cdot e + d_2$ with $d_1, d_2 \leq \ell$ (recall that $(\ell + 1) \cdot e = e^2 \geq n$).

$d \cdot e$: We have $d \cdot e = d_1 \cdot e^2 + d_2 \cdot e$ and the result follows by Claim 1.

$d \cdot t$: By (5), $t \leq 2 \cdot \ell < 2 \cdot e$. Thus, there are $t_1 \in \{0, 1\}$ and $t_2 \leq \ell$ with $t = t_1 \cdot e + t_2$. Therefore

$$d \cdot t = d_1 \cdot t_1 \cdot e^2 + d_1 \cdot t_2 \cdot e + d_2 \cdot t_1 \cdot e + d_2 \cdot t_2.$$

If $t_1 \neq 0$, then $d_1 \cdot t_1 \cdot e^2 = d_1 \cdot e^2$. As $d_1 \leq \ell$, this term is FO-definable by Claim 1. As $d_1 \cdot t_2$ and $d_2 \cdot t_1$ are $\leq n - 1$, the corresponding terms are FO-definable by the first part of this claim.

$d \cdot e^2$: Recall that $e^2 = (1, t)$. We have $(0, d) \times (1, t) = (0, d) \times (1, 0) + (0, d) \times (0, t)$ and we just saw how to FO-define $d \cdot t$. \dashv

Now we turn to the general case. Let $a, b \in A$. We may write $a = a_1 \cdot e + a_2$ and $b = b_1 \cdot e + b_2$ with $a_1, a_2, b_1, b_2 \leq \ell$. Thus,

$$a \cdot b = a_1 \cdot b_1 \cdot e^2 + a_1 \cdot b_2 \cdot e + a_2 \cdot b_1 \cdot e + a_2 \cdot b_2.$$

As the products $a_1 \cdot b_1$, $a_1 \cdot b_2$, $a_2 \cdot b_1$, and $a_2 \cdot b_2$ are all $\leq n - 1$, the result follows by Claim 2. \square

The following result, applied in Section 5, extends Proposition 3.3 to interpretations whose universe are definable initial segments of a Cartesian product.

Corollary 3.4. *Let τ contain $<, +, \times$ and assume that none of these symbols is in the vocabulary σ . Let $\mathbf{K} \subseteq \text{ARITHM}[\tau]$ and let I be an interpretation of σ -structures in the structures in \mathbf{K} with $\varphi_{\text{uni}}^I = \varphi_{\text{uni}}^I(x_1, \dots, x_s)$. Furthermore, assume that there is an FO-formula $\varphi_{\text{init}}(x_1, \dots, x_s)$ such that for all $\mathcal{A} \in \mathbf{K}$, there is a unique tuple in A^s , we denote it by (a_1, \dots, a_s) , such that*

$$\mathcal{A} \models \varphi_{\text{init}}(a_1, \dots, a_s) \quad \text{and} \quad (\varphi_{\text{uni}}^I)^{\mathcal{A}} = \{(b_1, \dots, b_s) \in A^s \mid (b_1, \dots, b_s) <_{\text{lex}} (a_1, \dots, a_s)\}$$

(here $<_{\text{lex}}$ denotes the lexicographic order with respect to $<^{\mathcal{A}}$). Then I can be extended to an interpretation of $\sigma \cup \{<, +, \times\}$ such that $J(\mathcal{A}) = (I(\mathcal{A}), <^{J(\mathcal{A})}, +^{J(\mathcal{A})}, \times^{J(\mathcal{A})})$ has built-in addition and multiplication for all $\mathcal{A} \in \mathbf{K}$.

4. Fagin definability

Let $\varphi(X)$ be an FO $[\tau]$ -formula which for a, say r -ary, second-order variable X may contain atomic formulas of the form $Xx_1 \dots x_r$. Then the parameterized problem $\text{FD}_{\varphi(X)}$ Fagin-defined by $\varphi(X)$ is the problem

<p>$\text{FD}_{\varphi(X)}$</p> <p style="margin-left: 20px;"><i>Input:</i> A τ-structure \mathcal{A}.</p> <p style="margin-left: 20px;"><i>Parameter:</i> $k \in \mathbb{N}$.</p> <p style="margin-left: 20px;"><i>Question:</i> Decide whether there is an $S \subseteq A^r$ with $S = k$ and $\mathcal{A} \models \varphi(S)$.</p>

The following metatheorem improves [10, Theorem 4.4].

Theorem 4.1. *Let $\varphi(X)$ be an $\text{FO}[\tau]$ -formula without first-order variables occurring free and in which X does not occur in the scope of an existential quantifier or negation symbol. Then $\text{FD}_{\varphi(X)} \in \text{XFO}_{\text{qr}}$ that is, $\text{FD}_{\varphi(X)}$ is slicewise definable with bounded quantifier rank.*

Recall that we view a hypergraph $\mathcal{G} := (V, E)$ as an $\{E_0, \varepsilon\}$ -structure $(V \cup E, E, \varepsilon^{\mathcal{G}})$, where E_0 is a unary relation symbol and ε is a binary relation symbol and

$$E_0^{\mathcal{G}} := E \quad \text{and} \quad \varepsilon^{\mathcal{G}} := \{(v, e) \mid v \in V, e \in E \text{ and } v \in e\}.$$

Fix $d \in \mathbb{N}$. For $k \in \mathbb{N}$ we have (assuming $|V| \geq k$)

$$(\mathcal{G}, k) \in p\text{-}d\text{-HITTING-SET} \iff \text{for some } S \text{ with } |S| = k \text{ we have } (V \cup E, E, \varepsilon^{\mathcal{G}}) \models \varphi(S),$$

where $\varphi(X) := \forall e (E_0 e \rightarrow \forall x_1 \dots \forall x_d ((\forall x (x \varepsilon e \leftrightarrow \bigvee_{i=1}^d x_i = x) \rightarrow (Xx_1 \vee \dots \vee Xx_d)))$. By Theorem 4.1 we know that $\text{FD}_{\varphi(X)} \in \text{XFO}_{\text{qr}}$. Hence, $p\text{-}d\text{-HITTING-SET} \in \text{XFO}_{\text{qr}}$, so we get the result of the previous section. However here, to prove Theorem 4.1 we use the result of the previous section.

Proof of Theorem 4.1: For simplicity, let us assume that X is unary. Without loss of generality we can assume that

$$\varphi(X) = \forall y_1 \dots \forall y_\ell \bigwedge_{i=1}^m \bigvee_{j=1}^p \psi_{ij},$$

where each ψ_{ij} either is Xy_q for some $q \in \{1, \dots, \ell\}$, or a first-order formula with free variables in $\{y_1, \dots, y_\ell\}$ in which X does not occur.

Let (\mathcal{A}, k) be an instance of $\text{FD}_{\varphi(X)}$. We construct an instance $(\mathcal{G}(\mathcal{A}), k)$ of $p\text{-}\ell\text{-HITTING-SET}$ such that

$$(\mathcal{A}, k) \in \text{FD}_{\varphi(X)} \iff (\mathcal{G}(\mathcal{A}), k) \in p\text{-}\ell\text{-HITTING-SET}. \quad (6)$$

As $(\mathcal{G}(\mathcal{A}), k)$ we take the hypergraph (V, E) with $V = A$ and where E contains the following hyperedges. Let $\bar{a} \in A^\ell$ and $i \in \{1, \dots, m\}$. If

$$\mathcal{A} \models \neg \bigvee_{\substack{j \in \{1, \dots, p\} \\ X \text{ does not occur in } \psi_{ij}}} \psi_{ij}(\bar{a}).$$

then E contains the hyperedge $\{a_{s_1}, \dots, a_{s_t}\}$ where $Xy_{s_1}, \dots, Xy_{s_t}$ are exactly the disjuncts of the form Xy_{\dots} in $\bigvee_{j=1}^p \psi_{ij}$. If $t = 0$ (for some $\bar{a} \in A^\ell$), we take as $\mathcal{G}(\mathcal{A})$ a fixed hypergraph chosen in advance such that $(\mathcal{G}(\mathcal{A}), k)$ is a NO instance of $p\text{-}\ell\text{-HITTING-SET}$.

Since $\mathcal{G}(\mathcal{A})$ can be defined from \mathcal{A} by an FO-interpretation and $p\text{-}\ell\text{-HITTING-SET} \in \text{XFO}_{\text{qr}}$, we get $\text{FD}_{\varphi(X)} \in \text{XFO}_{\text{qr}}$. \square

Some parameterized problems can be shown to be in para-FO by a simple application of this theorem, e.g., for every $\ell \geq 1$, the problem $p\text{-WSAT}(\Gamma_{1,\ell}^+)$, the restriction of $p\text{-DOMINATING-SET}$ to graphs of degree ℓ , and the problem $p\text{-}\ell\text{-MATRIX-DOMINATION}$. Let us consider one example in detail

p- ℓ -MATRIX-DOMINATION

Input: An $n \times n$ matrix M with entries from $\{0, 1\}$, which has in every row and in every column at most ℓ ones and $k \in \mathbb{N}$.

Parameter: k .

Question: Is there a set S of k nonzero entries in M that dominate all others, in the sense that every nonzero entry in M is in the same row or in the same column as some element of S ?

We assign to such a matrix M the structure $\mathcal{A}(M) := ([n], \text{One}^{\mathcal{A}(M)})$, where $\text{One}^{\mathcal{A}(M)}$, the interpretation of the binary relation symbol *One*, is

$$\text{One}^{\mathcal{A}(M)} = \{(i, j) \in [n] \times [n] \mid \text{the } (i, j)\text{th entry of } M \text{ is } 1\}.$$

Then for instances (M, k) (with $|\text{One}^{\mathcal{A}(M)}| \geq k$), we have

$$(M, k) \in p\text{-}\ell\text{-MATRIX-DOMINATION} \iff \mathcal{A}(M) \in \text{FD}_{\varphi(X)},$$

where $\varphi(X)$ with binary X is the following formula:

$$\begin{aligned} \forall x \forall y \left(\text{One}xy \rightarrow \forall y_1 \dots \forall y_\ell \forall x_1 \dots \forall x_\ell \left(\forall z (\text{One}xz \leftrightarrow \bigvee_{1 \leq i \leq \ell} z = y_i) \wedge \right. \right. \\ \left. \left. (\forall z (\text{One}zy \leftrightarrow \bigvee_{1 \leq i \leq \ell} z = x_i)) \right) \rightarrow \bigvee_{1 \leq i \leq \ell} (Xxy_i \vee Xx_iy) \right). \end{aligned}$$

5. para-AC⁰ = XFO_{qr}

The importance of the class XFO_{qr} from the point of view of complexity theory stems from the fact that it coincides with the class para-AC⁰, the class of parameterized problems that are in dlogtime-uniform AC⁰ after a precomputation. As dlogtime-uniform AC⁰ contains precisely the class of parameterized problems definable in first-order logic, the class para-AC⁰ corresponds to the class para-FO of parameterized problems definable in first-order logic after a precomputation on the parameter (see [7, 6]). We deal here with the class para-FO and thus in this section aim to show para-FO = XFO_{qr}.

To define the class para-FO we need a notion of union of two arithmetical structures.

Definition 5.1. Assume $\mathcal{A} \in \text{ARITHM}[\tau]$ and $\mathcal{A}' \in \text{ARITHM}[\tau']$ satisfy

$$A \cap A' = \emptyset \quad \text{and} \quad \tau \cap \tau' = \{<, +, \times\}.$$

Let U be a new unary relation symbol. We set $\tau \uplus \tau' := \tau \cup \tau' \cup \{U\}$. Then $\mathcal{A} \uplus \mathcal{A}'$ is the structure $\mathcal{B} \in \text{ARITHM}(\tau \uplus \tau')$ with

- $B := A \cup A'$;
- $U^{\mathcal{B}} = A'$;
- $<^{\mathcal{B}} := <^{\mathcal{A}} \cup <^{\mathcal{A}'} \cup \{(a, a') \mid a \in A \text{ and } a' \in A'\}$, that is, the order $<^{\mathcal{B}}$ extends the orders $<^{\mathcal{A}}$ and $<^{\mathcal{A}'}$, and in $<^{\mathcal{B}}$ every element of A precedes every element of A' ;
- $R^{\mathcal{B}} := R^{\mathcal{A}}$ for $R \in \tau$ and $R^{\mathcal{B}} := R^{\mathcal{A}'}$ for $R \in \tau'$.

If $A \cap A' \neq \emptyset$, then we pass to isomorphic structures with disjoint universes before defining $\mathcal{A} \uplus \mathcal{A}'$.

Definition 5.2. Let $Q \subseteq \text{ARITHM}[\tau] \times \mathbb{N}$ be a parameterized problem. Q is *first-order definable after a precomputation*, in symbols $Q \in \text{para-FO}$, if for some vocabulary τ' there is a computable function $pre : \mathbb{N} \rightarrow \text{ARITHM}[\tau']$, a *precomputation*, and a sentence $\varphi \in \text{FO}[\tau \uplus \tau']$ such that for all $(\mathcal{A}, k) \in \text{ARITHM}[\tau] \times \mathbb{N}$,

$$(\mathcal{A}, k) \in Q \iff \mathcal{A} \uplus pre(k) \models \varphi.$$

The main result of this section reads as follows. It is the modeltheoretic analogue of the equivalence between (i) and (ii) of [6, Proposition 6].²

Theorem 5.3. $\text{para-FO} = \text{XFO}_{\text{qr}}$.

In the proof we shall need the following lemma. Its proof uses the fact that every computable function may be defined on the natural numbers (with addition and multiplication) by a Σ_1 -sentence (that is, by an FO-sentence of the form $\exists x_1 \dots \exists x_n \psi$ with quantifier free ψ).

Lemma 5.4. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a computable function. Then there is an $\text{FO}[\{<, +, \times\}]$ -formula $\psi_f(x, y)$ and an increasing and computable function $g : \mathbb{N} \rightarrow \mathbb{N}$ with $g(m) > f(m)$ for $m \in \mathbb{N}$ such that for all $n, a \in \mathbb{N}$ with $n \geq g(a)$ and $b \in [n]$,

$$([n], <^{[n]}, +^{[n]}, \times^{[n]}) \models \psi_f(a, b) \iff f(a) = b.$$

The obvious generalization of this result to functions $f : \mathbb{N}^s \rightarrow \mathbb{N}$ for some $s \geq 1$ holds, too.

Proof of Theorem 5.3: Assume that $Q \in \text{para-FO}$. Hence, for some vocabulary τ' there is a computable function $pre : \mathbb{N} \rightarrow \text{ARITHM}[\tau']$ and a sentence $\varphi \in \text{FO}[\tau \uplus \tau']$ such that for all $(\mathcal{A}, k) \in \text{ARITHM}[\tau] \times \mathbb{N}$,

$$(\mathcal{A}, k) \in Q \iff \mathcal{A} \uplus pre(k) \models \varphi.$$

Clearly, then Q is decidable. Therefore, by Lemma 2.6, it suffices to show that for some $q \in \mathbb{N}$ the problem Q is eventually slicewise definable in FO_q , that is, that there are an increasing and computable function $g : \mathbb{N} \rightarrow \mathbb{N}$ and computable functions $k \mapsto m_k$ and $k \mapsto \psi_k \in \text{FO}_q[\tau \cup C(m_k)]$ such that for all $(\mathcal{A}, k) \in \text{ARITHM}[\tau] \times \mathbb{N}$ with $|A| \geq g(k)$ we have

$$\mathcal{A} \uplus pre(k) \models \varphi \iff \mathcal{A}_{C(m_k)} \models \psi_k. \quad (7)$$

The main idea: As the precomputation pre is computable, for $(\mathcal{A}, k) \in \text{ARITHM}[\tau] \times \mathbb{N}$ with sufficiently large $|A|$ compared with $|pre(k)|$, we can FO-define $pre(k)$ in $\mathcal{A}_{C(k+1)}$. Furthermore, from \mathcal{A} and from this FO-defined $pre(k)$ in $\mathcal{A}_{C(k+1)}$ we get (an isomorphic copy of) $\mathcal{A} \uplus pre(k)$ in $\mathcal{A}_{C(k+1)}$ by an FO-interpretation. Summing up, we can FO-interpret $\mathcal{A} \uplus pre(k)$ in $\mathcal{A}_{C(k+1)}$. This FO-interpretation yields the desired ψ_k satisfying (7).

Some details: Let τ' , the vocabulary of $pre(k)$, be the set $\{<, +, \times, R_1, \dots, R_m\}$, where R_i is of arity r_i . Recall that pre is computable. Thus there is a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ with

$$f(k) = |pre(k)|.$$

²Proposition 6 in [6] contains a third statement equivalent to (i) and (ii). The corresponding modeltheoretic analogue *decidable and eventually in FO* also characterizes XFO_{qr} .

We may assume that the universe of $pre(k)$ is $[f(k)]$ and $<, +, \times$ have their natural interpretations in $pre(k)$. For easier presentation, let us assume that the same holds for \mathcal{A} ; so, in particular, $[|A|]$ is the universe of \mathcal{A} .

For i with $1 \leq i \leq m$ let $h_i : \mathbb{N}^{1+r_i} \rightarrow \{0, 1\}$ be the computable function with

$$h_i(k, b_1, \dots, b_{r_i}) = 1 \iff \left(b_1, \dots, b_{r_i} < f(k) \quad \text{and} \quad R_i^{pre(k)} b_1, \dots, b_{r_i} \right).$$

As f and h_1, \dots, h_m are computable, (we know that they are FO-definable in arithmetic and) by Lemma 5.4, there is a computable and increasing function $g : \mathbb{N} \rightarrow \mathbb{N}$ with $g(k) > f(k)$ and there are FO-formulas $\psi_f(x, y)$ and $\psi_{h_i}(x, y_1, \dots, y_{r_i})$ such that for the relevant arguments, the formulas $\psi_f(x, y)$ and $\psi_{h_i}(x, y_1, \dots, y_{r_i})$ correctly define f and h_i in models with built-in addition and multiplication of size $\geq g(k)$. Clearly, once we have the values $f(k)$ and $h_i(k, b_1, \dots, b_{r_i})$ for $1 \leq i \leq m$ and $b_1, \dots, b_{r_i} < f(k)$, we can first-order define $pre(k)$, and hence $(\mathcal{A}_{C(k+1)}, R_1^{pre(k)}, \dots, R_m^{pre(k)})$, in $\mathcal{A}_{C(k+1)}$, whenever $|A| \geq g(k)$.

By Corollary 3.4 there is an FO-interpretation yielding the structure $\mathcal{A} \uplus pre(k)$ from the structure $(\mathcal{A}_{C(k+1)}, R_1^{pre(k)}, \dots, R_m^{pre(k)})$. Putting these interpretations together, we obtain an FO-interpretation yielding $\mathcal{A} \uplus pre(k)$ in $\mathcal{A}_{C(k+1)}$ assuming $|A| \geq g(k)$. Thus we obtain from φ an FO-sentence ψ_k satisfying the equivalence (7).

Now assume that $Q \in \text{XFO}_{\text{qr}}$. Then there is a $q \in \mathbb{N}$ and computable functions $k \mapsto m_k$ with $m_k \in \mathbb{N}$ and $k \mapsto \varphi_k$ with $\varphi_k \in \text{FO}_q[\tau \cup C(m_k)]$ such that for all $(\mathcal{A}, k) \in \text{ARITHM}[\tau] \times \mathbb{N}$,

$$(\mathcal{A}, k) \in Q \iff \mathcal{A}_{C(m_k)} \models \varphi_k.$$

We have to find a precomputation $pre : \mathbb{N} \rightarrow \text{ARITHM}[\tau']$ and an $\text{FO}[\tau \uplus \tau']$ -sentence φ such that for all $(\mathcal{A}, k) \in \text{ARITHM}[\tau] \times \mathbb{N}$,

$$\mathcal{A}_{C(m_k)} \models \varphi_k \iff \mathcal{A} \uplus pre(k) \models \varphi. \tag{8}$$

Essentially $pre(k)$ is the parse tree of φ_k and the sentence φ expresses that $\mathcal{A}_{C(m_k)}$ satisfies the sentence given by this parse tree, that is, the sentence φ_k .

We can assume that every sentence of quantifier rank $\leq q$ (and thus, every φ_k) has the variables among x_1, \dots, x_q and is written as a disjunction of conjunctions of atomic formulas and of formulas starting with a quantifier.

Let p_k be the number of nodes of the parse tree of φ_k . The structure $pre(k) \in \text{ARITHM}[\tau']$ has universe $[\max\{p_k, m_k\}]$. The binary relation symbol E is interpreted by the edge relation of the parse tree. Then, besides E , the vocabulary τ' among others, will contain unary relations *Exists*, *Forall*, X_1, \dots, X_q , *And*, *Or*, and *Neg*. Furthermore, for every relational symbol $R \in \tau$ (for simplicity, we consider a binary R) we need in τ' the unary relation symbols

$$At-R, VII-R, \dots, VIq-R, V21-R, \dots, V2q-R$$

and the binary relation symbols

$$C1-R, C2-R.$$

For example, for a node u , for $1 \leq j \leq q$, and for $i < m_k$ we have:

$Exists\ u$	\iff	the node u corresponds to an existentially quantified variable
$X_j\ u$	\iff	the quantifier in u binds the variable x_j
$Or\ u$	\iff	u corresponds to a disjunction
$At-R\ u$	\iff	u corresponds to an atomic formula with the relation symbol R
$V1j-R\ u$	\iff	u corresponds to an atomic formula of the form $Rx_j \cdot$
$V2j-R\ u$	\iff	u corresponds to an atomic formula of the form $R \cdot x_j$
$C2-R\ u\ i$	\iff	u corresponds to an atomic formula of the form $R \cdot \bar{i}$.

We leave it to the reader to write down a sentence φ satisfying (8). □

Corollary 5.5. *For every $d \in \mathbb{N}$, p - d -HITTING-SET is in para-FO (and hence in para-AC⁰).*

6. The hierarchy $(FO_q)_{q \in \mathbb{N}}$ on arithmetical structures

Let $\tau_0 := \{<, +, \times\}$ and let τ be a vocabulary with $\tau_0 \subseteq \tau$. For $q \in \mathbb{N}$ by $FO_q[\tau] \subsetneq FO_{q+1}[\tau]$ on arithmetical structures we mean that there is an $FO_{q+1}[\tau]$ -sentence which is not equivalent to any $FO_q[\tau]$ -sentence on all finite τ -structures with built-in addition and multiplication. We say that the hierarchy $(FO_q)_{q \in \mathbb{N}}$ is strict on arithmetical structures if there is a vocabulary $\tau \supseteq \tau_0$ such that $FO_q[\tau] \subsetneq FO_{q+1}[\tau]$ on arithmetical structures for every $q \in \mathbb{N}$.

Theorem 6.1. *The hierarchy $(FO_q)_{q \in \mathbb{N}}$ is strict on arithmetical structures.*

Some preparations are in order. First, we recall how structures are represented by strings. Let τ be a relational vocabulary and $n \in \mathbb{N}$. We encode a τ -structure \mathcal{A} with $A = [n]$ by a binary string $\text{enc}(\mathcal{A})$ of length

$$\ell_{\tau, n} := \sum_{R \in \tau} n^{\text{arity}(R)}.$$

For instance, assume $\tau = \{E, P\}$ with binary E and unary P , then

$$\text{enc}(\mathcal{A}) = i_0 i_1 \cdots i_{n^2-1} j_0 j_1 \cdots j_{n-1},$$

where for every $a, b \in [n]$, $(i_{a+b \cdot n} = 1 \iff (a, b) \in E^{\mathcal{A}})$ and $(j_a = 1 \iff a \in P^{\mathcal{A}})$.

$$\begin{aligned} i_{a+b \cdot n} = 1 &\iff (a, b) \in E^{\mathcal{A}}, \\ j_a = 1 &\iff a \in P^{\mathcal{A}}. \end{aligned}$$

Let \mathbf{K} be a class of τ -structures. A family of circuits $(C_n)_{n \in \mathbb{N}}$ decides \mathbf{K} if

- (1) every C_n has $\ell_{\tau, n}$ inputs,
- (2) for $n \in \mathbb{N}$ and every τ -structure \mathcal{A} with $A = [n]$, $(\mathcal{A} \in \mathbf{K} \iff C_n(\text{enc}(\mathcal{A})) = 1)$.

Recall that for $n \in \mathbb{N}$ the classes Σ_n and Π_n of formulas are defined as follows: Σ_0 and Π_0 are the class of quantifier free formulas. The class Σ_{n+1} (the class Π_{n+1}) is the class of formulas of the form $\exists x_1 \dots \exists x_k \varphi$ with $\varphi \in \Pi_n$ and arbitrary k (of the form $\forall x_1 \dots \forall x_k \varphi$ with $\varphi \in \Sigma_n$ and arbitrary k).

Lemma 6.2. *Every FO-formula of quantifier rank q is logically equivalent to a Σ_{q+1} -formula and to a Π_{q+1} -formula.*

Proof: The proof is by induction on q . For $q = 0$ the claim is trivial. The induction step follows from the facts:

- An FO-formula of quantifier rank $q + 1$ is a Boolean combination of formulas of the form $\exists x\psi$ and $\forall x\psi$, where ψ has quantifier rank $\leq q$. In formulas of the form $\exists x\psi$ we replace, using the induction hypothesis, the formula ψ by an equivalent Σ_{q+1} -formula, in formulas of the form $\forall x\psi$ we replace the formula ψ by an equivalent Π_{q+1} -formula.
- Boolean combinations of Σ_{q+1} -formulas and of Π_{q+1} -formulas are equivalent to both, a Σ_{q+2} -formula and to a Π_{q+2} -formula.

□

Lemma 6.3. *Let $q \in \mathbb{N}$. Then for every sentence $\varphi \in \text{FO}_q$ there is a family of circuits $(C_n)_{n \in \mathbb{N}}$ of depth $\leq q + 2$ and size $n^{O(1)}$ which decides $\text{Mod}(\varphi) = \{\mathcal{A} \mid \mathcal{A} \models \varphi\}$. Moreover, the output of C_n is an OR gate, and the bottom layer of gates in C_n has fan-in bounded by a constant which only depends on φ .*

Proof: To simplify the discussion, we assume $q = 3$. The other cases can be proved along the same lines. By Lemma 6.2 the sentence φ is equivalent to a Σ_4 -sentence

$$\psi = \exists x_{1,1} \cdots \exists x_{1,i_1} \forall x_{2,1} \cdots \forall x_{2,i_2} \exists x_{3,1} \cdots \exists x_{3,i_3} \forall x_{4,1} \cdots \forall x_{4,i_4} \bigwedge_{p \in I_\wedge} \bigvee_{q \in I_\vee} \chi_{pq},$$

where I_\wedge and I_\vee are index sets and every χ_{pq} is a literal.

For $n \in \mathbb{N}$ we construct the desired circuit $C = C_n$ using the standard translation from FO-sentences to AC^0 -circuits. That is, every existential (universal) quantifier corresponds to a \vee (\wedge) gate with fan-in n ; the conjunction is translated to a \wedge gate with fan-in $|I_\wedge|$ and the disjunctions to \vee gates with fan-in $|I_\vee|$. Next we merge consecutive layers of gates that are all \wedge , or that are all \vee . The resulting circuit C_n is of depth $q + 2$. It has an OR as output gate and bottom fan-in bounded by $|I_\vee|$. □

Key to our proof of Theorem 6.1 are the following Boolean functions, also known as *Sipser functions*.

Definition 6.4 ([15, 5]). Let $d \geq 1$ and $m_1, \dots, m_d \in \mathbb{N}$. For every $i_1 \in [m_1], i_2 \in [m_2], \dots, i_d \in [m_d]$ we introduce a Boolean variable X_{i_1, \dots, i_d} . Define

$$f_d^{m_1, \dots, m_d} := \bigwedge_{i_1 \in [m_1]} \bigvee_{i_2 \in [m_2]} \cdots \bigodot_{i_d \in [m_d]} X_{i_1, \dots, i_d}, \quad (9)$$

where \bigodot is \vee if d is even, and \wedge otherwise. For every $d \geq 2$ and $m \geq 1$ we set

$$\text{Sipser}_d^m := f_d^{m_1, \dots, m_d}$$

with $m_1 = \lceil \sqrt{m/\log m} \rceil$, $m_2 = \cdots = m_{d-1} = m$, and $m_d = \lceil \sqrt{d/2 \cdot m \cdot \log m} \rceil$.

Observe that the size of Sipser_d^m is bounded by $m^{O(d)}$.

The following lower bound for Sipser_d^m is proved in [11]. We use the version presented as Theorem 4.2 in [14].

Theorem 6.5. *Let $d \geq 2$. Then there exists a constant $\beta_d > 0$ so that if a depth $d + 1$, bottom fan-in k circuit with an OR gate as the output and at most S gates in levels 1 through d computes Sipser_d^m , then either $S \geq 2^{m^{\beta_d}}$ or $k \geq m^{\beta_d}$.*

Proof of Theorem 6.1: $\text{FO}_0 \subsetneq \text{FO}_1$ is trivial by considering the sentence $\exists x Ux$ where U is a unary relation symbol. We still need to show that for an appropriate vocabulary $\tau \supseteq \tau_0$ it holds $\text{FO}_q[\tau] \subsetneq \text{FO}_{q+1}[\tau]$ on arithmetical structures for every $q \geq 1$.

Let $d, m \in \mathbb{N}$. We identify the function Sipser_d^m with the circuit in (9) which computes it. Let E be a binary relation symbol and U a unary relation symbol. Then we view the underlying (directed) graph of Sipser_d^m as a $\{E, U\}$ -structure $\mathcal{A}_{d,m}$ with

$$\begin{aligned} A_{d,m} &:= \{v_g \mid g \text{ a gate in } \text{Sipser}_d^m\}, \\ E^{A_{d,m}} &:= \{(v_{g'}, v_g) \mid g' \text{ is an input to } g\}, \\ U^{A_{d,m}} &:= \{v_g \mid g \text{ is an input to the output gate}\}. \end{aligned}$$

Let P be a unary relation symbol. Every assignment B of (truth values to the input nodes of) Sipser_d^m can be identified with $P^{A_{d,m}} := \{g \mid g \text{ an input gate assigned to TRUE by } B\}$. For $\tau' := \{E, U, P\}$ we define an $\text{FO}[\tau']$ -sentence φ_d such that for all m ,

$$\text{Sipser}_d^m(P^{A_{d,m}}) = \text{TRUE} \iff (\mathcal{A}_{d,m}, P^{A_{d,m}}) \models \varphi_d. \quad (10)$$

Fix $q \geq 1$. Assume q is even and set $d := q + 1$ (the case of odd q is treated similarly). We define inductively $\text{FO}[\tau']$ -formulas $\psi_\ell(x)$ by

$$\psi_0(x) := Px, \quad \text{and} \quad \psi_{\ell+1}(x) := \begin{cases} \forall y (Eyx \rightarrow \psi_\ell(y)) & \text{if } \ell \text{ is even,} \\ \exists y (Eyx \wedge \psi_\ell(y)) & \text{if } \ell \text{ is odd.} \end{cases}$$

We set (recall the definition of $U^{A_{d,m}}$)

$$\varphi_{q+1} := \forall x (Ux \rightarrow \psi_q(x)).$$

It is straightforward to verify that $\text{qr}(\varphi_{q+1}) = q + 1$ and that φ_{q+1} satisfies (10) (for $d = q + 1$).

Let $\tau := \tau' \cup \{<, +, \times\} = \{E, U, P, <, +, \times\}$. We define

$$\text{SIPSER}_{q+1} := \{\mathcal{A} \in \text{ARITHM}[\tau] \mid \mathcal{A} \models \varphi_{q+1}\}.$$

By definition the class SIPSER_{q+1} is axiomatizable in $\text{FO}_{q+1}[\tau]$. We show that SIPSER_{q+1} is not axiomatizable in $\text{FO}_q[\tau]$. For a contradiction, assume that $\text{SIPSER}_{q+1} = \text{Mod}(\varphi)$ for some $\varphi \in \text{FO}_q[\tau]$. Then by Lemma 6.3 there exists a family of circuits $(C_n)_{n \in \mathbb{N}}$ such that the following conditions are satisfied.

(C1) Every C_n has $\ell_{\tau,n}$ inputs, depth $q + 2$, and size $\ell_{\tau,n}^{O(1)}$.

(C2) The output of C_n is an OR gate, and its bottom fan-in is bounded by a constant.

(C3) For every $n \in \mathbb{N}$ and every τ -structure \mathcal{A} with $A = [n]$

$$\mathcal{A} \in \text{SIPSER}_{q+1} \iff C_n(\text{enc}(\mathcal{A})) = 1.$$

Let $m \in \mathbb{N}$ and let n be the number of variables in Sipser_{q+1}^m , i.e.,

$$n = \left\lceil \sqrt{m/\log m} \right\rceil \cdot m^{q-1} \cdot \left\lceil \sqrt{(q+1)/2 \cdot m \cdot \log m} \right\rceil.$$

Consider the structure $\mathcal{A}_{q+1,m}$ associated with Sipser_{q+1}^m and expand it with $<, +, \times$. Thus for any assignment of the n inputs, identified with the unary relation $P^{\mathcal{A}_{q+1,m}}$, we have

$$\begin{aligned} \text{Sipser}_{q+1}^m(P^{\mathcal{A}_{q+1,m}}) = 1 &\iff (\mathcal{A}_{q+1,m}, <, +, \times, P^{\mathcal{A}_{q+1,m}}) \models \varphi \\ &\iff C_n(\text{enc}(\mathcal{A}_{q+1,m}, <, +, \times, P^{\mathcal{A}_{q+1,m}})) = 1. \end{aligned}$$

Here is the crucial observation. In the string $\text{enc}(\mathcal{A}_{q+1,m}, <, +, \times, P^{\mathcal{A}_{q+1,m}})$ only the last n bits depend on the assignment, that is, on $P^{\mathcal{A}_{q+1,m}}$. These are precisely the n input bits for the Sipser_{q+1}^m function. Thus we can simplify the circuit C_n by fixing the values of the first $\ell_{\tau,n} - n$ inputs according to $(\mathcal{A}_{q+1,m}, <, +, \times)$. Let C_n^* be the resulting circuit. We have

$$\text{Sipser}_{q+1}^m(P^{\mathcal{A}_{q+1,m}}) = 1 \iff C_n^*(P^{\mathcal{A}_{q+1,m}}) = 1.$$

By (C1), C_n^* has depth $q+2$ and size $n^{O(1)}$ (as $\ell_{\tau,n} = n^{O(1)}$). By (C2) its output is an OR gate, and its bottom fan-in is bounded by a constant. As $m \in \mathbb{N}$ is arbitrary, this clearly contradicts Theorem 6.5. \square

Proof of Theorem 1.2: Let $q \in \mathbb{N}$. By Theorem 6.1 we know that there is a vocabulary τ and an $\text{FO}_{q+1}[\tau]$ -sentence φ which is not equivalent to any $\text{FO}_q[\tau]$ -sentences on arithmetical structures. We claim that

$$Q := \{(\mathcal{A}, 0) \mid \mathcal{A} \in \text{ARITHM}[\tau] \text{ and } \mathcal{A} \models \varphi\}$$

is not slicewise definable in FO_q . As Q is slicewise definable in FO_{q+1} , this would give us the desired separation.

Assume otherwise, then, by Definition 2.4, there is a constant $m_0 \in \mathbb{N}$ and a sentence ψ in $\text{FO}_q[\tau \cup C(m_0)]$ such that for every $\mathcal{A} \in \text{ARITHM}[\tau]$

$$\mathcal{A} \models \varphi \iff \mathcal{A}_{C(m_0)} \models \psi.$$

This does not give us a contradiction immediately, since ψ might contain constants in $C(m_0)$. But it is easy to see that Lemma 6.2 and Lemma 6.3 both survive in the presence of constants. Thus almost the same proof of Theorem 6.1 shows that $\psi \in \text{FO}_q[\tau \cup C(m_0)]$ cannot exist. \square

7. Conclusions

We have shown that a few parameterized problems are slicewise definable in first-order logic with bounded quantifier rank. In particular, the k -vertex-cover problem, i.e., the k th slice of p -VERTEX-COVER, is definable in FO_{17} for every $k \in \mathbb{N}$. One natural follow-up question is whether this is optimal. Or can we show at least that p -VERTEX-COVER $\notin \text{XFO}_2$? Such a question is reminiscent of the recent quest for optimal algorithms for natural polynomial time solvable problems (see e.g., [2]). In our result p - d -HITTING-SET $\in \text{XFO}_q$ we have $q = O(d^2)$, and we conjecture that there is no universal constant q which works for every p - d -HITTING-SET. But so far, we do not know how to prove such a result.

It turns out that the class XFO_{qr} coincides with the parameterized circuit complexity class para-AC^0 which has been intensively studied in [3, 6]. Similar to [3], it seems that all the non-trivial examples in XFO_{qr} require the color-coding technique. It would be interesting to see whether other tools from parameterized complexity can be used to show membership in XFO_{qr} .

We have also established the strictness of $(XFO_q)_{q \in \mathbb{N}}$ by proving that $FO_q \subsetneq FO_{q+1}$ on arithmetical structures for every $q \in \mathbb{N}$. Our proof is built on a strict AC^0 -hierarchy on Sipser functions. We conjecture that the sentence

$$\exists x_1 \cdots \exists x_{q+1} \bigwedge_{1 \leq i < j \leq q+1} E_{x_i x_j},$$

which characterizes the existence of a $(q + 1)$ -clique, witnesses $FO_q \subsetneq FO_{q+1}$ on graphs with built-in addition and multiplication. Rossman [13] has shown that $(q + 1)$ -clique cannot be expressed in arithmetical structures with $\lfloor (q + 1)/4 \rfloor$ variables and hence not in $FO_{\lfloor (q+1)/4 \rfloor}$. This already shows that the hierarchy $(FO_q)_{q \in \mathbb{N}}$ does not collapse.

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