The Constant Inapproximability of the Parameterized Dominating Set Problem

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Abstract

We prove that there is no fpt-algorithm that can approximate the dominating set problem with any constant ratio, unless FPT $=$ W$[1]$. Our hardness reduction is built on the second author's recent W$[1]$-hardness proof of the biclique problem [23]. This yields, among other things, a proof without the PCP machinery that the classical dominating set problem has no polynomial time constant approximation under the exponential time hypothesis.

1. Introduction

The dominating set problem, or equivalently the set cover problem, was among the first problems proved to be NP-hard [21]. Moreover, it has been long known that the greedy algorithm achieves an approximation ratio $\approx \ln n$ [20, 32, 24, 9, 31]. And after a sequence of papers (e.g. [25, 30, 17, 1, 11]), this is proved to be best possible. In particular, Raz and Safra [30] showed that the dominating set problem cannot be approximated with ratio $c \cdot \log\, n$ for some constance $c \in \mathbb{N}$ unless P $=$ NP [30]. Under a stronger assumption NP $\not\subseteq$ DTIME $\left(n^{O(\log\, \log\, n)}\right)$ Feige proved that no approximation within $(1 - \varepsilon) \ln n$ is feasible [17]. Finally Dinur and Steuer established the same lower bound assuming only P $\neq$ NP [11]. However, it is important to note that the approximation ratio $\ln n$ is measured in terms of the size of an input graph G, instead of $\gamma(G)$, i.e., the size of its minimum dominating set. As a matter of fact, the standard examples for showing the $\Theta(\log\, n)$ greedy lower bound have constant-size dominating sets. Thus, the size of the greedy solutions cannot be bounded by any function of $\gamma(G)$. So the question arises whether there is an approximation algorithm $\mathbb{A}$ that always outputs a dominating set whose size can be bounded by $\rho(\gamma(G)) \cdot \gamma(G)$, where the function $\rho : \mathbb{N} \to \mathbb{N}$ is known as the approximation ratio of $\mathbb{A}$. The constructions in [17, 1] indeed show that we can rule out $\rho(x) \le \ln x$. For any $c < 1/2$ and $\delta_c(x) = 1/(\log\, \log\, x)^c$, in [28] under the assumption that SAT cannot be solved in time $2^{O(2^{\log^{1-\delta_c(n)} n})}$, Nelson proved that the set cover problem has no approximation within $2^{\log^{1-\delta_c(m)} m}$, where m is the number of given sets. This clearly implies that, under the same assumption, for the dominating set problem the approximation ratio $\rho(\gamma(G))$ cannot be bounded by $2^{\log^{1-\delta_c(\gamma(G))} \gamma(G)}$ either. In particular, this rules out polylogarithmic approximation. To the best of our knowledge, it is not known whether this bound is tight. For instance, it is conceivable that there is *no* a polynomial time algorithm that always outputs a dominating set of size at most $2^{2^{\gamma(G)}}$.

Other than looking for approximate solutions, parameterized complexity [13, 18, 29, 14, 10] approaches the dominating set problem from a different perspective. With the expectation that in practice we are mostly interested in graphs with relatively small dominating sets, algorithms of running time

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$2^{\gamma(G)} \cdot |G|^{O(1)}$ can still be considered efficient. Unfortunately, it turns out that the parameterized dominating set problem is complete for the second level of the so-called W-hierarchy [12], and thus fixed-parameter intractable unless FPT $=$ W$[2]$. So one natural follow-up question is whether the problem can be approximated in fpt-time. More precisely, we aim for an algorithm with running time $f(\gamma(G)) \cdot |G|^{O(1)}$ which always outputs a dominating set of size at most $\rho(\gamma(G)) \cdot \gamma(G)$. Here, $f : \mathbb{N} \to \mathbb{N}$ is an arbitrary computable function. The study of parameterized approximability was initiated in [5, 7, 15]. Compared to the classical polynomial time approximation, the area is still in its very early stage with few known positive and even less negative results.

Our results. We prove that any constant-approximation of the parameterized dominating set problem is W$[1]$-hard.

Theorem 1.1. *For any constant $c \in \mathbb{N}$ there is* no fpt-*algorithm $\mathbb{A}$ such that on every input graph G the algorithm $\mathbb{A}$ outputs a dominating set of size at most $c \cdot \gamma(G)$, unless* FPT $=$ W$[1]$ *(which implies that the exponential time hypothesis (*ETH*) fails).*

In the above statement, clearly we can replace "fpt-algorithm" by "polynomial time algorithm," thereby obtaining the classical constant-inapproximability of the dominating set problem. But let us mention that our result is not comparable to the classical version, even if we restrict ourselves to polynomial time tractability. The assumption FPT $\neq$ W$[1]$ or ETH is apparently much stronger than P $\neq$ NP, and in fact ETH implies both NP $\not\subseteq$ DTIME $\left(n^{O(\log\log n)}\right)$ used in aforementioned Feige's result and the assumption in Nelson's result. But on the other hand, our lower bound applies even in case that we know in advance that a given graph has small dominating sets.

Corollary 1.2. *Let $\beta : \mathbb{N} \to \mathbb{N}$ be a nondecreasing and unbounded computable function. Consider the following promise problem.*

MIN-DOMINATING-SET$_\beta$	
Instance:	A graph $G = (V, E)$ with $\gamma(G) \le \beta(|V|)$.
Solution:	A dominating set D of G.
Cost:	$|D|$.
Goal:	min.

Then there is no polynomial time constant approximation algorithm for MIN-DOMINATING-SET$_\beta$*, unless* FPT $=$ W$[1]$.

The proof of Theorem 1.1 is crucially built on a recent result of the second author [23] which shows that the parameterized biclique problem is W$[1]$-hard. We exploit the gap created in its hardness reduction (see Section 2.1 for more details). In the known proofs of the classical inapproximability of the dominating set problem, one always needs the PCP theorem in order to have such a gap, which makes those proofs highly non-elementary. More importantly, it can be verified that reductions based on the PCP theorem produce instances with optimal solutions of relatively large size, e.g., a graph $G = (V, E)$ with $\gamma(G) \ge |V|^{\Theta(1)}$. This is inevitable, since otherwise we might be able to solve every NP-hard problem in subexponential time. As an example, if it is possible to reduce an NP-hard problem to the approximation of MIN-DOMINATING-SET$_\beta$ for $\beta(n) = \log\log\log n$, then by brute-force searching for a minimum dominating set, we are able to solve the problem in time $n^{O(\log\log\log n)}$. It implies NP $\subseteq$ DTIME $\left(n^{O(\log\log\log n)}\right)$. Because of this, Corollary 1.2, and hence also Theorem 1.1, is unlikely provable following the traditional approach.

Using a result of Chen et.al. [6] the lower bound in Theorem 1.1 can be further sharpened.

Theorem 1.3. *Assume* ETH *holds. Then there is* no fpt-*algorithm which on every input graph* G *outputs a dominating set of size at most* $\sqrt[4+\varepsilon]{\log(\gamma(G))} \cdot \gamma(G)$ *for every* $0 < \varepsilon < 1$.

Related work. The existing literature on the dominating set problem is vast. The most relevant to our work is the classical approximation upper and lower bounds as explained in the beginning. But as far as the parameterized setting is concerned, what was known is rather limited.

Downey et. al proved that there is no *additive* approximation of the the parameterized dominating set problem [16]. In the same paper, they also showed that the independent dominating set problem has no fpt approximation with any approximation ratio. Recall that an independent dominating set is a dominating set which is an independent set at the same time. With this additional requirement, the problem is no longer *monotone,* i.e., a superset of a solution is not necessarily a solution. Thus it is unclear how to reduce the independent dominating set problem to the dominating set problem by an approximation-preserving reduction.

In [8, 19] it is proved under ETH that there is no $c\sqrt{\log \gamma(G)}$-approximation algorithm for the dominating set problem[1] with running time $2^{O(\gamma(G)^{(\log \gamma(G))^d})}|G|^{O(1)}$, where c and d are some appropriate constants. With the additional *Projection Game Conjecture* due to [27] and some of its further strengthening, the authors of [8, 19] are able to even rule out $\gamma(G)^c$-approximation algorithms with running time almost doubly exponential in terms of $\gamma(G)$. Clearly, these lower bounds are against far better approximation ratio than those of Theorem 1.1 and Theorem 1.3, while the drawback is that the dependence of the running time on $\gamma(G)$ is not an arbitrary computable function.

Using the so-called *linear PCP Conjecture (LPC)* Bonnet et al. showed [4] that the parameterized dominating set problem has no fpt approximation for *some specific constant ratio*. LPC is the statement that 3SAT has a PCP verifier which only uses log $n + O(1)$ random bits, improving the classical $O(\log n)$ random bits. It is worth noting that the status of LPC is far from being clear.

The dominating set problem can be understood as a special case of the weighted satisfiability problem of CNF-formulas, in which all literals are positive. The weighted satisfiability problems for various fragments of propositional logic formulas, or more generally circuits, play very important roles in parameterized complexity. In particular, they are complete for the W-classes. In [7] it is shown that they have no fpt approximation of any possible ratio, again by using the non-monotoncity of the problems. Marx strengthened this result significantly in [26] by proving that the weighted satisfiability problem is not fpt approximable for circuits of depth 4 without negation gates, unless FPT $=$ W$[2]$. Our result can be viewed as an attempt to improve Marx's result to depth-2 circuits, although at the moment we are only able to rule out fpt approximations with constant ratio.

As far as *upper bounds* are concerned, there is no known fpt-algorithm which always computes a dominating set of size at most $\rho(\gamma(G)) \cdot \gamma(G)$, where $\rho : \mathbb{N} \to \mathbb{N}$ is an arbitrary function.

Organization of the paper. We fix our notations in Section 2. In the same section we also explain the result in [23] key to our proof. Using a weighted version of the dominating set problem, we explain the underlying idea of our gap reduction in Section 3. To help readability, we first prove that the dominating set problem is not fpt approximable with ratio smaller than $3/2$ in Section 4. In the case of the clique problem, once we have inapproximability for a particular constant ratio, it can be easily improved to any constant by gap-amplification via graph products. But dominating sets for general graph products are notoriously hard to understand (see e.g. [22]). So to prove Theorem 1.1, Section 5 presents a modified reduction which contains a tailor-made graph product. Section 6 discusses some

[1]The papers actually address the set cover problem, which is equivalent to the dominating set problem as mentioned in the beginning.

consequences of our results. We conclude in Section 7.

2. Preliminaries

We assume familiarity with basic combinatorial optimizations and parameterized complexity, so we only introduce those notions and notations central to our purpose. The reader is referred to the standard textbooks (e.g., [3] and [13, 18]) for further background.

$\mathbb{N}$ and $\mathbb{N}^+$ denote the sets of natural numbers (that is, nonnegative integers) and positive integers, respectively. For every $n \in \mathbb{N}$ we let $[n] := \{1, \dots, n\}$. $\mathbb{R}$ is the set of real numbers, and $\mathbb{R}_{\geq 1} := \big\{r \in \mathbb{R} \bigm| r \geq 1\big\}$. For a function $f : A \to B$ we can extend it to sets and vectors by defining $f(S) := \{f(x) \mid x \in S\}$ and $f(\boldsymbol{v}) := \big(f(v_1), f(v_2), \cdots, f(v_k)\big)$, where $S \subseteq A$ and $\boldsymbol{v} = (v_1, v_2, \cdots, v_k) \in A^k$ for some $k \in \mathbb{N}^+$.

Graphs $G = (V, E)$ are always simple, i.e., undirected and without loops and multiple edges. Here, V is the vertex set and E the edge set, respectively. The *size* of G is $|G| := |V| + |E|$. A subset $D \subseteq V$ is a *dominating set* of G, if for every $v \in V$ either $v \in D$ or there exists a $u \in D$ with $\{u, v\} \in E$. In the second case, we might say that v is dominated by u, and this can be easily generalized to v dominated by a set of vertices. The *domination number* $\gamma(G)$ of G is the size of a smallest dominating set. The classical *minimum dominating set problem* is to find such a dominating set:

> MIN-DOMINATING-SET
> *Instance:* A graph $G = (V, E)$.
> *Solution:* A dominating set D of G.
> *Cost:* $|D|$.
> *Goal:* min.

The decision version of MIN-DOMINATING-SET has an additional input $k \in \mathbb{N}$. Thereby, we ask for a dominating set of size at most k instead of $\gamma(G)$. But it is well known that two versions can be reduced to each other in polynomial time. In parameterized complexity, we view the input k as the parameter and thus obtain the standard parameterization of MIN-DOMINATING-SET:

> p-DOMINATING-SET
> *Instance:* A graph G and $k \in \mathbb{N}$.
> *Parameter:* k.
> *Problem:* Decide whether G has a dominating set of size at most k.

As mentioned in the Introduction, p-DOMINATING-SET is complete for the parameterized complexity class W[2], the second level of the W-hierarchy. We will need another important parameterized problem, the *parameterized clique problem*

> p-CLIQUE
> *Instance:* A graph G and $k \in \mathbb{N}$.
> *Parameter:* k.
> *Problem:* Decide whether G has a clique of size at most k.

which is complete for W[1]. Recall that a subset $S \subseteq V$ is a *clique* in $G = (V, E)$, if for every $u, v \in S$ we have either $u = v$ or $\{u, v\} \in E$.

Those W-classes are defined by weighted satisfiability problems for propositional formulas and circuits. As they will be used only in Section 6, we postpone their definition until then.

Parameterized approximability. We follow the general framework of [7]. However, to lessen the notational burden we restrict our attention to the approximation of the dominating set problem.

Definition 2.1. Let $\rho : \mathbb{N} \to \mathbb{R}_{\geq 1}$. An algorithm $\mathbb{A}$ is a *parameterized approximation algorithm* for p-DOMINATING-SET with approximation ratio ρ if for every graph G and $k \in \mathbb{N}$ with $\gamma(G) \leq k$ the algorithm $\mathbb{A}$ computes a dominating set D of G such that

$$|D| \leq \rho(k) \cdot k.$$

If the running time of $\mathbb{A}$ is bounded by $f(k) \cdot |G|^{O(1)}$ where $f : \mathbb{N} \to \mathbb{N}$ is computable, then $\mathbb{A}$ is an fpt approximation algorithm.

One might also define parameterized approximation directly for MIN-DOMINATING-SET by taking $\gamma(G)$ as the parameter. The next result shows that essentially this leads to the same notion.

Proposition 2.2 ([7, Proposition 5])**.** *Let $\rho : \mathbb{N} \to \mathbb{R}_{\geq 1}$ be a function such that $\rho(k) \cdot k$ is nondecreasing. Then the following are equivalent.*

(1) p-DOMINATING-SET *has an* fpt *approximation algorithm with approximation ratio ρ.*

(2) There exists a computable function $g : \mathbb{N} \to \mathbb{N}$ and an algorithm A that on every graph G computes a dominating set D of G with $|D| \leq \rho(\gamma(G)) \cdot \gamma(G)$ in time $g(\gamma(G)) \cdot |G|^{O(1)}$.

The Color-Coding.

Lemma 2.3 ([2])**.** *For every $n, k \in \mathbb{N}$ there is a family $\Lambda_{n,k}$ of polynomial time computable functions from $[n]$ to $[k]$ such that for every k-element subset X of $[n]$, there is an $h \in \Lambda_{n,k}$ such that h is injective on X. Moreover, $\Lambda_{n,k}$ can be computed in time $2^{O(k)} \cdot n^{O(1)}$.*

2.1. One side gap for the biclique problem. Our starting point is the following theorem proved in [23] which states that, on input a bipartite graph, it is W[1]-hard to distinguish whether there exist k vertices with large number of common neighbors or every k-vertex set has small number of common neighbors.

Theorem 2.4 ([23, Theorem 1.3])**.** *There is a polynomial time algorithm $\mathbb{A}$ such that for every graph G with n vertices and $k \in \mathbb{N}$ with $\left\lceil n^{\frac{6}{k+6}} \right\rceil > (k+6)!$ and $6 \mid k+1$ the algorithm $\mathbb{A}$ constructs a bipartite graph $H = (A \dot\cup B, E)$ satisfying:*

(1) if G contains a clique of size k, i.e., $K_k \subseteq G$, then there are s vertices in A with at least $\left\lceil n^{\frac{6}{k+1}} \right\rceil$ common neighbors in B;

(2) otherwise $K_k \nsubseteq G$, every s vertices in A have at most $(k+1)!$ common neighbors in B,

where $s = \binom{k}{2}$.

In our reductions from p-CLIQUE to p-DOMINATING-SET, we use the following procedure to ensure that the instance (G, k) of p-CLIQUE satisfies $6 \mid k + 1$.

Preprocessing. On input a graph G and $k \in \mathbb{N}^+$, if 6 does not divide $k + 1$, let k' be the minimum integer such that $k' \geq k$ and $6 \mid k' + 1$. We construct a new graph G' by adding a clique with $k' - k$ vertices into G and making every vertex of this clique adjacent to other vertices in G. It is easy to see that $k' \leq k + 5$, and G contains a k-clique if and only if G' contains a k'-clique. Then we proceed with $G \leftarrow G'$ and $k \leftarrow k'$.

3. Overview of Our Reduction

To give a brief overview of our reduction, let us consider the weighted version of the minimum dominating set problem.

MIN-WEIGHTED-DOMINATING-SET
Instance: A graph G and $w : V(G) \to \mathbb{R} \cup \{\infty\}$.
Solution: $D \subseteq V(G)$ is a dominating set of G.
Cost: $w(D) := \sum_{v \in D} w(v)$.
Goal: min.

For a given weight function w, let $\gamma_w(G)$ be the minimum weight of a dominating set of G.

We show that approximating the minimum weighted dominating set problem to a ratio close to 2 is W[1]-hard. The starting point is Theorem 2.4, which reduces an instance (G, k) of p-CLIQUE to a bipartite graph $H = (A \mathbin{\dot\cup} B, E)$ such that one of the following conditions is satisfied, depending on whether (G, k) is a yes instance.

(yes) There are s vertices in A with d common neighbors in B.

(no) Every s vertices in A have at most ℓ common neighbors in B.

Here, $s = \binom{k}{2}$, and d, ℓ both depend on k with d being far larger than ℓ, in particular

$$\varepsilon\sqrt{d} \gg \ell, \tag{1}$$

where ε is some small constant.

Our goal is to construct a graph G' and a weight function $w : V(G') \to \mathbb{R} \cup \{\infty\}$ such that, in the cases of (yes), we have $\gamma_w(G') \leq (1 + \varepsilon)d$; otherwise $\gamma_w(G') \geq (1 - \varepsilon)2d$ for the cases of (no). This reduction is illustrated in Figure 1.

We partition B into d disjoint subsets $B_1, \ldots, B_d$. In the above (yes) cases, using color-coding, each B_i is supposed to contain exactly one of the d common neighbors of an s-vertex set. Moreover, for each $i \in [d]$, we introduce a vertex set W_i whose purpose will become clear shortly.

Let $t := d^{1-1/2s}$ and we assign to the vertices in A weight t. On the other hand, the vertices in W_i's have weight ∞, and vertices in B_i's have weight 1. As a consequence, every finite-weighted dominating set contains no vertices from W_i's. Finally, we add edges between A and W_i's, and between each W_i and B_i in such a way that for every finite-weighted set D dominating all the vertices in W_i,

(C1) either D contains one vertex in B_i and its s neighbors in A,

(C2) or D contains at least c ($c = 2$) vertices in B_i.

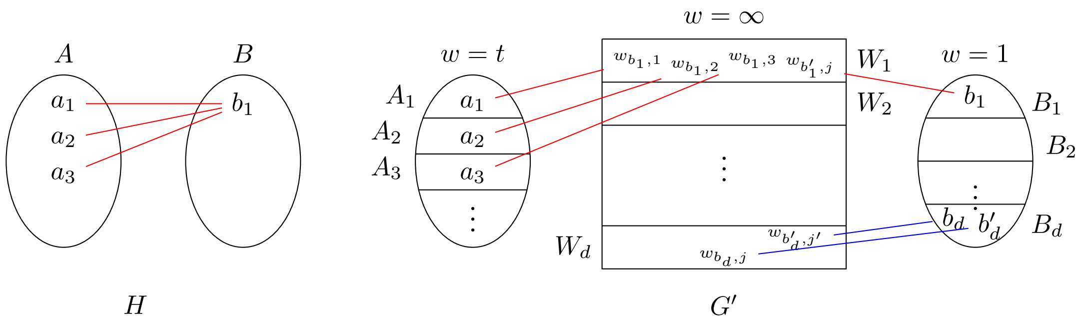

Figure 1. The fpt-reduction for MIN-WEIGHTED-DOMINATING-SET creating a gap close to 2

To achieve these, again using color-coding, we partition $A = \dot{\bigcup}_{i \in [s]} A_i$ such that in the (yes) cases each A_i contains one of the stated s vertices. For every $i \in [d]$ we let

$$W_i := \{ w_{b,j} \mid b \in B_i \text{ and } j \in [s] \}.$$

Then we add edges between every $b \in B_i$ and $w_{b',j}$ provided $b \neq b'$, and edges between every $w_{b,j}$ and every $a \in A_j$ if $\{a, b\}$ is an edge in H. Now (C1) and (C2) follow immediately. In Figure 1, the vertices with red edges are the case (C1), and the vertices with blue edges are the second case (C2). Now we argue that this construction produces a desired gap.

Completeness. If there exists a set $X \subseteq A$ with $|X| = s$ and that X has at least d common neighbors in B, then we can choose this s-vertex set X and its d common neighbors as a dominating set. The weight of this dominating set is $st + d$. As d is sufficiently large and $t = d^{1-1/2s} = o(d)$, it holds that $st + d \leq (1 + \varepsilon)d$.

Soundness. Assume every set $X \subseteq A$ with $|X| = s$ has at most ℓ common neighbors. Let D be a dominating set. Then we have two possibilities.

- Either there are at least $(1-\varepsilon)$-fraction of $i \in [d]$ with $|D \cap B_i| \geq c = 2$, i.e., (C1). Then, the weight of D is $w(D) \geq (1 - \varepsilon)2d$;
- Or for at least εd distinct $i \in [d]$, we use one vertex $v_i \in D \cap B_i$ and its s neighbors in $D \cap A$ to dominate W_i, i.e., (C2). Assume that $w(D) \leq 2d$, otherwise we are done. It follows that $|D \cap A| \leq 2d/t$. Thus, there are at least εd vertices in $D \cap B$, each having at least s neighbors in $|D \cap A| \leq 2d/t = O(d^{1/2s})$. Observe that the number of s-tuples of $D \cap A$ is bounded by $(2d/t)^s = O(\sqrt{d})$. By the pigeonhole principle and (1), at least

$$\frac{\varepsilon d}{O(\sqrt{d})} > \varepsilon \cdot \Omega(\sqrt{d}) > \ell$$

vertices in $D \cap B$ must be adjacent to the same s-tuples of $D \cap A$. This contradicts to the fact that every s-vertex set in A has at most ℓ common neighbors in B.

In order to remove the weight in the above construction, the natural idea is to make many copies of A and W_i's. In Section 4 we show that such duplication indeed works with only a little loss of the hardness factor, i.e., from 2 to $3/2$. The intuitive reason for this loss is that we cannot make infinite copies of W_i's, although their vertices have weight ∞.[2] Observe that the gap 2 of our reduction

[2] In fact, by a slightly more involved construction, we can get very close to 2. But this would further complicate the reduction in Section 5.

comes from the parameter c in (C2). If for any constant $c \geq 3$, we were able to add edges between A and W_i and between W_i and B_i such that either (C1) or (C2) for parameter c holds, then we could establish any constant-inapproximability of the dominating set problem. Unfortunately, at the time of writing, we do not know how to do that. So, in Section 5, we take another approach to amplify the gap via some special graph products. It is known that the standard graph product does not always increase the size of a minimum dominating set polynomially [22].[3] This is hardly a surprise, given the sharp $(1-\varepsilon)\ln n$-approximation lower bound. Otherwise, the greedy $\ln n$-approximation for the dominating set problem, composed with any such polynomial-time amplification, would beat the $(1-\varepsilon)\ln n$ bound. So, basically we construct a product directly on the graphs produced in Section 4. After some carefully analysis, it turns out to suit our purpose nicely.

4. The Case $\rho < 3/2$

Now we follow the strategy outlined in Section 3 to show that p-DOMINATING-SET cannot be fpt approximated within ratio $< 3/2$. This serves as a stepping stone to the general constant-inapproximability of the problem.

Theorem 4.1. *Let $\rho < 3/2$. Then there is no* fpt *approximation of the parameterized dominating set problem achieving ratio ρ unless* FPT $=$ W[1].

Proof : We fix some $\varepsilon, \delta \in \mathbb{R}$ with $0 < \varepsilon < 1$, $0 < \delta < 1/2$, and

$$\frac{3/2-\delta}{1+\varepsilon} > \rho. \tag{2}$$

Let G be a graph with n vertices and $k \in \mathbb{N}$ a parameter. We set $s := \binom{k}{2}$,

$$d := \left\lceil \frac{s}{\varepsilon} \right\rceil^{2s}, \quad \text{and} \quad t := \left\lceil \left(\frac{1}{2}-\delta\right) \cdot d^{1-1/2s} \right\rceil.$$

As a consequence, when k and n are sufficiently large, we have

$$st < \varepsilon d, \quad \left(\frac{1}{2}-\delta\right) \cdot \frac{d}{t} \leq \sqrt[2s]{d}, \quad (k+1)! < 2\delta\sqrt{d}-1, \quad \text{and} \quad d \leq \lceil n^{\frac{6}{k+1}} \rceil. \tag{3}$$

Here, $(k+1)!$ plays the role of ℓ as in Section 3, hence it is easy to verify that (1) holds.

By Theorem 2.4 (and the preprocessing) we can compute in fpt-time a bipartite graph $H_0 = (A_0 \dot\cup B_0, E_0)$ such that:

- if $K_k \subseteq G$, then there are s vertices in A_0 with d common neighbors in B_0;
- if $K_k \nsubseteq G$, then every s vertices in A_0 have at most $(k+1)!$ common neighbors in B_0.

Then we need to partition A_0 and B_0 similarly as the partitioning of A and B in Figure 1. Using the color-coding in Lemma 2.3, again in fpt-time, we construct two function families $\Lambda_A := \Lambda_{|A_0|,s}$ and $\Lambda_B := \Lambda_{|B_0|,d}$ such that

- for every s-element subset $X \subseteq A_0$ there is an $h \in \Lambda_A$ with $h(X) = [s]$;
- for every d-element subset $Y \subseteq B_0$ there is an $h \in \Lambda_B$ with $h(Y) = [d]$.

[3]For example, we do not have $\gamma(G^2) = (\gamma(G))^2$ for every graph G.

Define the bipartite graph $H = \big(A(H) \mathbin{\dot\cup} B(H), E(H)\big)$ by

$$A(H) := A_0 \times \Lambda_A \times \Lambda_B, \qquad B(H) := B_0 \times \Lambda_A \times \Lambda_B$$
$$E(H) := \Big\{ \big\{(u, h_1, h_2), (v, h_1, h_2)\big\} \;\Big|\; u \in A_0, v \in B_0, h_1 \in \Lambda_A, h_2 \in \Lambda_B, \text{ and } \{u, v\} \in E_0 \Big\}.$$

Moreover, define two colorings $\alpha : A(H) \to [s]$ and $\beta : B(H) \to [d]$ by

$$\alpha(u, h_1, h_2) := h_1(u) \quad \text{and} \quad \beta(v, h_1, h_2) := h_2(v).$$

It is straightforward to verify that

(H1) if $K_k \subseteq G$, then there are s vertices of distinct α-colors in $A(H)$ with d common neighbors of distinct β-colors in $B(H)$;

(H2) if $K_k \nsubseteq G$, then every s vertices in $A(H)$ have at most $(k + 1)!$ common neighbors in $B(H)$.

Now from H, α, and β we construct a new graph $G' = \big(V(G'), E(G')\big)$ as follows. First, its vertex set is defined by

$$V(G') := B(H) \mathbin{\dot\cup} \big\{x_i, y_i \;\big|\; i \in [d]\big\} \mathbin{\dot\cup} C \mathbin{\dot\cup} W,$$

where

$$C := A(H) \times [t] \quad \text{and} \quad W := \Big\{ w_{b,j,i} \;\Big|\; b \in B(H), i \in [t], j \in [s] \Big\}.$$

Thereby, the duplication of $A(H)$-vertices simulates their weight being t, so does that of W-vertices (instead of weight being ∞), as in Figure 1. Then we need to define the edge set of G'.

(E1) $\{b, b'\} \in E(G')$ with $b, b' \in B(H)$, $b \neq b'$, and $\beta(b) = \beta(b')$ $\big($i.e., all vertices in $B(H)$ with the same color under β form a clique in $G'\big)$.

(E2) Let $b \in B(H)$ and $c := \beta(b)$. Then $\{x_c, b\}, \{y_c, b\} \in E(G')$.

(E3) Let $b, b' \in B(H)$ with $\beta(b) = \beta(b')$ and $b \neq b'$. Then $\big\{w_{b,j,i}, b'\big\} \in E(G')$ for every $i \in [t]$ and $j \in [s]$.

(E4) $\big\{(a, i), w_{b,j,i}\big\} \in E(G')$ for every $\{a, b\} \in E(H)$, $j = \alpha(a)$ and $i \in [t]$.

(E5) Let $a, a' \in A(H)$ with $a \neq a'$ and $i \in [t]$. Then $\big\{(a, i), (a', i)\big\} \in E(G')$.

To ease presentation, for every $c \in [d]$ we set

$$B_c := \big\{b \in B(H) \;\big|\; \beta(b) = c\big\} \cup \{x_c, y_c\}.$$

Claim 1. If D is a dominating set of G', then $D \cap B_c \neq \emptyset$ for every $c \in [d]$.

Proof of the claim. We observe that every x_c is only adjacent to vertices in B_c. $\dashv$

Claim 2. If G contains a k-clique, then $\gamma(G') < (1 + \varepsilon)d$.

Proof of the claim. By (H1) the bipartite graph H has a $K_{s,d}$ biclique K with $\alpha(A(H) \cap K) = [s]$ and $\beta(B(H) \cap K) = [d]$. It is then easy to verify that

$$\big(B(H) \cap K\big) \,\dot\cup\, \big((A(H) \cap K) \times [t]\big)$$

is a dominating set of G', whose size is $d + s \cdot t < (1 + \varepsilon)d$ by (3). $\dashv$

The next claim follows directly from (H2) and (3).

Claim 3. If G contains no k-clique, then every s-vertex set of $A(H)$ has at most $(k+1)! < 2\delta\sqrt{d} - 1$ common neighbors in $B(H)$.

Claim 4. If G contains no k-clique, then

$$\gamma(G') > \left(\frac{3}{2} - \delta\right) \cdot d.$$

Proof of the claim. Let D be a dominating set of G'. By Claim 1 we have $D \cap B_c \neq \emptyset$ for every $c \in [d]$. Define

$$e := \Big|\big\{c \in [d] \;\big|\; |D \cap B_c| \geq 2\big\}\Big|.$$

If $e > (1/2 - \delta) \cdot d$ then $|D| > d + e > (3/2 - \delta) \cdot d$ and we are done.

So let us consider $e \leq (1/2 - \delta) \cdot d$ and without loss of generality $|D \cap B_c| = 1$ for every $c \leq (1/2 + \delta) \cdot d$. Fix such a c and assume $D \cap B_c = \{b_c\}$. Recall $x_c, y_c \in V(G')$ are not adjacent to any vertex outside B_c, and there is no edge between them, thus $b_c \in B_c \setminus \{x_c, y_c\} = \big\{b \in B(H) \;\big|\; \alpha(b) = c\big\}$. Let

$$W_1 := \Big\{w_{b_c,j,i} \;\Big|\; i \in [t], j \in [s], \text{ and } c \leq (1/2 + \delta) \cdot d\Big\} \subseteq W.$$

(E3) implies that every $w_{b_c,j,i} \in W_1$ is not dominated by any vertex in $D \cap \bigcup_{c \in [d]} B_c$. Therefore, it has to be dominated by or included in $D \cap (C \cup W)$.

If $|D \cap W_1| > (1/2 - \delta) \cdot d$, then again we are done. So suppose $|D \cap W_1| \leq (1/2 - \delta) \cdot d$. Without loss of generality let

$$W_2 := \Big\{w_{b_c,j,i} \;\Big|\; i \in [t], j \in [s], \text{ and } c \leq 2\delta d\Big\} \subseteq W_1$$

and assume $W_2 \cap D = \emptyset$. Thus W_2 has to be dominated by $D \cap C$. For later purpose, let

$$Y := \big\{b_c \;\big|\; c \leq 2\delta d\big\}.$$

Obviously, $|Y| \geq 2\delta d - 1$.

Again we only need to consider the case $|D \cap C| \leq (1/2 - \delta) \cdot d$. Recall $C = A(H) \times [t]$. Thus there is an $i \in [t]$ such that

$$\Big|D \cap \big(A(H) \times \{i\}\big)\Big| \leq \left(\frac{1}{2} - \delta\right) \cdot \frac{d}{t}.$$

Let $X := \big\{a \in A(H) \;\big|\; (a, i) \in D\big\}$, and in particular, $|X| \leq (1/2 - \delta) \cdot d/t$. Since W_2 is dominated by $D \cap C$, we have for all $b \in Y$ and $j \in [s]$ there exists an $a \in X$ such that $\big\{(a, i), w_{b,j,i}\big\} \in E(G')$, which means that $\{a, b\} \in E(H)$ and $\alpha(a) = j$. It follows that in the graph H every vertex of Y has at least s neighbors in X. Recall that $(1/2 - \delta) \cdot d/t \leq \sqrt[2s]{d}$ by (3). There are at most $\sqrt{d}$ different types of s-vertex sets in X, i.e.,

$$\left| \binom{X}{s} \right| \leq \binom{(1/2 - \delta) \cdot d/t}{s} \leq \left(\sqrt[2s]{d} \right)^s = \sqrt{d}.$$

By the pigeonhole principle, there exists an s-vertex set of $X \subseteq A(H)$ having at least $|Y|/\sqrt{d} \geq 2\delta\sqrt{d} - 1$ common neighbors in $Y \subseteq B(H)$, which contradicts Claim 3. $\dashv$

Claim 2 and Claim 4 indeed imply that there is an fpt-reduction from the clique problem to the dominating set problem which creates a gap great than

$$\frac{3/2 - \delta}{1 + \varepsilon}.$$

So if there is a ρ-approximation of the dominating set problem, by (2) we can decide the clique problem in fpt time. $\square$

5. The Constant-Inapproximibility of p-DOMINATING-SET

Theorem 1.1 is a fairly direct consequence of the following theorem.

Theorem 5.1 (Main)**.** *There is an algorithm $\mathbb{A}$ such that on input a graph G, $k \geq 3$, and $c \in \mathbb{N}$ the algorithm $\mathbb{A}$ computes a graph G_c such that*

(i) if $K_k \subseteq G$, then $\gamma(G_c) < 1.1 \cdot d^c$;

(ii) if $K_k \nsubseteq G$, then $\gamma(G_c) > c \cdot d^c/3$,

where $d = \left(30 \cdot c^2 \cdot (k+1)^2\right)^{4 \cdot k^3 + 3c}$. Moreover the running time of $\mathbb{A}$ is bounded by $f(k, c) \cdot |G|^{O(c)}$ for a computable function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

Proof of Theorem 1.3: Suppose for some $\varepsilon > 0$ there is an fpt-algorithm $\mathbb{A}(G)$ which outputs a dominating set of G of size at most $\sqrt[4+\varepsilon]{\log (\gamma(G))} \cdot \gamma(G)$. Of course we can further assume that $\varepsilon < 1$. Then on input a graph G and $k \in \mathbb{N}$, let

$$c := \left\lceil k^{1 - \varepsilon/5} \right\rceil = o(k) \quad \text{and} \quad d := \left(30 \cdot c^2 \cdot (k+1)^2\right)^{4 \cdot k^3 + 3c}.$$

We have

$$\sqrt[4+\varepsilon]{\log (1.1 \cdot d^c)} = O\left(\sqrt[4+\varepsilon]{c \cdot k^3 \cdot \log k} \right) = o\left(k^{\frac{4}{4+\varepsilon}} \right) = o(c).$$

By Theorem 5.1, we can construct a graph G_c with properties (i) and (ii) in time

$$f(k, c) \cdot |G|^{O(c)} = h(k) \cdot |G|^{o(k)}$$

for an appropriate computable function $h : \mathbb{N} \to \mathbb{N}$. Thus, G contains a clique of size k if and only if $\mathbb{A}(G_c)$ returns a dominating set of size at most

$$1.1 \cdot d^c \cdot \sqrt[4+\varepsilon]{\log (1.1 \cdot d^c)} = o\left(c \cdot d^c\right) < \frac{c \cdot d^c}{3},$$

where the inequality holds for sufficiently large k $\left(\text{and hence sufficiently large } c \cdot d^c\right)$.

Therefore we can determine whether G contains a k-clique in time $g(k) \cdot |G|^{o(k)}$ for some computable $g : \mathbb{N} \to \mathbb{N}$. This contradicts a result in Chen et.al. [6, Theorem 4.4] under ETH. $\square$

5.1. Proof of Theorem 5.1. We start by showing a variant of Theorem 2.4. Besides giving proper colors to all vertices in the constructed bipartite graphs H, it also creates an *additive gap* on the left-hand side $A(H)$ of H.

Theorem 5.2. *Let $\Delta \in \mathbb{N}^+$ be a constant and $d : \mathbb{N}^+ \to \mathbb{N}^+$ a computable function. Then there is an* fpt*-algorithm that on input a graph G and a parameter $k \in \mathbb{N}$ with $6 \mid k+1$ constructs a bipartite graph $H = \big(A(H) \,\dot\cup\, B(H), E(H)\big)$ together with two colorings*

$$\alpha : A(H) \to [\Delta s] \quad \textit{and} \quad \beta : B(H) \to [d(k)]$$

such that:

(H1) if $K_k \subseteq G$, then there are Δs vertices of distinct α-colors in $A(H)$ with $d(k)$ common neighbors of distinct β-colors in $B(H)$;

(H2) if $K_k \not\subseteq G$, then every $\Delta(s-1)+1$ vertices in $A(H)$ have at most $(k+1)!$ common neighbors in $B(H)$,

where $s = \binom{k}{2}$.

Proof : Let G be a graph with n vertices and $k \in \mathbb{N}$. Assume without loss of generality

$$\left\lceil n^{\frac{6}{k+6}} \right\rceil > (k+6)! \quad \text{and} \quad \left\lceil n^{\frac{6}{k+1}} \right\rceil \geq d(k).$$

By Theorem 2.4 we can construct in polynomial time a bipartite graph $H_0 = (A_0 \,\dot\cup\, B_0, E_0)$ such that for $s := \binom{k}{2}$:

- if $K_k \subseteq G$, then there are s vertices in A_0 with at least $d(k)$ common neighbors in B_0;
- if $K_k \not\subseteq G$, then every s vertices in A_0 have at most $(k+1)!$ common neighbors in B_0.

Define

$$A_1 := A_0 \times [\Delta],\ B_1 := B_0, \text{ and } E_1 := \big\{\{(u,i),v\} \bigm| (u,i) \in A_0 \times [\Delta], v \in B_0, \text{and } \{u,v\} \in E_0\big\}.$$

It is easy to verify that in the bipartite graph $(A_1 \,\dot\cup\, B_1, E_1)$

- if $K_k \subseteq G$, then there are Δs vertices in A_1 with at least $d(k)$ common neighbors in B_2;
- if $K_k \not\subseteq G$, then every $\Delta(s-1)+1$ vertices in A_1 have at most $(k+1)!$ common neighbors in B_1.

Applying Lemma 2.3 on

$$\big(n \leftarrow |A_1|, k \leftarrow \Delta s\big) \quad \text{and} \quad \big(n \leftarrow |B_1|, k \leftarrow d(k)\big)$$

we obtain two function families $\Lambda_A := \Lambda_{|A_1|,\Delta s}$ and $\Lambda_B := \Lambda_{|B_1|,d(k)}$ with the stated properties. Finally the desired bipartite graph H is defined by $\Big((A_1 \times \Lambda_A \times \Lambda_B) \,\dot\cup\, (B_1 \times \Lambda_A \times \Lambda_B), E)\Big)$ with

$$E := \Big\{\big\{(u,h_1,h_2),(v,h_1,h_2)\big\} \Bigm| u \in A_1, v \in B_1, h_1 \in \Lambda_A, h_2 \in \Lambda_B, \text{and } \{u,v\} \in E_1\Big\}$$

and the colorings

$$\alpha(u, h_1, h_2) := h_1(u) \quad \text{and} \quad \beta(v, h_1, h_2) := h_2(v). \qquad \square$$

Setting the parameters. Let $\Delta := 2$. Recall that $k \geq 3$, $s = \binom{k}{2} \geq 3$, and $c \in \mathbb{N}^+$. We first define

$$d := d(k) := \left(30 \cdot c^2 \cdot (k+1)^2\right)^{4 \cdot k^3 + 3c}.$$

It is easy to check that:

(i) $d^{\frac{1}{2} - \frac{1}{2s}} > c \cdot s^c \left(= c \cdot \binom{k}{2}^c\right)$.

(ii) $d > \left(3(k+1)!\right)^{2s}$.

(iii) $d > \left(10\Delta s \cdot c^2\right)^{2\Delta s}$.

Then let

$$t := c \cdot d^{c - \frac{1}{2\Delta s}}.^4 \tag{4}$$

From (ii), (iii), and (4) we conclude

$$\Delta sct < 0.1 \cdot d^c, \quad \frac{c \cdot d^c}{3t} \leq \sqrt[2\Delta s]{d}, \quad \text{and } (k+1)! < \frac{\sqrt[2s]{d}}{3}. \tag{5}$$

Moreover by (i) and $\Delta = 2$ we have

$$c \cdot d^c + c\Delta^c s^c d^{c - \frac{1}{2} + \frac{1}{2s}} < 2\Delta^c d^c. \tag{6}$$

Construction of G_c. We invoke Theorem 5.2 to obtain $H = (A \,\dot\cup\, B, E)$, α, and β. Then we construct a new graph $G_c = \big(V(G_c), E(G_c)\big)$ as follows. First, the vertex set of G_c is given by

$$V(G_c) := \bigcup_{\boldsymbol{i} \in [d]^c} V_{\boldsymbol{i}} \,\dot\cup\, C \,\dot\cup\, W,$$

where

$$V_{\boldsymbol{i}} := \left\{ \boldsymbol{v} \in B^c \;\middle|\; \beta(\boldsymbol{v}) = \boldsymbol{i} \right\} \quad \text{for every } \boldsymbol{i} \in [d]^c,$$

$$C := A \times [c] \times [t], \quad \text{and} \quad W := \left\{ w_{\boldsymbol{v}, \boldsymbol{j}, i} \;\middle|\; \boldsymbol{v} \in V_{\boldsymbol{i}} \text{ for some } \boldsymbol{i} \in [d]^c, \boldsymbol{j} \in [\Delta s]^c \text{ and } i \in [t] \right\}.$$

Thus, $\bigcup_{\boldsymbol{i} \in [d]^c} V_{\boldsymbol{i}}$ is the c-fold Cartesian product of B. Moreover, G_c contains the following types of edges.

(E1) For each $\boldsymbol{i} \in [d]^c$, $V_{\boldsymbol{i}}$ forms a clique.

(E2) Let $\boldsymbol{i} \in [d]^c$ and $\boldsymbol{v}, \boldsymbol{v}' \in V_{\boldsymbol{i}}$. If for all $\ell \in [c]$ we have $\boldsymbol{v}(\ell) \neq \boldsymbol{v}'(\ell)$ then $\{w_{\boldsymbol{v}, \boldsymbol{j}, i}, \boldsymbol{v}'\} \in E(G_c)$ for every $i \in [t]$ and $\boldsymbol{j} \in [\Delta s]^c$.

(E3) Let $i \in [t]$. Then $\big\{(u, \ell, i), w_{\boldsymbol{v}, \boldsymbol{j}, i}\big\} \in E(G_c)$ if $\{u, \boldsymbol{v}(\ell)\} \in E$ and $\boldsymbol{j}(\ell) = \alpha(u)$.

(E4) Let $u, u' \in A$ with $u \neq u'$, $\ell \in [c]$, and $i \in [t]$. Then $\big\{(u, \ell, i), (u', \ell, i)\big\} \in E(G_c)$.

[4] Here, we assume $d^{c - \frac{1}{2\Delta s}}$ is an integer. Otherwise, let $d \leftarrow d^{2\Delta s}$ which maintains (i)– (iii).

Theorem 5.1 then follows from the completeness and the soundness of this reduction.

Lemma 5.3 (Completeness)**.** *If G contains k-clique, then $\gamma(G_c) < 1.1d^c$.*

Lemma 5.4 (Soundness)**.** *If G contains no k-clique then $\gamma(G_c) > c \cdot d^c/3$.*

We first show the easier completeness.

Proof of Lemma 5.3: By (H1) in Theorem 5.2, if G contains a subgraph isomorphic to K_k, then the bipartite graph H has a $K_{\Delta s,d}$-subgraph K such that $\alpha(A \cap K) = [\Delta s]$ and $\beta(B \cap K) = [d]$. Let

$$D := (B \cap K)^c \,\dot\cup\, \big((A \cap K) \times [c] \times [t]\big).$$

Obviously, $|D| = d^c + \Delta sct < 1.1 \cdot d^c$ by (5). And (E1) and (E4) imply that D dominates every vertex in C and every vertex in $V_{\boldsymbol{i}}$ for all $\boldsymbol{i} \in [d]^c$.

To see that D also dominates W, let $w_{\boldsymbol{v},\boldsymbol{j},i}$ be a vertex in W. First consider the case where $\boldsymbol{v}(\ell) \notin B \cap K$ for all $\ell \in [c]$. Since $\beta\big((B \cap K)^c\big) = [d]^c$, there exists a vertex $\boldsymbol{v}' \in (B \cap K)^c$ with $\beta(\boldsymbol{v}') = \beta(\boldsymbol{v})$ and $\boldsymbol{v}(\ell) \neq \boldsymbol{v}'(\ell)$ for all $\ell \in [c]$. Then $w_{\boldsymbol{v},\boldsymbol{j},i}$ is dominated by $\boldsymbol{v}'$ because of (E2).

Otherwise assume $\boldsymbol{v}(\ell) \in B \cap K$ for some $\ell \in [c]$, then $A \cap K \subseteq N^H(\boldsymbol{v}(\ell)) = \big\{u \in A \bigm| \{u, \boldsymbol{v}(\ell)\} \in E\big\}$. There exists a vertex $u \in A \cap K$ such that $\alpha(u) = \boldsymbol{j}(\ell)$ and $\big\{\boldsymbol{v}(\ell), u\big\} \in E$. By (E3), $w_{\boldsymbol{v},\boldsymbol{j},i}$ is adjacent to (u, ℓ, i). □

5.2. Soundness.

Lemma 5.5. *Suppose $c, \Delta, t \in \mathbb{N}^+$ and $\Delta < t$. Let $V \subseteq [t]^c$. If there exists a function $\theta : V \to [c]$ such that for all $i \in [c]$ we have*

$$\Big|\big\{\mathbf{v}(i) \bigm| \mathbf{v} \in V \text{ and } \theta(\mathbf{v}) = i\big\}\Big| \leq t - \Delta, \tag{7}$$

then $|V| \leq t^c - \Delta^c$.

Proof : When $c = 1$, we have $|V| \leq t - \Delta$ by (7). Suppose the lemma holds for $c \leq n$ and consider $c = n + 1$. Given $V \subseteq [t]^{n+1}$ and θ, let

$$C_{n+1} := \big\{\boldsymbol{v}(n+1) \bigm| \boldsymbol{v} \in V \text{ and } \theta(\boldsymbol{v}) = n+1\big\}.$$

By (7), $|C_{n+1}| \leq t - \Delta$. If $|C_{n+1}| < t - \Delta$, we add $\big(t - \Delta - |C_{n+1}|\big)$ arbitrary integers from $[t] \setminus C_{n+1}$ to C_{n+1}. So we have $|C_{n+1}| = t - \Delta$. Let $A := \big\{\boldsymbol{v} \in V \bigm| \boldsymbol{v}(n+1) \in C_{n+1}\big\}$ and $B := V \setminus A$. It follows that

$$|A| \leq (t - \Delta)t^{c-1}, \tag{8}$$

$\Big|\big\{\boldsymbol{v}(n+1) \bigm| \boldsymbol{v} \in B\big\}\Big| = \Big|\big\{\boldsymbol{v}(n+1) \bigm| \boldsymbol{v} \in V \text{ and } \boldsymbol{v}(n+1) \notin C_{n+1}\big\}\Big| \leq \Delta$, and $\theta(\boldsymbol{v}) \in [c-1]$ for all $\boldsymbol{v} \in B$. Let

$$V' := \big\{(v_1, v_2, \cdots, v_n) \bigm| \exists v_{n+1} \in [t], (v_1, v_2, \cdots, v_n, v_{n+1}) \in B\big\}.$$

We define a function $\theta' : V' \to [c-1]$ as follows. For all $\boldsymbol{v}' \in V'$, choose $\boldsymbol{v} \in B$ with the minimum $\boldsymbol{v}(c)$ such that for all $i \in [c-1]$ it holds $\boldsymbol{v}'(i) = \boldsymbol{v}(i)$. By the definition of V', such a $\boldsymbol{v}$ must exist, and we let $\theta'(\boldsymbol{v}') := \theta(\boldsymbol{v})$. By (7), $\Big|\big\{\boldsymbol{v}'(i) \bigm| \boldsymbol{v}' \in V' \text{ and } \theta'(\boldsymbol{v}') = i\big\}\Big| \leq t - \Delta$ for all $i \in [c-1]$. Applying the induction hypothesis, we get $|V'| \leq t^{c-1} - \Delta^{c-1}$. Obviously,

$$|B| \leq \Delta|V'| \leq \Delta t^{c-1} - \Delta^c. \tag{9}$$

From (8) and (9), we deduce that $|V| = |A| + |B| \leq (t - \Delta)t^{c-1} + \Delta t^{c-1} - \Delta^c \leq t^c - \Delta^c$. □

We are now ready to prove the soundness of our reduction.

Proof of Lemma 5.4: Let D be a dominating set of G_c. Define

$$a := \left|\left\{\boldsymbol{i} \in [d]^c \;\middle|\; |D \cap V_{\boldsymbol{i}}| \geq c+1\right\}\right|.$$

If $a > d^c/3$, then $|D| \geq (c+1)a > c \cdot d^c/3$ and we are done.

So let us consider $a \leq d^c/3$. Thus, the set

$$I := \left\{\boldsymbol{i} \in [d]^c \;\middle|\; |D \cap V_{\boldsymbol{i}}| \leq c\right\}$$

has size $|I| \geq 2d^c/3$. Let $\boldsymbol{i} \in I$ and assume that $D \cap V_{\boldsymbol{i}} = \left\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{c'}\right\}$ for some $c' \leq c$. We define a $\boldsymbol{v}_{\boldsymbol{i}} \in V_{\boldsymbol{i}}$ as follows. If $c' = 0$, we choose an arbitrary $\boldsymbol{v}_{\boldsymbol{i}} \in V_{\boldsymbol{i}}$.[5] Otherwise, let

$$\boldsymbol{v}_{\boldsymbol{i}}(\ell) := \begin{cases} \boldsymbol{v}_\ell(\ell) & \text{for all } \ell \in [c']; \\ \boldsymbol{v}_1(\ell) & \text{for all } c' < \ell \leq c. \end{cases}$$

Obviously, $\beta(\boldsymbol{v}_{\boldsymbol{i}}) = \boldsymbol{i}$.

(E2) implies that for every $\boldsymbol{j} \in [\Delta s]^c$ and every $i \in [t]$, the vertex $w_{\boldsymbol{v}_{\boldsymbol{i}},\boldsymbol{j},i}$ is not dominated by $D \cap V_{\boldsymbol{i}}$. Observe that $w_{\boldsymbol{v}_{\boldsymbol{i}},\boldsymbol{j},i}$ cannot be dominated by other $D \cap V_{\boldsymbol{i}'}$ with $\boldsymbol{i}' \neq \boldsymbol{i}$ either, by (E2) and (E3). Therefore every vertex in the set

$$W_1 := \left\{w_{\boldsymbol{v}_{\boldsymbol{i}},\boldsymbol{j},i} \;\middle|\; \boldsymbol{i} \in I, \boldsymbol{j} \in [\Delta s]^c, \text{ and } i \in [t]\right\}$$

is *not* dominated by $D \cap \bigcup_{\boldsymbol{i} \in [d]^c} V_{\boldsymbol{i}}$. As a consequence, W_1 has to be dominated by or included in $D \cap (C \cup W)$.

If $|D \cap W_1| > c \cdot d^c/3$, then again we are done. So suppose $|D \cap W_1| \leq c \cdot d^c/3$ and let $W_2 := W_1 \setminus D$. It follows that W_2 has to be dominated by $D \cap C$. Once again we only need to consider the case $|D \cap C| \leq c \cdot d^c/3$, and hence there is an $i' \in [t]$ such that

$$\left|D \cap \left(A \times [c] \times \{i'\}\right)\right| \leq \frac{c \cdot d^c}{3t}. \tag{10}$$

Then we define

$$Z := \left\{w_{\boldsymbol{v},\boldsymbol{j},i} \in W_2 \;\middle|\; i = i'\right\} = \left\{w_{\boldsymbol{v}_{\boldsymbol{i}},\boldsymbol{j},i'} \;\middle|\; \boldsymbol{i} \in I, \boldsymbol{j} \in [\Delta s]^c, \text{ and } w_{\boldsymbol{v}_{\boldsymbol{i}},\boldsymbol{j},i'} \notin D\right\}.$$

So Z has to be dominated by $D \cap C$, and in particular those vertices of the form $(u, \ell, i') \in D \cap C$. Moreover,

$$|Z| \geq \Delta^c s^c |I| - |D \cap W_1| \geq \Delta^c s^c |I| - c \cdot d^c/3. \tag{11}$$

Our next step is to upper bound $|Z|$. To that end, let

$$X := \left\{u \in A \;\middle|\; (u, \ell, i') \in D \text{ for some } \ell \in [c]\right\}.$$

Thus Z is dominated by those vertices (u, ℓ, i') with $u \in X$. And by (10)

$$|X| \leq \frac{c \cdot d^c}{3t}.$$

[5] Since the coloring β is obtained by the color-coding used in the proof of Theorem 5.2, for every $b \in [d]$ it holds that $\{v \in B \mid \beta(v) = b\} \neq \emptyset$, hence $V_{\boldsymbol{i}} \neq \emptyset$.

Set

$$Y := \Big\{ v \in B \;\Big|\; \big|N^H(v) \cap X\big| > \Delta(s-1) \Big\}.$$

Recall that $c \cdot d^c/(3t) \leq \sqrt[2\Delta s]{d}$ by (5). Hence X has at most $\sqrt{d}$ different subsets of size $\Delta(s-1)+1$, i.e.,

$$\left| \binom{X}{\Delta(s-1)+1} \right| \leq |X|^{\Delta(s-1)+1} \leq |X|^{\Delta s} \leq \sqrt{d}.$$

We should have

$$|Y| \leq \sqrt{d} \cdot (k+1)! \leq \frac{d^{\frac{1}{2}+\frac{1}{2s}}}{3}, \tag{12}$$

where the second inequality is by (5). Otherwise, by the pigeonhole principle, there exists a $(\Delta(s-1)+1)$-vertex set of $X \subseteq A(H)$ having at least $|Y|/\sqrt{d} > (k+1)!$ common neighbors in $Y \subseteq B(H)$. However, if G contains no k-clique, then by (H2) every $\big(\Delta(s-1)+1\big)$-vertex set of $A(H)$ has at most $(k+1)!$ common neighbors in $B(H)$, and we obtain a contradiction.

Let

$$\begin{aligned} Z_1 :=& \big\{ w_{\mathbf{v},\boldsymbol{j},i'} \in Z \;\big|\; \text{there exists an } \ell \in [c] \text{ with } \boldsymbol{v}(\ell) \in Y \big\} \quad \big(\subseteq Z\big) \\ =& \big\{ w_{\mathbf{v}_{\boldsymbol{i}},\boldsymbol{j},i'} \;\big|\; \boldsymbol{i} \in I, \boldsymbol{j} \in [\Delta s]^c,\, w_{\mathbf{v}_{\boldsymbol{i}},\boldsymbol{j},i'} \notin D, \text{ and there exists an } \ell \in [c] \text{ with } \mathbf{v}_{\boldsymbol{i}}(\ell) \in Y \big\} \\ \text{and} \qquad Z_2 :=& Z \setminus Z_1 = \big\{ w_{\mathbf{v}_{\boldsymbol{i}},\boldsymbol{j},i'} \;\big|\; \boldsymbol{i} \in I, \boldsymbol{j} \in [\Delta s]^c,\, w_{\mathbf{v}_{\boldsymbol{i}},\boldsymbol{j},i'} \notin D, \text{ and } \mathbf{v}_{\boldsymbol{i}}(\ell) \notin Y \text{ for all } \ell \in [c] \big\}. \end{aligned}$$

Moreover, let $I_1 := \{\boldsymbol{i} \in I \mid \text{there exists a } w_{\mathbf{v}_{\boldsymbol{i}},\boldsymbol{j},i'} \in Z_1\}$. From the definition, we can deduce that

$$\text{for all } \boldsymbol{i} \in I_1 \text{ there exists an } \ell \in [c] \text{ such that } \boldsymbol{i}(\ell) \in \beta(Y).$$

Then $|I_1| \leq c|Y|d^{c-1}$ and hence

$$|Z_1| \leq |I_1|\Delta^c s^c \leq c|Y|d^{c-1}\Delta^c s^c.$$

To estimate $|Z_2|$, let us fix an $\boldsymbol{i} \in I$ and thus fix the tuple $\mathbf{v}_{\boldsymbol{i}} \in B^c$, and consider the set

$$J_{\boldsymbol{i}} := \big\{\, \boldsymbol{j} \in [\Delta s]^c \;\big|\; w_{\mathbf{v}_{\boldsymbol{i}},\boldsymbol{j},i'} \in Z_2 \big\}.$$

Recall that Z is dominated by those vertices (u, ℓ, i') with $u \in X$, so for every $\boldsymbol{j} \in J_{\boldsymbol{i}}$ the vertex $w_{\mathbf{v}_i,\boldsymbol{j},i'}$ is adjacent to some (u, ℓ, i') in the dominating set D with $u \in X$. Moreover, for every $\ell \in [c]$, in the original graph H the vertex $\mathbf{v}_{\boldsymbol{i}}(\ell) \in B$ has at most $\Delta(s-1)$ neighbors in X, by the fact that $\mathbf{v}_{\boldsymbol{i}}(\ell) \notin Y$ and our definition of the set Y.

Define a function $\theta : J_{\boldsymbol{i}} \to [c]$ such that for each $\boldsymbol{j} \in J_{\boldsymbol{i}}$, if $w_{\mathbf{v}_{\boldsymbol{i}},\boldsymbol{j},i'}$ is adjacent to a vertex $(u, \ell, i') \in D$ with $u \in X$, then $\theta(\boldsymbol{j}) = \ell$. As argued above, such a (u, ℓ, i') must exist, and if there are more than one such, choose an arbitrary one.

Let $\boldsymbol{j} \in J_{\boldsymbol{i}}$ and $\ell := \theta(\boldsymbol{j})$. By (E3), in the graph H the vertex $\mathbf{v}_{\boldsymbol{i}}(\ell)$ is adjacent to some vertex $u \in X$ with $\alpha(u) = \boldsymbol{j}(\ell)$. It follows that for each $\ell \in [c]$ we have

$$\Big| \big\{\, \boldsymbol{j}(\ell) \;\big|\; \boldsymbol{j} \in J_{\boldsymbol{i}} \text{ and } \theta(\boldsymbol{j}) = \ell \big\} \Big| \leq \Big| \big\{ \alpha(u) \;\big|\; u \in X \text{ adjacent to } \mathbf{v}_{\boldsymbol{i}}(\ell) \big\} \Big| \leq \Delta(s-1).$$

Applying Lemma 5.5, we obtain

$$\big|J_{\boldsymbol{i}}\big| \leq \Delta^c s^c - \Delta^c.$$

Then

$$\left|Z_2\right| = \sum_{\boldsymbol{i}\in I} \left|J_{\boldsymbol{i}}\right| \leq |I|(\Delta^c s^c - \Delta^c).$$

By (11) and the definition of Z_1 and Z_2, we should have

$$\Delta^c s^c |I| - c \cdot d^c/3 \leq |Z| = |Z_1| + |Z_2| \leq c|Y|d^{c-1}\Delta^c s^c + |I|(\Delta^c s^c - \Delta^c).$$

That is,

$$c \cdot d^c/3 + c|Y|d^{c-1}\Delta^c s^c \geq \Delta^c |I| \geq 2\Delta^c d^c/3.$$

Combined with (12), we have

$$c \cdot d^c + c\Delta^c s^c d^{c-\frac{1}{2}+\frac{1}{2s}} \geq 2\Delta^c d^c,$$

which contradicts the equation (6). □

6. Some Consequences

Proof of Corollary 1.2: Let $c \in \mathbb{N}^+$, and assume that $\mathbb{A}$ is a polynomial time algorithm which on input a graph $G = (V, E)$ with $\gamma(G) \leq \beta(|V|)$ outputs a dominating set D with $|D| \leq c \cdot \gamma(G)$. Without loss of generality, we further assume that given $0 \leq k \leq n$ it can be tested in time $n^{O(1)}$ whether $k > c \cdot \beta(n)$.

Now let G be an arbitrary graph. We first simulate $\mathbb{A}$ on G, and there are three possible outcomes of $\mathbb{A}$.

– $\mathbb{A}$ does not output a dominating set. Then we know $\gamma(G) > \beta(|V|)$. So in time

$$2^{O(|V|)} \leq 2^{O(\beta^{-1}(\gamma(G)))}$$

we can exhaustively search for a minimum dominating set D of G.

– $\mathbb{A}$ outputs a dominating set D_0 with $|D_0| > c \cdot \beta(|V|)$. We claim that again $\gamma(G) > \beta(|V|)$. Otherwise, the algorithm $\mathbb{A}$ would have behaved correctly with

$$|D_0| \leq c \cdot \gamma(G) \leq c \cdot \beta(|V|).$$

So we do the same brute-force search as above.

– $\mathbb{A}$ outputs a dominating set D_0 with $|D_0| \leq c \cdot \beta(|V|)$. If $|D_0| > c \cdot \gamma(G)$, then

$$c \cdot \beta(|V|) \geq |D_0| > c \cdot \gamma(G), \quad \text{i.e., } \beta(|V|) > \gamma(G),$$

which contradicts our assumption for $\mathbb{A}$. Hence, $|D_0| \leq c \cdot \gamma(G)$ and we can output $D := D_0$.

To summarize, we can compute a dominating set D with $|D| \leq c \cdot \gamma(G)$ in time $f(\gamma(G)) \cdot |G|^{O(1)}$ for some computable $f : \mathbb{N} \to \mathbb{N}$. This is a contradiction to Theorem 1.1. □

Now we come to the approximability of the monotone circuit satisfiability problem.

MONOTONE-CIRCUIT-SATISFIABILITY *Instance:* A monotone circuit C. *Solution:* A satisfying assignment S of C. *Cost:* The weight of $\|S\|$. *Goal:* min.

Recall that a Boolean circuit C is *monotone* if it contains no negation gates; and the *weight* of an assignment is the number of inputs assigned to 1.

As mentioned in the Introduction, Marx showed [26] that MONOTONE-CIRCUIT-SATISFIABILITY has no fpt approximation with any ratio ρ for circuits of depth 4, unless FPT $=$ W$[2]$.

Corollary 6.1. *Assume* FPT $\neq$ W$[1]$. *Then* MONOTONE-CIRCUIT-SATISFIABILITY *has no constant fpt approximation for circuits of depth 2.*

Proof : This is an immediate consequence of Theorem 1.1 and the following well-known approximation-preserving reduction from MONOTONE-CIRCUIT-SATISFIABILITY to MIN-DOMINATING-SET. Let $G = (V, E)$ be a graph. We define a circuit

$$C(G) = \bigwedge_{v \in V} \bigvee_{\{u,v\} \in E} X_u.$$

There is a one-one correspondence between a dominating set in G of size k and a satisfying assignment of $C(G)$ of weight k. □

Remark 6.2. Of course the constant ratio in Corollary 6.1 can be improved according to Theorem 1.3.

7. Conclusions

We have shown that p-DOMINATING-SET has *no* fpt approximation with any constant ratio, and in fact with a ratio slightly super-constant. The immediate question is whether the problem has fpt approximation with *some* ratio $\rho : \mathbb{N} \to \mathbb{N}$, e.g., $\rho(k) = 2^{2^k}$. We tend to believe that it is not the case.

Our proof does not rely on the deep PCP theorem, instead it exploits the gap created in the W$[1]$-hardness proof of the parameterized biclique problem in [23]. In the same paper, the second author has already proved some inapproximability result which was shown by the PCP theorem before. Except for the derandomization using algebraic geometry in [23] the proofs are mostly elementary. Of course we are working under some stronger assumptions, i.e., ETH and FPT $\neq$ W$[1]$. It remains to be seen whether we can take full advantage of such assumptions to prove lower bounds matching those classical ones or even improve them as in Corollary 1.2.

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