Capture Complexity by Partition

Yijia Chen*[1] and Enshao Shen**[2]

[1] Université de Paris-Sud, Laboratoire de Recherche en Informatique, Bâtiment 490, F-91405 Orsay Cedex, France,
Yijia.Chen@lri.fr

[2] Shanghai Jiao Tong University, Department of Computer Science, Shanghai 200030, China,
shen-es@cs.sjtu.edu.cn

Abstract. We show in this paper a special extended logic, partition logic based on so called partition quantifiers, is able to capture some important complexity classes **NP**, **P** and **NL** by its natural fragments. The Fagin's Theorem and Immerman-Vardi's Theorem are rephrased and strengthened into a uniform partition logic setting. Also the dual operators for the partition quantifiers are introduced to expose some of their important model-theoretic properties. In particular they enable us to show a 0-1 law for the partition logic, even when finite variable infinitary logic is adjunct to it. As a consequence, partition logic cannot count without built-in ordering on structures. Considering its better theoretical properties and tools than those of second order logic, partition logic may provide us with an alternative, yet uniform insight for descriptive complexity.

1 Introduction

From finite model theory, or more precisely, the theory of descriptive complexity, we know that all important complexity classes have their own natural logic counterparts. In other words, for each of these complexity classes, there exists a logical language capable for defining exactly those problems effectively checkable in this complexity class. The first of such correspondence is due to Fagin[4], which equates nondeterministic polynomial time with existential second order logic Σ_1^1. Some of the major results are summarized by the following table:

Complexity Class	Logic
NP	Existential Second Order Logic
P	Least Fixed Point Logic
NL	Transitive Closure Logic

Aside from the assurance of the machine-independent description of complexity classes, these results pave the way for logical approach to complexity

* supported by Calife, a project of "Réseau National de Recherche en Télécommunications".

** supported in part by National Foundation of Natural Science of China.

issues, culminating in Immerman's famous proof of **NL** = co-**NL**[10]. Nevertheless compared with the Turing Machine, the unified machine model behind complexity classes, those logics seem more or less incoherent. For instance, we have different ways to reach them from first order logic: by adding higher order quantifiers (e.g. second order logic) or recursive operators (e.g. least fixed-point and transitive closure logic), and the latter enjoys an inductive flavor explicitly. Some effort has already been made to unify the logic theories, one is by Grädel[6], he identified some fragments of second order logic, say second order Horn and Krom logics, to capture **P** and **NL** respectively. Other approach is to augment the first order logic by a series of Lindström quantifiers which are based on some particular complete problems[1], such as using *Hamiltonian Path Operators* to capture **NP**[18].

Partition Logic arises from the ubiquitous mathematical operation as: partition a set into several disjoint union, over each partition subset certain property is satisfied homogeneously. One typical example is the congruent relation. H.-D. Ebbinghaus first in 1990's introduced the *Partition Quantifiers* to mimic such phenomena logically. This idea may go further back to Maltiz, yet with some extra infinite cardinality constraint[13], which is well beyond finite model theory. As it can define the connectivity of graph and some other non-first-order properties, partition logic is surely second order in nature. However it possesses some nice model-theoretic properties which are not shared by second order logic, such as the downward Löwenheim-Skolen-Tarski theorem and the Tarski Chain theorem[16]. In the meantime partition quantifiers can be looked as a special kind of monotone Lindström ones, so their Ehrenfeucht-Fraissé game is more elegant and tractable than that of second order logic[15]. Therefore in a sense, partition logic is locates in the lower level (near first order logic) of the fragment-spectrum of second order logic.

The attempt to apply partition logic in computer science started at a series of papers [14, 15, 17], in which it was proved that on word and tree structures, monadic fragment of partition logic is equivalent to monadic second order logic by the second author. He also found a natural fragment of partition logic equivalent to transitive closure logic, while Imhof showed another sublogic of it corresponds to bounded fixed point logic[8]. All these facts demonstrate that partition logic incorporates the recursive mechanism in a succinct form. In this paper, we show that partition logic may serve as a uniform platform to accommodate the most important part of complexity spectrum, i.e. **NP**, **P** and **NL**. Our main theorem provides a unified characterization of **NP** and **P** in partition logic on finite ordered structures, in which some key parameters of the machine is explicitly related to those of the partition quantifier, thereby giving the Turing machine a clearer logical reflection. Meanwhile we will also prove a 0-1 law for partition logic, thus without ordering, it even cannot define some very simple counting problem like Parity over arbitrary finite structures.

The paper is organized as follows: we give the definition of partition logic in Section 2, some examples are also provided. Section 3 reviews its relation with transitive closure logic and least fixed point logic by means of the dual operators of partition quantifiers, while the capturing of **NP** and **P** by partition logic is demonstrated in section 4. Section 5 is devoted to its 0-1 law.

We assume the reader has some basic knowledge of finite model theory, especially those results compiled in the preceding table, comprehensive details and references can be found in [3, 11].

2 Preliminary

In this paper, only finite relational vocabularies are considered. Unless otherwise declared, structures are not necessarily finite but with two elements at least. We use FO, SO to denote first order and second order logics respectively. $\mathsf{FV}(\psi)$ is the set of free variables in ψ of a certain logic. $|A|$ is the cardinality of the set A, also we overload $|\overline{a}|$ for the length of a vector or a sequence of elements $\overline{a}$.

The language of partition logic is the enlargement of FO by a new formation rule: for any $k, m, n > 0$, if $\varphi(\overline{x}, \overline{y})$ is a well-formed formula with $|\overline{x}| = mk$ and $|\overline{y}| = nk$, then $\overset{k}{\mathsf{P}}{}^{m,n}_{\overline{x};\overline{y}}\varphi(\overline{x}, \overline{y})$ is also well-formed, and in which $\overline{x}, \overline{y}$ are bound.

Definition 1. *Given a structure $\mathcal{A}$ and $\overline{e} \in A^{|\overline{z}|}$, $\mathcal{A} \models \overset{k}{\mathsf{P}}{}^{m,n}_{\overline{x};\overline{y}}\varphi(\overline{x}, \overline{y}, \overline{z})[\overline{e}]$ where $\mathsf{FV}(\varphi) \subseteq \{\overline{x}, \overline{y}, \overline{z}\}$, iff there is a partition of A^k: $A^k = U \dot\cup V$, $U \neq \emptyset \neq V$, such that $\mathcal{A} \models \varphi[\overline{a}_0 \dots \overline{a}_{m-1}, \overline{b}_0 \dots \overline{b}_{n-1}, \overline{e}]$ for arbitrary $\overline{a}_0, \dots, \overline{a}_{m-1} \in U$ and $\overline{b}_0, \dots, \overline{b}_{n-1} \in V$.*

Obviously partition logic is a fragment of SO and also a monotone Lindström logic. For convenience, we will write $\mathsf{P}^{m,n}_{\overline{x};\overline{y}}\varphi$ in lieu of $\overset{1}{\mathsf{P}}{}^{m,n}_{\overline{x};\overline{y}}\varphi$, and these quantifiers are particularly named monadic partition quantifiers[17]. For any $k, m, n > 0$, let $\mathrm{FO}(\overset{k}{\mathsf{P}}{}^{m,n})$ be the extension of FO with $\overset{k}{\mathsf{P}}{}^{m,n}$ only. We abbreviate

$$\mathrm{FO}(\overset{\omega}{\mathsf{P}}{}^{1,1}) = \bigcup_{k<\omega} \mathrm{FO}(\overset{k}{\mathsf{P}}{}^{1,1}),$$

$$\mathrm{FO}(\overset{\omega}{\mathsf{P}}{}^{\omega,1}) = \bigcup_{k,m<\omega} \mathrm{FO}(\overset{k}{\mathsf{P}}{}^{m,1}),$$

$$\mathrm{FO}(\overset{\omega}{\mathsf{P}}{}^{\omega,\omega}) = \bigcup_{k,m,n<\omega} \mathrm{FO}(\overset{k}{\mathsf{P}}{}^{m,n}).$$

It is not very hard to show whenever $k \geq k', m \geq m'$ and $n \geq n'$, $\mathrm{FO}(\overset{k}{\mathsf{P}}{}^{m,n}) \geq \mathrm{FO}(\overset{k'}{\mathsf{P}}{}^{m',n'})$, so for example, $(\overset{3}{\mathsf{P}}{}^{1,5}_{\overline{x};\overline{y}}R\overline{xy}) \wedge (\overset{2}{\mathsf{P}}{}^{4,2}_{\overline{x}';\overline{y}'}R'\overline{x}'\overline{y}') \in \mathrm{FO}(\overset{3}{\mathsf{P}}{}^{4,5}) \leq \mathrm{FO}(\overset{\omega}{\mathsf{P}}{}^{\omega,\omega})$. Meanwhile let $\mathrm{FO}(\mathrm{pos}\ \overset{\omega}{\mathsf{P}}{}^{\omega,\omega})$ denote the sublogic of $\mathrm{FO}(\overset{\omega}{\mathsf{P}}{}^{\omega,\omega})$ consisting of the formulae in which all partition quantifiers occur positively, i.e. within the scope of an even number of negation signs.

Example 1. One of the simplest properties that partition logic can deal with is the connectivity of (directed) graphs which is undefinable in FO,

$$\underline{\mathbf{Conn}} := \neg\mathsf{P}^{1,1}_{x;y}\neg Exy, \quad \text{where } E \text{ is a binary relation symbol.}$$

Namely, the graph can not be divided into two parts between which there is no cross edge. So $\mathcal{A} = (A, E^{\mathcal{A}})$ is strongly connected iff $(A, E^{\mathcal{A}}) \models$ **Conn**. Meanwhile the reachability of two vertices can be characterized by:

$$\underline{\mathbf{Path}}(u,v) := \neg \mathsf{P}^{1,1}_{x;y}[\neg \mathrm{E}xy \wedge y \neq u \wedge x \neq v],$$

which states that we can not divide the graph into two parts without cross edge to separate u and v. Then e is reachable to f in $(A, E^{\mathcal{A}})$ iff $(A, E^{\mathcal{A}}) \models \underline{\mathbf{Path}}[e,f]$. Surely we have

$$\models \underline{\mathbf{Conn}} \leftrightarrow \forall uv[u \neq v \rightarrow \underline{\mathbf{Path}}(u,v)].$$

Example 2. Though the definition of partition quantifiers concerns only bipartitions, $\mathrm{FO}(\overset{\omega}{\mathsf{P}}^{\omega,\omega})$ can also deal with properties built upon multi-partitions.

$$\begin{aligned}
\underline{\mathbf{4\text{-}Color}} &:= \exists u_0u_1u_2u_3 \, \overset{2}{\mathsf{P}}^{4,4}_{\overline{x}_0\overline{x}_1\overline{x}_2\overline{x}_3;\overline{y}_0\overline{y}_1\overline{y}_2\overline{y}_3}[\psi_1 \wedge \psi_2],\\
\psi_1 := &\quad (x_{00} = 0 \rightarrow (x_{01} \neq u_2 \wedge x_{01} \neq u_3))\\
&\wedge(y_{00} = 0 \rightarrow (y_{01} \neq u_0 \wedge y_{01} \neq u_1))\\
&\wedge(x_{00} = 1 \rightarrow (x_{01} \neq u_1 \wedge y_{01} \neq u_3))\\
&\wedge(y_{00} = 1 \rightarrow (y_{01} \neq u_0 \wedge y_{01} \neq u_2)),\\
\psi_2 := &\quad (x_{00} = 0 \wedge x_{10} = 1 \wedge x_{01} = x_{11}\\
&\quad \wedge x_{20} = 0 \wedge x_{30} = 1 \wedge x_{21} = x_{31}) \rightarrow \neg E x_{01}x_{21}\\
&\wedge(x_{00} = 0 \wedge y_{00} = 1 \wedge x_{01} = y_{01}\\
&\quad \wedge x_{10} = 0 \wedge y_{10} = 1 \wedge x_{11} = y_{11}) \rightarrow \neg E x_{01}x_{11}\\
&\wedge(y_{00} = 0 \wedge x_{00} = 1 \wedge y_{01} = x_{01}\\
&\quad \wedge y_{10} = 0 \wedge x_{10} = 1 \wedge y_{11} = x_{11}) \rightarrow \neg E y_{01}y_{11}\\
&\wedge(y_{00} = 0 \wedge y_{10} = 1 \wedge y_{01} = y_{11}\\
&\quad \wedge y_{20} = 0 \wedge y_{30} = 1 \wedge y_{21} = y_{31}) \rightarrow \neg E y_{01}y_{21},
\end{aligned}$$

where each $\overline{x}_i = x_{i0}x_{i1}$ and similar for $\overline{y}_i$. Note $0, 1$ are the boolean constants which can be easily eliminated by first order existential quantification. A graph $\mathcal{A} \models$ **4-color** iff $\mathcal{A}$ can be 4 colored in a way such that each color is used at least once, which in case $|A| \geq 4$, is equivalent to 4-colorability problem. Assume $U|V$ is the partition that makes **4-color** satisfied, let $U_i = \{e \mid \langle i, e\rangle \in U\}$ and $V_i = \{e \mid \langle i, e\rangle \in V\}$ for $i = 0, 1$, then it induces 4 disjoint partition subsets of A, $U_0 \cap V_0$, $U_0 \cap V_1$, $U_1 \cap V_0$ and $U_1 \cap V_1$. While ψ_1 ensures that each of them has a non-empty witness u_i, and any two points in the same subset being not adjacent is expressed by ψ_2. Clearly 3-colorability can be defined likewise.

3 Dual Operators of Partition Quantifiers

The classical extended logic capturing the graph reachability is the transitive closure logic, i.e. FO(TC)[9]. Example 1 invokes the following relation between $\mathrm{FO}(\overset{\omega}{\mathsf{P}}^{1,1})$ and FO(TC).

Theorem 1. [14] *The "duality" between* $\overset{k}{\mathsf{P}}{}^{1,1}$ *and* TC*,*

$$\models [\mathrm{TC}^k_{\overline{x},\overline{y}}\psi(\overline{x},\overline{y})](\overline{u},\overline{v}) \leftrightarrow \neg \overset{k}{\mathsf{P}}{}^{1,1}_{\overline{x};\overline{y}}[\overline{x}\neq\overline{v}\wedge\overline{y}\neq\overline{u}\wedge\neg\psi(\overline{x},\overline{y})],$$

$$\models \overset{k}{\mathsf{P}}{}^{1,1}_{\overline{x};\overline{y}}\psi(\overline{x},\overline{y}) \leftrightarrow \exists\overline{u}\,\overline{v}[\neg(\mathrm{TC}^k_{\overline{x},\overline{y}}\neg\psi(\overline{x},\overline{y}))(\overline{u},\overline{v})].$$

Hence $\mathrm{FO}(\overset{\omega}{\mathsf{P}}{}^{1,1}) = \mathrm{FO}(\mathrm{TC})$, and as a result, $\mathrm{FO}(\overset{\omega}{\mathsf{P}}{}^{1,1})$ captures **NL** on finite ordered structures. The above "duality" also inspires the next modification of partition quantifiers to so called *pseudo transitive closure operators* TP.

Definition 2. *Let* $\psi(\overline{u},\overline{v},\overline{z}) = [\overset{k}{\mathsf{TP}}{}^{m,n}_{\overline{x};\overline{y}}\varphi(\overline{x},\overline{y},\overline{z})](\overline{u},\overline{v})$*, where* $|\overline{u}| = |\overline{v}| = k$*,* $|\overline{x}| = mk$*,* $|\overline{y}| = nk$ *and* $\mathsf{FV}(\varphi) \subseteq \{\overline{x},\overline{y},\overline{z}\}$*. Given* $\overline{e},\overline{f} \in A^k$ *and* $\overline{g} \in A^{|\overline{z}|}$*, if for any partition* $U|V$ *of* A^k *with* $\overline{e} \in U$ *and* $\overline{f} \in V$*, there exist* $\overline{a}_0,\ldots,\overline{a}_{m-1} \in U$ *and* $\overline{b}_0,\ldots,\overline{b}_{n-1} \in V$ *such that* $\mathcal{A} \models \varphi[\overline{a}_0\ldots\overline{a}_{m-1},\overline{b}_0\ldots\overline{b}_{n-1},\overline{g}]$*, then we say* $\mathcal{A} \models \psi[\overline{e},\overline{f},\overline{g}]$*.*

The correspondence between $\overset{k}{\mathsf{P}}{}^{m,n}$ and $\overset{k}{\mathsf{TP}}{}^{m,n}$ is superficial, and we can see $\overset{k}{\mathsf{TP}}{}^{1,1}$ is exactly k-dimensional TC by Theorem 1:

$$\models [\overset{k}{\mathsf{TP}}{}^{m,n}_{\overline{x};\overline{y}}\psi(\overline{x},\overline{y})](\overline{u},\overline{v}) \leftrightarrow \neg \overset{k}{\mathsf{P}}{}^{m,n}_{\overline{x};\overline{y}}[\overline{x}\neq\overline{v}^{3}\wedge\overline{y}\neq\overline{u}\wedge\neg\psi(\overline{x},\overline{y})],$$

$$\models \overset{k}{\mathsf{P}}{}^{m,n}_{\overline{x};\overline{y}}\psi(\overline{x},\overline{y}) \leftrightarrow \exists\overline{u}\,\overline{v}[\neg(\overset{k}{\mathsf{TP}}{}^{m,n}_{\overline{x};\overline{y}}\neg\psi(\overline{x},\overline{y}))(\overline{u},\overline{v})].$$

The following lemma shows some essential similarities between TP and TC, which was proved in [16], yet without introducing TP.

Lemma 1.
(1) *For any* $\mathcal{A}$ *and* $\overline{e},\overline{f},\overline{g} \in A^k$*, if both* $\mathcal{A} \models \overset{k}{\mathsf{TP}}{}^{m,n}_{\overline{x};\overline{y}}\psi[\overline{e},\overline{f}]$ *and* $\mathcal{A} \models \overset{k}{\mathsf{TP}}{}^{m,n}_{\overline{x};\overline{y}}\psi[\overline{f},\overline{g}]$*, then we have* $\mathcal{A} \models \overset{k}{\mathsf{TP}}{}^{m,n}_{\overline{x};\overline{y}}\psi[\overline{e},\overline{g}]$*.*
(2) *Given* $\mathcal{A} \subseteq \mathcal{B}$ *and* $\overline{e},\overline{f} \in A^k$ *with* $\mathcal{A} \models \overset{k}{\mathsf{TP}}{}^{m,n}_{\overline{x};\overline{y}}\psi[\overline{e},\overline{f}]$*, for any* $\overline{u} \in A^{mk}$ *and* $\overline{v} \in A^{nk}$*,* $\mathcal{A} \models \psi[\overline{u},\overline{v}]$ *implies* $\mathcal{B} \models \psi[\overline{u},\overline{v}]$*, then* $\mathcal{B} \models \overset{k}{\mathsf{TP}}{}^{m,n}_{\overline{x};\overline{y}}\psi[\overline{e},\overline{f}]$*.*
(3) *Let* $\tau = \{R\}$*, where* R *is a* $k(m+n)$*-ary relation symbol, for a* τ*-structure* $\mathcal{A}$*,* $\mathcal{A} \models [\overset{k}{\mathsf{TP}}{}^{m,n}_{\overline{x};\overline{y}}\mathrm{R}\overline{x}\,\overline{y}][\overline{e},\overline{f}]$ *where* $\overline{e},\overline{f} \in A^k$*, and* D *is the diagram of* $\mathcal{A}$*, there exists a finite subset* $D_{\overline{e},\overline{f}}$ *of* D *satisfying* $D_{\overline{e},\overline{f}} \models [\overset{k}{\mathsf{TP}}{}^{m,n}_{\overline{x};\overline{y}}\mathrm{R}\overline{x}\,\overline{y}](\underline{\overline{e}},\underline{\overline{f}})$*, where* $\underline{\overline{e}}$*,* $\underline{\overline{f}}$ *are new constant symbol sequences interpreted by* $\overline{e}$ *and* $\overline{f}$ *respectively.*

Intuitively, (1) guarantees the transitivity of TP, (2) means TP is closed under extensions, and in (3), $D_{\overline{e},\overline{f}}$ bears witness to the satisfaction of TP on $\overline{e},\overline{f}$, which may be imagined as the finite "path" in $\mathcal{A}$ that connects $\overline{e}$ and $\overline{f}$. The last property plays a crucial role in the proof of certain model-theoretic theorems of

[3] Here $\overline{x}\neq\overline{v}$ stands for $x_{00}\neq v_0 \vee x_{01}\neq v_1 \vee\cdots\vee x_{0k-1}\neq v_{k-1}$.

$\mathrm{FO}(\overset{\omega}{\mathsf{P}}{}^{\omega,\omega})$[16], also it can be used to embed partition logic into $\mathcal{L}_{\omega_1\omega}$. But as Example 2 shows that 3-colorability is axiomatizable in it, $\mathrm{FO}(\overset{\omega}{\mathsf{P}}{}^{\omega,\omega})$ is not inside finite variable infinitary logic $\mathcal{L}^{\omega}_{\infty\omega}$ by a result of Dawar[7].

The naive model-checking algorithm derived directly from the definition of $\overset{k}{\mathsf{P}}{}^{m,n}$ is unavoidably of exponential time, but Theorem 1 has already implied some **NL** algorithm for $\overset{k}{\mathsf{P}}{}^{1,1}$. To step forward more, we can design a **P** algorithm for any $\overset{k}{\mathsf{P}}{}^{m,1}$ by the following embedding result of $\overset{k}{\mathsf{P}}{}^{m,1}$ in the least fixed point logic, FO(LFP).

Proposition 1. *Given a structure $\mathcal{A}$ and $\overline{e}, \overline{f} \in A^k$,*
(1) $\mathcal{A} \models [\overset{k}{\mathsf{TP}}{}^{m,1}_{\overline{x};\overline{y}}\varphi(\overline{x},\overline{y})][\overline{e},\overline{f}]$ *iff* $\mathcal{A} \models \mathrm{LFP}_{\overline{x},\overline{y},X}(\overline{x} = \overline{y} \vee \exists \overline{w} X\overline{x}\overline{w}_0 \wedge \cdots \wedge X\overline{x}\overline{w}_{m-1} \wedge \varphi(\overline{w},\overline{y}))[\overline{e},\overline{f}]$, *where* $\overline{w} = \overline{w}_0,\ldots,\overline{w}_{m-1}$.
(2) $\mathcal{A} \models [\mathrm{LFP}_{\overline{y},Y}(\varphi_0(\overline{y}) \vee \exists \overline{x}(Y\overline{x}_0 \wedge \ldots \wedge Y\overline{x}_{m-1} \wedge \varphi_1(\overline{x}_0\ldots\overline{x}_{m-1},\overline{y})))][\overline{e}]$, *iff* $\mathcal{A} \models [\neg \overset{k}{\mathsf{P}}{}^{m,1}_{\overline{x};\overline{y}} \neg\varphi_1(\overline{x},\overline{y}) \wedge \overline{x} \neq \overline{u} \wedge \neg\varphi_0(\overline{y})][\overline{e}]$, *where* $\overline{x} = \overline{x}_0,\ldots,\overline{x}_{m-1}$.

Proof. Routine. □

The LFP-formulae in above proposition indeed fall into Bounded Fixed-Point Logic, FO(BFP)[3], which allows the LFP operator only if there is bounded $m \geq 1$ such that the tuple in a new stage is already witnessed by a set of at most m many tuples of the preceding stage, i.e. all fixed-point formulae are of the form: $\mathrm{LFP}_{\overline{y},Y}(\varphi_0(\overline{y}) \vee \exists\overline{x}_0 \ldots \exists\overline{x}_{m-1}(Y\overline{x}_0 \wedge \ldots \wedge Y\overline{x}_{m-1} \wedge \varphi_1(\overline{x}_0\ldots\overline{x}_{m-1},\overline{y}))$. Thus by Proposition 1 and the equivalence between $\overset{k}{\mathsf{P}}{}^{m,1}$ and $\overset{k}{\mathsf{TP}}{}^{m,1}$,

Theorem 2. ([8]) $\mathrm{FO}(\overset{\omega}{\mathsf{P}}{}^{\omega,1}) = \mathrm{FO(BFP)}$.

We know that FO(TC) < FO(BFP) < FO(LFP)[3], and 3-colorability can separate $\mathrm{FO}(\overset{\omega}{\mathsf{P}}{}^{\omega,\omega})$ and $\mathcal{L}^{\omega}_{\infty\omega}$, henceforth

Corollary 1.

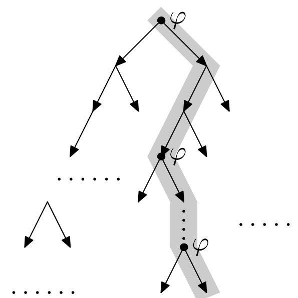
$$\mathrm{FO}(\overset{\omega}{\mathsf{P}}{}^{1,1}) < \mathrm{FO}(\overset{\omega}{\mathsf{P}}{}^{\omega,1}) < \mathrm{FO}(\overset{\omega}{\mathsf{P}}{}^{\omega,\omega}),$$
$$\mathrm{FO}(\overset{\omega}{\mathsf{P}}{}^{\omega,1}) < \mathrm{FO(LFP)}.$$

4 Characterizing NP machine

In this section, we will characterize **NP** Turing machines by $\mathrm{FO}(\overset{\omega}{\mathsf{P}}{}^{\omega,\omega})$ on finite ordered structures. The main technique and convention used here follow [3]. First a simple warming-up example is given, then we sketch the proof idea of the main theorem enlightened by this example, while full-length proof is omitted due to the lack of space.

Example 3. Let τ be the vocabulary for binary tree, i.e. $\tau = \{\epsilon, S_1, S_2\}$, where ϵ is the constant symbol for the root node, and S_1,S_2 are respectively the left and right successor relation symbols. The sentence Θ defined left below asserts that there exists a path from the root such that if a node in the path satisfies the formula $\varphi(x)$ then its right successor must lie in this path too, which may be depicted by the figure right below, where the solid nodes are those satisfying φ in the path.

$$\begin{aligned}\Theta := \quad \mathsf{P}^{2,2}_{x,x';y,y'}\ & \neg(\epsilon = y) && (1)\\ \wedge\ & \neg(S_1yx \vee S_2yx) && (2)\\ \wedge\ & \neg(S_1xy \wedge S_2xy') && (3)\\ \wedge\ & \neg\exists z(S_1zx \wedge S_2zx') && (4)\\ \wedge\ & \neg(\varphi(x) \wedge S_2xy) && (5)\end{aligned}$$



Now for any finite tree structure $\mathcal{A} = \langle A, \epsilon^{\mathcal{A}}, S_1^{\mathcal{A}}, S_2^{\mathcal{A}}\rangle \models \Theta$, there exists a partition $U|V$ of A such that the inner formula is satisfied homogeneously on this $U|V$. We claim that U is the required path. First by (1), $\epsilon^{\mathcal{A}} \in U$, since for any $y \in V$, $\epsilon^{\mathcal{A}} \neq y$. To verify that U is closed under predecessor, note if there is some node x in U such that its predecessor node is not in U, i.e. some $y \in V$ with $S_1^{\mathcal{A}}yx$ or $S_2^{\mathcal{A}}yx$, then these x and y will refute the second conjunct. The subformula (3) implies that for any $x \in U$, at least one of its successors will also lie in U, assume contrarily one x's left successor y and right successor y' are both in V, then (3) can not be satisfied homogeneously. On the other hand, for any node $z \in U$, at most one of its successors is inside U, which is ensured by (4). Henceforth, (1)-(4) guarantee that the first partition subset U is indeed a path of $\mathcal{A}$. Finally by (5), no element $y \in V$ is the right successor of an $x \in U$ satisfying φ. The reverse direction is trivial.

Next we fix a vocabulary $\tau = \tau_0 \dot\cup \tau_1$, where $\tau_0 = \{<, S, \min, \max\}$ and $\tau_1 = \{R_1, \ldots, R_k\}$, while all τ-structures under consideration have the interpretation of $<$, S, min and max as the ordering, successor relation, minimum and maximum elements, i.e. ordered structures. For an **NP** Turing machine M which is time-bounded by n^d to accept τ-structures using $k+1$ input tapes for coding the structure, and some other m work tapes for intermediate computation, let br be the maximum number of choices that M can face each time, and note $br = 1$ if and only if M is deterministic. It is a standard technique to code any computation run of M by a $(2d+2)$-ary relation on the input structure: the first d-ary part is the "time stamp", the rest will code the actual configuration of M at a particular time, that is to say, if $|t|$ is the n-adic representation of time t, then the $(d+2)$-ary relation $R|t|$ fully describes M's configuration at time t, including the state, the inscriptions of each cell on each input or work tape, and also the position of each reading head on those tapes.

The Fagin's theorem relies on the observation that it is possible to define a first order formula $\varphi(X, Y)$ saying that Y is a valid configuration after M makes a move from the original configuration X, and in the meantime we can introduce two simple formulae $\psi_{\text{init}}(X)$ and $\psi_{\text{end}}(X)$ expressing X is the initial configuration and a final configuration with an accepting state respectively. Thus a Σ_1^1 sentence

$$\begin{aligned} \Theta = \exists R[\psi_{\mathrm{init}}(R|t_{\mathrm{init}}|) \wedge \psi_{\mathrm{end}}(R|t_{\mathrm{end}}|) \\ \wedge \forall |t_0|\ |t_1|(\text{“}t_1 = t_0 + 1\text{”} \rightarrow \varphi(R|t_0|, R|t_1|))] \end{aligned}$$

where $|t_{\mathrm{init}}|$ and $|t_{\mathrm{end}}|$ are FO definable constant vectors representing the initial and the final times of the computation run, can be constructed. Clearly for any finite structure $\mathcal{A} \models \Theta$ iff R can be interpreted as a accepting run of M on $\mathcal{A}$, i.e. M recognizes $\mathcal{A}$.

Shifted to partition logic, we aim to devise a sentence $\overset{2d+2}{\mathsf{P}}\chi$, such that once it is satisfied in a structure $\mathcal{A}$, the first partition subset will be interpreted as an accepting run. Surely the above φ, ψ_{init} and ψ_{end} can not be directly applied, because we are deprived of the explicit use of second order variables. Though the overall idea is similar to the previous tree example, much more deliberation is needed. Our description of one step computation is divided into two phases: firstly M chooses an instruction according to the current configuration, then M changes its configuration following the instruction. For the first phase, we add a new element into each $R|t|$ to indicate which instruction will be actually carried out concerning all nondeterministic choices that M can make over $R|t|$. Note the set of all possible instructions is fully determined by the state of M at time t together with the symbols read by those heads on input and work tapes at that moment, which are reflected by a finite number of elements in $R|t|$. Therefore we are in a similar situation like Example 3 which must regulate any nonterminal point of the path has one and only one successor also lying in the path, while the choice between left and right successor can be nondeterministic. Once the instruction is chosen for time t, the configuration of time $t+1$ is totally determined. It is crucial that each element in $R|t+1|$ only depends on a bounded number of elements in $R|t|$: the new state and the new inscriptions on those positions originally the heads were pointing at are determined by the chosen instruction; the new head positions are determined by the instruction and the heads' original positions; the inscriptions of rest positions remain unchanged, thus determined by their original inscriptions. So $R|t+1|$ can be characterized in the same fashion as the tree example requires those nodes satisfying φ must have right successor in the path. Thus a careful and tedious elaboration will yield,

Theorem 3. *if a class of finite ordered τ-structures K is accepted by M, then K is axiomatizable in* $\mathrm{FO}(\overset{\omega}{\breve{\mathsf{P}}}^{\omega,\omega})$ *by a sentence* $\Theta = \overset{2d+2}{\mathsf{P}}{}^{3+k+2m,br}\chi$, *where χ is a quantifier-free formula.*

Conversely, we can effectively construct a **NP** machine for any given $\varphi \in \mathrm{FO}(\mathrm{pos}\ \overset{\omega}{\breve{\mathsf{P}}}^{\omega,\omega})$ to check whether $\mathcal{A} \models \varphi$ for each $\mathcal{A}$, so

Theorem 4. *if K is a* $\mathrm{FO}(\mathrm{pos}\ \overset{\omega}{\breve{\mathsf{P}}}^{\omega,\omega})$ *definable τ-structure class, then* $K \in \mathbf{NP}$.

Combining Theorems 2, 3 and 4, we obtain

Corollary 2. *On finite ordered structures,*

$$\begin{array}{lcl}\mathrm{FO}(\text{pos } \overset{\omega}{\mathsf{P}}^{\omega,\omega}) & = & \mathbf{NP}, \\ \mathrm{FO}(\overset{\omega}{\mathsf{P}}^{\omega,1}) & = & \mathbf{P}.\end{array}$$

5 0-1 Law for Partition Logic

In this section, we are only interested in the labeled 0-1 law under the uniform probability measure, i.e. each structure of cardinality n has $\{0, \dots, n-1\}$ as its underlying universe, and with same probability. To prove the 0-1 law for partition logic, we would rather focus on TP instead of the original quantifier P, making substantial use of its preservation over extensions, i.e. Lemma 1.2. There have been several results concerning logics that deal with properties closed under extensions, or equivalently closed under substructures[5, 2, 12]. In particular [2] proved that the 0-1 law is retained in FO augmented with those generalized quantifiers defined over classes of structures that are closed under extensions. Later we will see TP can be regarded as such generalized quantifier. First we revise the definition of generalized quantifiers. Fix a vocabulary $\sigma = \{R_1, \dots, R_s\}$, where each R_i is r_i-ary relation symbol. Now upon a class of σ-structures K which is closed under isomorphisms, if for $1 \le i \le s$, $\psi_i(\overline{x}_i, \overline{y})$ is a formula with $\mathsf{FV}(\psi_i) \subseteq \{\overline{x}_i, \overline{y}\}$, then $Q_K \overline{x}_1, \dots, \overline{x}_s[\psi_1, \dots, \psi_s]$ is also a formula which bounds all $\overline{x}_i$. For any τ-structure $\mathcal{A}$ and $\overline{a} \in A^{|\overline{y}|}$,

$$\mathcal{A} \models Q_K \overline{x}_1, \dots, \overline{x}_s[\psi_1, \dots, \psi_s][\overline{a}], \text{ iff } (A, \psi_1^{\mathcal{A}}(\_\_, \overline{a}), \dots, \psi_s^{\mathcal{A}}(\_\_, \overline{a})) \in K,$$

where each $\psi_i^{\mathcal{A}}(\_\_, \overline{a})$ stands for $\{\overline{b} \in A^{r_i} \mid \mathcal{A} \models \psi_i[\overline{b}, \overline{a}]\}$. For a set of generalized quantifiers $\mathcal{Q}$, $\mathrm{FO}(\mathcal{Q})$ and $\mathcal{L}^t_{\infty\omega}(\mathcal{Q})$ denote the extension of FO and t-variable infinitary logic with $\mathcal{Q}$ respectively, while $\mathcal{L}^\omega_{\infty\omega}(\mathcal{Q}) = \bigcup_{t<\omega} \mathcal{L}^t_{\infty\omega}(\mathcal{Q})$.

Next we detail some notions concerning the class of structures that are used to define generalized quantifiers of which we will prove the 0-1 law.

Definition 3. *Let K be a class of structures as defined above, K is said to be* closed under extensions *iff whenever $\mathcal{A} \in K$ and $\mathcal{A} \subseteq \mathcal{A}'$, we have $\mathcal{A}' \in K$. While a* finitely witnessed *K means for any infinite $\mathcal{A}$, if $\mathcal{A} \in K$, then there exists a finite $\mathcal{A}'$ such that $\mathcal{A}' \subset \mathcal{A}$ and $\mathcal{A}' \in K$, and we can say $\mathcal{A}'$ finitely witnesses $\mathcal{A} \in K$. K is* finitely based *if it is both closed under extensions and finitely witnessed. A Q_K is called closed under extensions, finitely witnessed or finitely based if K is respectively so.*

Example 4.
(1) To define first order quantifier $\exists$, assume $K = \{(A, U) \mid U \subseteq A \text{ and } U \neq \emptyset\}$, then $\models \exists x \varphi(x) \leftrightarrow Q_K x[\varphi(x)]$. Counting quantifier $\exists^{\ge l}$ can be regarded as Q_K where $K = \{(A, U) \mid |U| \ge l\}$.
(2) For each $\overset{k}{\mathsf{TP}}^{m,n}$, let $\sigma = \{R, P\}$ where R and P are respectively $k(m+n)$-ary

and $2k$-ary relation symbols, set $K = \{\mathcal{A} \mid \mathcal{A} \models \exists\overline{uv}\{[\overset{k}{\mathsf{TP}}{}^{m,n}_{\overline{x};\overline{y}} R\overline{xy}](\overline{uv}) \wedge P\overline{uv}\}\}$. We have

$$\models [\overset{k}{\mathsf{TP}}{}^{m,n}_{\overline{x};\overline{y}} \psi(\overline{x},\overline{y})](\overline{uv}) \leftrightarrow Q_K\overline{xyx}'\overline{y}'[\psi, \overline{x}' = \overline{u} \wedge \overline{y}' = \overline{v}],$$

$$\models Q_K\overline{xyx}'\overline{y}'[\psi_1(\overline{x},\overline{y}), \psi_2(\overline{x}',\overline{y}')] \leftrightarrow \exists\overline{uv}\{[\overset{k}{\mathsf{TP}}{}^{m,n}_{\overline{x};\overline{y}} \psi_1(\overline{x},\overline{y})](\overline{uv}) \wedge \psi_2(\overline{u},\overline{v})\}.$$

In the above examples, all Q_Ks are finitely based. Specifically, Lemma 1.2 guarantees TP is closed under extensions, while in Lemma 1.3 $\mathcal{A} \restriction D_{\overline{e},\overline{f}}$ finitely witnesses $\mathcal{A} \models [\overset{k}{\mathsf{TP}}{}^{m,n}_{\overline{x};\overline{y}} \mathrm{R}\overline{x}\,\overline{y}][\overline{e},\overline{f}]$. As a consequence, one result in [2] implies that $\mathrm{FO}(\overset{\omega}{\mathsf{P}}{}^{\omega,\omega})$ has the 0-1 law. Furthermore in the following we will strengthen it to

Theorem 5. *$\mathcal{L}^{\omega}_{\infty\omega}(\overset{\omega}{\mathsf{P}}{}^{\omega,\omega})$ has the labeled 0-1 law.*

The proof method we adopt here is rather traditional, i.e. based on a *transfer property* (Theorem 6) , compared with [2] of which we see no easy extension to $\mathcal{L}^{\omega}_{\infty\omega}$.

Let ϵ_i be the conjunction of finitely many r-extension axioms with $r \leq i$, and $\mathrm{T}_{\mathrm{rand}}$ is the set of all extension axioms, i.e. $\mathrm{T}_{\mathrm{rand}} = \bigwedge_{i>0} \epsilon_i$. Furthermore $\mathcal{A}_{\mathrm{rand}}$ is the unique countable random structure up to isomorphism, i.e. $\mathcal{A}_{\mathrm{rand}} \models \mathrm{T}_{\mathrm{rand}}$. We will rely on the following lemma heavily later.

Lemma 2. *Given two structures $\mathcal{A}, \mathcal{B}$, particularly $\mathcal{A} \models \epsilon_i$ and $h : \overline{a} \longmapsto \overline{b}$ is a partial isomorphism between $\mathcal{A}$ and $\mathcal{B}$ with finite domain $|\{\overline{a}\}| \leq i$, then for any finite subset $S \subseteq B$ with $|S| \leq i - |\{\overline{a}\}|$, there exists a finite subset $S' \subseteq A$, such that h can be extended to some larger partial isomorphism $h' : S', \overline{a} \longmapsto S, \overline{b}$.*

Proof. Easy. □

For any structure $\mathcal{A}$ and $\overline{a} \in A^*$, define a first order formula

$$\varphi^0_{\mathcal{A},\overline{a}} = \bigwedge\{\psi(\overline{x}) \mid \psi \text{ atomic or negated atomic, and } \mathcal{A} \models \psi[\overline{a}]\},$$

and it follows that $\models \varphi^0_{\mathcal{A},\overline{a}} \leftrightarrow \varphi^0_{\mathcal{B},\overline{b}}$ iff $\overline{a} \longmapsto \overline{b}$ is a partial isomorphism between $\mathcal{A}$ and $\mathcal{B}$. Later on we will write $\varphi^0_{\mathcal{A},\overline{a}} = \varphi^0_{\mathcal{B},\overline{b}}$ instead of $\models \varphi^0_{\mathcal{A},\overline{a}} \leftrightarrow \varphi^0_{\mathcal{B},\overline{b}}$, if no ambiguity arises. Obviously an equivalence relation over A^* can be induced: $\overline{a} \equiv^0_{\mathcal{A}} \overline{b}$ iff $\varphi^0_{\mathcal{A},\overline{a}} = \varphi^0_{\mathcal{A},\overline{b}}$, so each $\varphi^0_{\mathcal{A},\overline{a}}$ defines an equivalence class. Next lemma will show in $\mathcal{A}_{\mathrm{rand}}$, this equivalence relation holds for arbitrary higher level formulae.

Lemma 3. *For any $\psi \in \mathcal{L}[\tau]$ and $\overline{a}, \overline{b} \in A^{|\mathsf{FV}(\psi)|}_{\mathrm{rand}}$ with $\varphi^0_{\mathcal{A}_{\mathrm{rand}},\overline{a}} = \varphi^0_{\mathcal{A}_{\mathrm{rand}},\overline{b}}$, $\mathcal{A}_{\mathrm{rand}} \models \psi[\overline{a}]$ iff $\mathcal{A}_{\mathrm{rand}} \models \psi[\overline{b}]$. The above $\mathcal{L}$ could be any logical system whose satisfaction relation is closed under isomorphisms and permits substitution.*

Proof. By $\varphi^0_{\mathcal{A}_{\text{rand}},\overline{a}} = \varphi^0_{\mathcal{A}_{\text{rand}},\overline{b}}$, $h : \overline{a} \longmapsto \overline{b}$ is a partial automorphism over $\mathcal{A}_{\text{rand}}$. Then the fact of $\mathcal{A}_{\text{rand}}$ satisfying T_{rand} and being countable ensures that h can be enlarged to an automorphism h' on $\mathcal{A}_{\text{rand}}$ via a back and forth process. Thereby

$$\begin{aligned} & \mathcal{A}_{\text{rand}} \models \psi[\overline{a}] \\ \text{iff} \quad & h'(\mathcal{A}_{\text{rand}}) \models \psi[h'(\overline{a})] \\ \text{iff} \quad & \mathcal{A}_{\text{rand}} \models \psi[h(\overline{a})] \\ \text{iff} \quad & \mathcal{A}_{\text{rand}} \models \psi[\overline{b}]. \end{aligned}$$

□

So it makes sense to introduce the following canonization function δ on $\mathcal{A}_{\text{rand}}$. Let θ be a choice function over the equivalence classes of $\equiv^0_{\mathcal{A}_{\text{rand}}}$, i.e. $\theta([\overline{a}]_{\equiv^0_{\mathcal{A}_{\text{rand}}}}) \in [\overline{a}]_{\equiv^0_{\mathcal{A}_{\text{rand}}}}$, define $\delta : A^*_{\text{rand}} \longrightarrow A^*_{\text{rand}}$ with $\delta(\overline{a}) = \theta([\overline{a}]_{\equiv^0_{\mathcal{A}_{\text{rand}}}})$. Surely $\varphi^0_{\mathcal{A}_{\text{rand}},\overline{a}} = \varphi^0_{\mathcal{A}_{\text{rand}},\delta(\overline{a})}$, so by Lemma 3, $\mathcal{A}_{\text{rand}} \models \psi[\overline{a}]$ iff $\mathcal{A}_{\text{rand}} \models \psi[\delta(\overline{a})]$ for arbitrary ψ. Note for any fixed i, $|\{\delta(\overline{a}) \mid \overline{a} \in A^i_{\text{rand}}\}| = |\{[\overline{a}]_{\equiv^0_{\mathcal{A}_{\text{rand}}}} \mid \overline{a} \in A^i_{\text{rand}}\}| < \omega$ due to the finiteness of τ.

Clearly Lemma 3 exhibits the extreme symmetry of $\mathcal{A}_{\text{rand}}$, furthermore it implies the following technical result which gives a bound on the finite witness of any Q_K over $\mathcal{A}_{\text{rand}}$.

Lemma 4. *Given a finitely witnessed Q_K and $t \in \mathbb{N}$, we can find a fixed n such that: for any $\varphi = Q_K\overline{x}_1, \ldots, \overline{x}_s[\psi_1, \ldots, \psi_s] \in \mathcal{L}[\tau]$ with $|\mathsf{FV}(\varphi)| \le t$ and $\overline{a} \in A^{|\mathsf{FV}(\varphi)|}_{\text{rand}}$, if $\mathcal{A}_{\text{rand}} \models \varphi[\overline{a}]$, then there exists a finite set $D \subset A_{\text{rand}}$ with $|D| \le n$ such that*

$$(D, \psi_1^{\mathcal{A}_{\text{rand}}}(\_\_, \overline{a}) \upharpoonright D, \ldots, \psi_s^{\mathcal{A}_{\text{rand}}}(\_\_, \overline{a}) \upharpoonright D) \in K.$$

Note $\mathcal{L}$ is the same as in Lemma 2.

Proof. First observe that for any $\psi \in \mathcal{L}[\tau]$,

$$\mathcal{A}_{\text{rand}} \models \psi \leftrightarrow \bigvee_{\mathcal{A}_{\text{rand}} \models \psi[\delta(\overline{a})], \overline{a} \in A^{|\mathsf{FV}(\psi)|}_{\text{rand}}} \varphi^0_{\mathcal{A}_{\text{rand}},\delta(\overline{a})}$$

by Lemma 3, and the finiteness of $\delta(\overline{a})$ ensures the above conjunction is finite, i.e. in FO. So we can define an equivalent translation $[\![\_\_]\!] : \mathcal{L}[\tau] \to \mathrm{FO}[\tau]$ over $\mathcal{A}_{\text{rand}}$,

$$[\![\psi]\!] = \bigvee_{\mathcal{A}_{\text{rand}} \models \psi[\delta(\overline{a})], \overline{a} \in A^{|\mathsf{FV}(\psi)|}_{\text{rand}}} \varphi^0_{\mathcal{A}_{\text{rand}},\delta(\overline{a})}.$$

It is important that the number of possible $\bigvee_{\mathcal{A}_{\text{rand}} \models \psi[\delta(\overline{a})], \overline{a} \in A^{|\mathsf{FV}(\psi)|}_{\text{rand}}} \varphi^0_{\mathcal{A}_{\text{rand}},\delta(\overline{a})}$ is finite, when ψ ranges over all $\mathcal{L}[\tau]$-formulae with a bounded number of free variables.

Given $\varphi = Q_K\overline{x}_1,\ldots,\overline{x}_s[\psi_1,\ldots,\psi_s] \in \mathrm{FO}(\{Q_K\})[\tau]$ and $\overline{a} \in A_{\mathrm{rand}}^{|\mathsf{FV}(\varphi)|}$, if $\mathcal{A}_{\mathrm{rand}} \models \varphi[\overline{a}]$, since Q_K is finitely witnessed, there exists a finite $D \subset A_{\mathrm{rand}}$ to fulfill

$$(D, \psi_1^{\mathcal{A}_{\mathrm{rand}}}(\_\_, \overline{a}) \upharpoonright D, \ldots, \psi_s^{\mathcal{A}_{\mathrm{rand}}}(\_\_, \overline{a}) \upharpoonright D) \in K.$$

Such D may not be unique, so fix one specific $D^{\overline{a}}_{\psi_1,\ldots,\psi_s}$.

Then we set

$$\begin{aligned} n = \max\{|D^{\delta(\overline{a})}_{[\![\psi_1]\!],\ldots,[\![\psi_s]\!]}| \mid\ & \mathcal{A}_{\mathrm{rand}} \models Q_K\overline{x}_1,\ldots,\overline{x}_s[[\![\psi_1]\!],\ldots,[\![\psi_s]\!]][\delta(\overline{a})], \\ & \text{where } \varphi = Q_K\overline{x}_1,\ldots,\overline{x}_s[\psi_1,\ldots,\psi_s] \in \mathcal{L}[\tau] \\ & \text{with } |\mathsf{FV}(\varphi)| \le t, \text{ hence } |\mathsf{FV}(\psi_i)| \le t + r_i, \\ & \text{and } \overline{a} \in A_{\mathrm{rand}}^{|\mathsf{FV}(\varphi)|}\ \}. \end{aligned}$$

By the discussion in the beginning and the finiteness of $\delta(\overline{a})$, the right-hand set is also finite, so n is well defined.

Now for any $\varphi = Q_K\overline{x}_1,\ldots,\overline{x}_s[\psi_1,\ldots,\psi_s] \in \mathcal{L}[\tau]$ with $|\mathsf{FV}(\varphi)| \le t$ and $\overline{a} \in A_{\mathrm{rand}}^{|\mathsf{FV}(\varphi)|}$, assume $\mathcal{A}_{\mathrm{rand}} \models \varphi[\overline{a}]$, by $[\![\_\_]\!]$ is an equivalent translation and Lemma 3, we have $\mathcal{A}_{\mathrm{rand}} \models Q_K\overline{x}_1,\ldots,\overline{x}_s[[\![\psi_1]\!],\ldots,[\![\psi_s]\!]][\delta(\overline{a})]$. Hence $D^{\delta(\overline{a})}_{[\![\psi_1]\!],\ldots,[\![\psi_s]\!]}$ exists, i.e.

$$\begin{aligned} (D^{\delta(\overline{a})}_{[\![\psi_1]\!],\ldots,[\![\psi_s]\!]}, [\![\psi_1]\!]^{\mathcal{A}_{\mathrm{rand}}}(\_\_, \delta(\overline{a})) \upharpoonright D^{\delta(\overline{a})}_{[\![\psi_1]\!],\ldots,[\![\psi_s]\!]}, \\ \ldots, [\![\psi_s]\!]^{\mathcal{A}_{\mathrm{rand}}}(\_\_, \delta(\overline{a})) \upharpoonright D^{\delta(\overline{a})}_{[\![\psi_1]\!],\ldots,[\![\psi_s]\!]}) \in K. \end{aligned}$$

Moreover $|D^{\delta(\overline{a})}_{[\![\psi_1]\!],\ldots,[\![\psi_s]\!]}| \le n$. Like the proof of Lemma 3, $\delta(\overline{a}) \longmapsto \overline{a}$ can be extended to an automorphism $h : A_{\mathrm{rand}} \to A_{\mathrm{rand}}$ with $h(\delta(\overline{a})) = \overline{a}$. Then we deduce

$$\begin{aligned} & (D^{\delta(\overline{a})}_{[\![\psi_1]\!],\ldots,[\![\psi_s]\!]}, \psi_1^{\mathcal{A}_{\mathrm{rand}}}(\_\_, \delta(\overline{a})) \upharpoonright D^{\delta(\overline{a})}_{[\![\psi_1]\!],\ldots,[\![\psi_s]\!]}, \\ & \qquad \ldots, \psi_s^{\mathcal{A}_{\mathrm{rand}}}(\_\_, \delta(\overline{a})) \upharpoonright D^{\delta(\overline{a})}_{[\![\psi_1]\!],\ldots,[\![\psi_s]\!]}) \in K, \\ & \qquad \text{by } [\![\_\_]\!] \text{ is an equivalent translation,} \\ \Rightarrow\ & (h(D^{\delta(\overline{a})}_{[\![\psi_1]\!],\ldots,[\![\psi_s]\!]}), \psi_1^{\mathcal{A}_{\mathrm{rand}}}(\_\_, \overline{a}) \upharpoonright h(D^{\delta(\overline{a})}_{[\![\psi_1]\!],\ldots,[\![\psi_s]\!]}), \\ & \qquad \ldots, \psi_s^{\mathcal{A}_{\mathrm{rand}}}(\_\_, \overline{a}) \upharpoonright h(D^{\delta(\overline{a})}_{[\![\psi_1]\!],\ldots,[\![\psi_s]\!]})) \in K, \\ & \qquad \text{as } K \text{ is closed under isomorphisms.} \end{aligned}$$

So $\mathcal{A}_{\mathrm{rand}} \upharpoonright h(D^{\delta(\overline{a})}_{[\![\psi_1]\!],\ldots,[\![\psi_s]\!]})$ witnesses $\mathcal{A}_{\mathrm{rand}} \models Q_K\overline{x}_1,\ldots,\overline{x}_s[\psi_1,\ldots,\psi_s][\overline{a}]$ with required $|h(D^{\delta(\overline{a})}_{[\![\psi_1]\!],\ldots,[\![\psi_s]\!]})| \le n$. □

Now we say a set of generalized quantifiers $\mathcal{Q}$ is closed under extensions, finitely witnessed or finitely based if each $Q_K \in \mathcal{Q}$ is respectively so.

Theorem 6. *Assume a finite set of generalized quantifiers $\mathcal{Q}$ is finitely based, and $t \in \mathbb{N}$. There exists an $i \in \mathbb{N}$ such that for any $\varphi \in \mathcal{L}^t_{\infty\omega}(\mathcal{Q})[\tau]$ and $\overline{a} \in A_{\mathrm{rand}}^{|\mathsf{FV}(\varphi)|}$, if $\mathcal{A}_{\mathrm{rand}} \models \varphi[\overline{a}]$, then for any finite $\mathcal{A} \models \epsilon_i$ and $\overline{a}' \in A^{|\mathsf{FV}(\varphi)|}$ with $\varphi^0_{\mathcal{A},\overline{a}'} = \varphi^0_{\mathcal{A}_{\mathrm{rand}},\overline{a}}$, we have $\mathcal{A} \models \varphi[\overline{a}']$.*

Proof. Let n be the maximum such as in Lemma 4, when Q_K ranges over the finite set $\mathcal{Q}$. Take $i = n + t$. First for any φ, we can use De Morgan Law to push all its negation symbols either the atomic level or right before a Q_K. Then we proceed by induction on the structure of such transformed φ. Note as mentioned before $\exists$ can be treated as a finitely based Q_K, so the cases of first order quantifiers are absorbed in the discussion of general finitely based Q_K.
(i) φ is atomic or negated atomic, trivial.
(ii) The proof for $\varphi = \bigwedge_{j\in J} \varphi_j$ or $\bigvee_{j\in J} \varphi_j$ is an easy induction argument.
(iii) $\varphi = Q_K \overline{x}_1, \dots, \overline{x}_s[\psi_1, \dots, \psi_s]$ where $Q_K \in \mathcal{Q}$ and clearly $|\mathsf{FV}(\varphi)| \leq t$. By definition $\mathcal{A}_{\text{rand}} \models \varphi[\overline{a}]$ is equivalent to

$$(A_{\text{rand}}, \psi_1^{\mathcal{A}_{\text{rand}}}(\_\_, \overline{a}), \dots, \psi_s^{\mathcal{A}_{\text{rand}}}(\_\_, \overline{a})) \in K.$$

As Q_K is finitely witnessed, for some finite $D \subset A_{\text{rand}}$, $(D, \psi_1^{\mathcal{A}_{\text{rand}}}(\_\_, \overline{a}) \restriction D, \dots, \psi_s^{\mathcal{A}_{\text{rand}}}(\_\_, \overline{a}) \restriction D) \in K$. Moreover by Lemma 4, D can be chosen in a way such that $|D| \leq n$.

Now for any finite $\mathcal{A} \models \epsilon_i$ and $\overline{a}' \in A^{|\mathsf{FV}(\varphi)|}$ with $\varphi^0_{\mathcal{A},\overline{a}'} = \varphi^0_{\mathcal{A}_{\text{rand}},\overline{a}}$, since $|\{\overline{a}\}| \leq |\mathsf{FV}(\varphi)| \leq t$, so $i = n + t \geq |D| + |\{\overline{a}\}|$, thus we can enlarge the partial isomorphism $\overline{a}' \longmapsto \overline{a}$ to $h : S, \overline{a}' \longmapsto D, \overline{a}$ between $\mathcal{A}$ and $\mathcal{A}_{\text{rand}}$ by Lemma 2 for some $S \subseteq A$. We claim

$$\begin{aligned} &h((S, \psi_1^{\mathcal{A}}(\_\_, \overline{a}') \restriction S, \dots, \psi_s^{\mathcal{A}}(\_\_, \overline{a}') \restriction S)) \\ &= (D, \psi_1^{\mathcal{A}_{\text{rand}}}(\_\_, \overline{a}) \restriction D, \dots, \psi_s^{\mathcal{A}_{\text{rand}}}(\_\_, \overline{a}) \restriction D). \end{aligned} \tag{1}$$

Then it follows that $(A, \psi_1^{\mathcal{A}}(\_\_, \overline{a}'), \dots, \psi_s^{\mathcal{A}}(\_\_, \overline{a}')) \in K$, since K is closed under isomorphisms and extensions, that is to say $\mathcal{A} \models \varphi[\overline{a}']$.

Indeed (1) is equivalent to for any $1 \leq j \leq s$ and $\overline{e} \in S^{r_j}$,

$$\mathcal{A} \models \psi_j[\overline{e}\overline{a}'] \text{ iff } \mathcal{A}_{\text{rand}} \models \psi_j[h(\overline{e})\overline{a}] \quad \text{i.e.} \quad \mathcal{A}_{\text{rand}} \models \psi_j[h(\overline{e}\overline{a}')].$$

If $\mathcal{A}_{\text{rand}} \models \psi_j[h(\overline{e}\overline{a}')]$, observe $\varphi^0_{\mathcal{A},\overline{e}\overline{a}'} = \varphi^0_{\mathcal{A}_{\text{rand}},h(\overline{e}\overline{a}')}$, so by induction hypothesis, $\mathcal{A} \models \psi_j[\overline{e}\overline{a}']$. The case for $\mathcal{A}_{\text{rand}} \models \neg\psi_j[h(\overline{e}\overline{a}')]$ is the same.
(iv) $\varphi = \neg Q_K \overline{x}_1, \dots, \overline{x}_s[\psi_1, \dots, \psi_s]$, $\mathcal{A}_{\text{rand}} \models \varphi[\overline{a}]$ i.e.

$$(A_{\text{rand}}, \psi_1^{\mathcal{A}_{\text{rand}}}(\_\_, \overline{a}), \dots, \psi_s^{\mathcal{A}_{\text{rand}}}(\_\_, \overline{a})) \notin K.$$

Given any finite $\mathcal{A}$ with $\mathcal{A} \models \epsilon_i$ and $\overline{a}' \in A^{|\mathsf{FV}(\varphi)|}$ with $\varphi^0_{\mathcal{A},\overline{a}'} = \varphi^0_{\mathcal{A}_{\text{rand}},\overline{a}}$, i.e. $\overline{a}' \longmapsto \overline{a}$ is a partial isomorphism between $\mathcal{A}$ and $\mathcal{A}_{\text{rand}}$, moreover it can be extended to an isomorphic embedding $h : A \to A_{\text{rand}}$ with $h(\overline{a}') = \overline{a}$, by Lemma 2 for $\mathcal{A}_{\text{rand}} \models \epsilon_{|A|}$. Similar to (iii), we shall prove

$$\begin{aligned} &h((A, \psi_1^{\mathcal{A}}(\_\_, \overline{a}'), \dots, \psi_s^{\mathcal{A}}(\_\_, \overline{a}'))) \\ &\subset (A_{\text{rand}}, \psi_1^{\mathcal{A}_{\text{rand}}}(\_\_, \overline{a}), \dots, \psi_s^{\mathcal{A}_{\text{rand}}}(\_\_, \overline{a})). \end{aligned} \tag{2}$$

Then assume contrarily $\mathcal{A} \models \neg\varphi[\overline{a}']$, i.e. $(A, \psi_1^{\mathcal{A}}(\_\_, \overline{a}'), \dots, \psi_s^{\mathcal{A}}(\_\_, \overline{a}')) \in K$. Because K is closed under isomorphisms and extensions, we would have

$$(A_{\text{rand}}, \psi_1^{\mathcal{A}_{\text{rand}}}(\_\_, \overline{a}), \dots, \psi_s^{\mathcal{A}_{\text{rand}}}(\_\_, \overline{a})) \in K$$

by (2), a contradiction. Therefore $\mathcal{A} \models \varphi[\overline{a}']$.

To establish (2), it amounts to show that for any $1 \le j \le s$ and $\overline{e} \in A^{r_j}$,

$$\mathcal{A} \models \psi_j[\overline{ea}'] \text{ iff } \mathcal{A}_{\text{rand}} \models \psi_j[h(\overline{e})\overline{a}] \quad \text{i.e.} \quad \mathcal{A}_{\text{rand}} \models \psi_j[h(\overline{ea})],$$

of which the proof is identical to the last part of (iii). □

Corollary 3. *For a finite set of generalized quantifiers $\mathcal{Q}$, $\mathcal{L}^{\omega}_{\infty\omega}(\mathcal{Q})$ satisfies the labeled 0-1 law, if $\mathcal{Q}$ is closed under extensions.*

Proof.
To make Theorem 6 applicable, $\mathcal{Q}$ must be also finitely witnessed. Set

$$\mathcal{Q}^{\uparrow}_{\text{fin}} = \{Q_{K^{\uparrow}_{\text{fin}}} \mid Q_K \in \mathcal{Q}, K^{\uparrow}_{\text{fin}} = \{\mathcal{A} \mid \text{for some finite } \mathcal{A}' \in K, \mathcal{A}' \subseteq \mathcal{A}\}\}.$$

It is easy to verify that $\mathcal{Q}^{\uparrow}_{\text{fin}}$ is finitely based for any $\mathcal{Q}$, and more importantly $\mathcal{Q}^{\uparrow}_{\text{fin}}$ behaves exactly the same as $\mathcal{Q}$ on all finite structures provided $\mathcal{Q}$ is closed under extensions. So we can safely assume $\mathcal{Q}$ is finitely based. Then the result follows from the fact that each ϵ_i has the asymptotic probability 1 and Theorem 6. □

When $\mathcal{Q}$ is an infinite set of generalized quantifiers closed under extensions, for any logic $\mathcal{L} < \mathcal{L}^{\omega}_{\infty\omega}$ with finitary syntax like FO, LFP and PFP, that is, each sentence in $\mathcal{L}(\mathcal{Q})$ only involves finitely many Q_Ks, we can argue in the same way as if $\mathcal{Q}$ were finite like the above corollary, so the 0-1 law still holds in $\mathcal{L}(\mathcal{Q})$. But for $\mathcal{L}^{\omega}_{\infty\omega}$ itself, consider the sentence

$$\varphi = \bigvee_{l \text{ is even.}} \exists^{\ge l} x(x = x) \wedge \neg\exists^{\ge l+1} x(x = x),$$

surely it defines the Parity property which has no asymptotic probability, while each $\exists^{\ge l}$ is finitely based, so Theorem 6 and Corollary 3 can by no means be extended to infinite $\mathcal{Q}$. Nevertheless finite many of generalized quantifiers usually do not suffice. One typical situation of infinite many quantifiers is the *vectorization*, which extends a given quantifier to finite Cartesian product of the universe of the original structure. For instance, $\overset{k}{\mathsf{TP}}$ can be viewed as the k-vectorization of monadic TP. Another extension of much interest is the *relativization* which closes the Lindström logic under definable unary set. Combine them together, we have for any Q_K, its relativized k-vectorization $\overset{k}{Q}{}^{\text{rel}}_K$ is the $Q_{K'}$ for $K' = \{(\mathcal{A}, U) \mid U \subseteq A^k \text{ and } (U, R^{\mathcal{A}}_1 \restriction U, \ldots, R^{\mathcal{A}}_s \restriction U) \in K\}$. It is easy to prove being closed under extensions, finitely witnessed and finitely based are all preserved under relativization and vectorization. Observe that each $\overset{k}{Q}{}^{\text{rel}}_K$ must consume $k + \sum_{1 \le i \le s} kr_i$ distinct variables, so there are only finite many types of $\overset{k}{Q}{}^{\text{rel}}_K$ that are valid to appear in a $\mathcal{L}^{t}_{\infty\omega}$ formula for any given t, thus we have

Corollary 4. *For a finite set $\mathcal{Q}$ of Q_K closed under extensions, $\mathcal{L}^{\omega}_{\infty\omega}(\overset{\omega}{\mathcal{Q}}{}^{\text{rel}})$ has the labeled 0-1 law, where $\overset{\omega}{\mathcal{Q}}{}^{\text{rel}} = \{\overset{k}{Q}{}^{\text{rel}}_K \mid Q_K \in \mathcal{Q}, k \in \mathbb{N}\}$.*

A similar argument can be applied to $\mathcal{L}^{\omega}_{\infty\omega}(\tilde{\mathsf{P}}^{\omega,\omega})$, so Theorem 5 holds. Note one of its immediate consequences is that Hamiltonicity can not be defined in partition logic, for FO[Ham], the minimal regular logic capturing Hamiltonicity does not have a 0-1 law[2].

Acknowledgments. We are grateful to H.-D. Ebbinghaus for suggestions on an earlier draft of the paper. An anonymous referee's comments also help to improve the presentation.

References

1. A. Dawar. Generalized quantifiers and logical reducibilities. *Journal of Logic and Computation*, 5(1995), 213-226.
2. A. Dawar and E. Grädel. Generalized quantifiers and 0-1 laws. in Proceedings of 10th IEEE Symposium on Logic in Computer Science, 54-64, 1995.
3. H.-D. Ebbinghaus and J. Flum, *Finite Model Theory*. Springer-Verlag, 1995.
4. R. Fagin. Generalized first-order spectra and polynomial-time recognizable sets. In R.M. Karp, editor, *Complexity of Computation, SIAM-AMS Proceedings, Vol. 7*, 43-73, 1974.
5. G. Fayolle, S. Grumbach, and C. Tollu. Asymptotic probabilities of languages with generalized quantifiers, in Proceedings of 8th IEEE Symposium on Logic in Computer Science, 199-207, 1993.
6. E. Grädel. Capturing complexity classes by fragments of second-order logic. *Theoretical Computer Science*, 101(1992), 35–57.
7. E. Grädel, *et al.* Problems in finite model theory. Available at: http://www-mgi.informatik.rwth-aachen.de/FMT/, June 2000.
8. H. Imhof. Logiken mit Partitionsquantoren. *Diplomarbeit*, Freiburg University, 1992.
9. N. Immerman. Languages that capture complexity classes. *SIAM Journal on Computing*, 16(1987), 760-778.
10. N. Immerman. Nondeterministic space is closed under complementation. *SIAM Journal on Computing*, 17(1988), 935-938.
11. N. Immerman. *Descriptive Complexity*. Graduate Texts in Computer Science, Springer-Verlag, 1999.
12. T. Lacoste. 0-1 laws by preservation. *Theoretical Computer Science*, 184(1-2):237-245, 30, 1997.
13. J. Malitz. Downward transfer of satisfiability for sentences of $L^{1,1}$. *Journal Symbolic Logic*, 48(1984), 1146–1150.
14. E. Shen. Partition logics and transitive closure logics (Abstract). *Bulletin of Science* (Chinese), 38(1993), 1271–1272.
15. E. Shen. Partition logics and colored pebble games. *Prog. Maths.* (Chinese), 24(6)(1996), 540–546.
16. E. Shen and Y. Chen. The downward transfer of elementary satisfiability of partition logics. *Mathematical Logic Quarterly*, 46, 477-487, No. 4, 2000.
17. E. Shen and Q. Tian. Monadic partition logics and finite automata. *Theoretical Computer Science*, 166(1996), 63–81.
18. I. A. Stewart. Using the Hamiltonian path operator to capture NP. *Journal of Computer and System Sciences*, 45, 127-151(1992).