# The Downward Transfer of Elementary Satisfiability of Partition Logics *

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**§1. Basic Notions and Definability Examples.**

The following operation has frequently occurred in mathematics and its applications: partition a set into several disjoint non-empty subsets, so that the elements in each partition subset are homogeneous or indistinguishable with respect to some given properties. H.-D.Ebbinghaus first (in 1991) distilled from this phenomenon, which is a monadic second order property in nature, a special kind of quantifiers, and augmented first order logic $\mathcal{L}_{\omega\omega}$ with them to obtain a family of extended logics, called monadic partition logics. Interesting applications have been found outside mathematics, especially in computer science [8, 10]. Ebbinghaus quantifiers (i.e. monadic partition quantifiers) will reduce to Malitz quantifiers [6] if we restrict them with some infinite cardinality requirements. However the latter one appeared earlier and their backgrounds are also different.

There are several types of partition quantifiers, such as 2-partition or multi-partition, monadic or non-monadic type. When augmenting $\mathcal{L}_{\omega\omega}$ with all the monadic partition quantifiers, we get the extended logic $\mathcal{L}(\mathrm{MP})$; while $\mathcal{L}_P$ denotes the extended logic obtained by adding all sorts of partition quantifiers to $\mathcal{L}_{\omega\omega}$ [8, 9, 10]. First we introduce the semantic interpretation of monadic partition quantifiers. As a typical example we look at a special case of type, $P^{2,1}$.

**Definition 1.** $\mathfrak{A} \models P^{2,1}_{x,x';y}\varphi(x,x',y)$ iff there is a partition of the universe $A(|A| \geq 2)$ of $\mathfrak{A}$: $A = A'\dot{\cup}A''$, $A' \neq \emptyset \neq A''$ [1], such that for all $a, a' \in A', b \in A''$, $\mathfrak{A} \models \varphi[a,a',b]$.

Obviously $\mathcal{L}(\mathrm{MP})$ can be embedded into $MSO$ (monadic second order logic).

The partition quantifiers of non-monadic type concern the partition of the Cartesian product of the universe, i.e. the partition of multi-dimensional space. For instance $\mathfrak{A} \models P^{1,1}_2\varphi(x,y,x',y')$ iff there is a partition of $A^2$: $A^2 = U_0\dot{\cup}U_1$, $U_0 \neq \emptyset \neq U_1$, such that for all $(a_0,b_0) \in U_0, (a_1,b_1) \in U_1$, $\mathfrak{A} \models \varphi[a_0,b_0,a_1,b_1]$. Surely it can be translated into $SO$ (second order logic), so $\mathcal{L}_P \leq SO$.

**Example 1.** After introducing the partition quantifier, the universal and existential quantifiers can be absorbed:

$$\forall x\psi(x) \equiv \neg P^{1,1}_{x,y}[\neg\psi(x) \wedge y = y], \quad \exists x\psi(x) \equiv P^{1,1}_{x,y}[\psi(x) \wedge y = y].$$

Also, if $m_0 < m_1, n_0 < n_1$, then clearly $\mathcal{L}(P^{m_0,n_0}) \leq \mathcal{L}(P^{m_1,n_1})$.

**Example 2.** *Peano induction principle* can be defined easily in arithmetic languages: let $S$ be the unary symbol of successor function, define:

$$\underline{\mathbf{Ind}} := \neg P^{1,1}_{x,y}[0 \neq y \wedge S(x) \neq y].$$

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[1] Such partition will be called *non-trivial* in the sequel.

**Example 3.** Connectivity is a global property of the directed graph, using the partition quantifier, it can be characterized in a local way:

$$\underline{\mathbf{Conn}} := \neg P^{1,1}_{x,y} \neg E(x,y), \quad \text{where } E \text{ is a binary relation symbol.}$$

Meanwhile the reachability of two points of the graph can be characterized through:

$$\underline{\mathbf{Path}}(u,v) := \neg P^{1,1}_{x,y}[\neg E(x,y) \wedge y \neq u \wedge x \neq v],$$

Surely we have

$$\models \underline{\mathbf{Conn}} \leftrightarrow \forall uv[u \neq v \rightarrow \underline{\mathbf{Path}}(u,v)]. \tag{1}$$

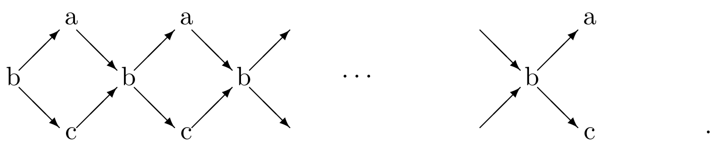
More generally the monadic transitive closure operator $TC^1$, which characterizes reflexive transitive closure over binary relations [5], can be depicted by $P^{1,1}$, and vice versa.

$$\begin{aligned} [TC^1_{u,v}\psi(u,v)](x,y) &\equiv \neg P^{1,1}_{u,v}[x \neq v \wedge y \neq u \wedge \neg\psi(u,v)]; \\ P^{1,1}_{u,v}\psi(u,v) &\equiv \exists xy[\neg(TC^1_{u,v}\neg\psi(u,v))(x,y)]. \end{aligned}$$

Note that the above "duality" is not in a strict form, for $P^{1,1}$ is a *generalized quantifier*, it binds some free variables of the inner formula; while $TC^1$, as a *predicate transformer*, does not [7, 8].

The following example combines reachability with parity in partial ordering, say Mazurkiewicz traces.

**Example 4.** Consider the concurrent alphabet$(A, I)$ where $A = \{a, b, c\}$ and $I = \{(a,c),(c,a)\}$. Let $T_1$ be the set of traces with dependency graphs of the following form:




$T_1$ is first-order definable. Let $T_2$ be the set of traces as in $T_1$ with extra requirement that there is an odd number of labels $b$. Then $T_2$ is MSO-definable [12]. Now we show that it is also $\mathcal{L}(P^{1,1})$ definable by adding:

$$\begin{aligned} \underline{\mathbf{Odd}}_{trace} &:= P^{1,1}_{x,y}[\min \neq y \wedge \max_b \neq y \wedge Q_b(x) \\ &\quad\ \wedge \forall u(\varphi^2(x,u) \rightarrow u \neq y) \wedge (\text{n-max}_b \neq x)]; \\ \text{where } \varphi(x,y) &:= x < y \wedge Q_b(x) \wedge Q_b(y) \wedge \forall z(x < z < y \rightarrow Q_a(z) \vee Q_c(z)); \\ \text{and } \varphi^2(x,y) &:= \exists z(\varphi(x,z) \wedge \varphi(z,y)), \end{aligned}$$

where $n - max_b$ stands for the second last occurrence of $b$ in the dependency graphs,which is unique and FO-definable, and may be used as the (reduced) constant. [2] Similarly for constants $max_b$ and min.

Furthermore, even one positive occurrence of $P^{1,1}$ can capture cyclic counting (here w.r.t. vertices), take **Mod 3** as an example,

$$\begin{aligned} \psi_1 &:= \min \neq y \wedge \forall u[\delta(\min, u) \rightarrow u \neq y]; \\ \psi_2 &:= \max \neq x; \\ \psi_3 &:= \forall u[\exists vw(\delta(u,v) \wedge \delta(v,w) \wedge \delta(w,y)) \rightarrow u \neq x]; \\ \underline{\mathbf{Mod\ 3}} &:= P^{1,1}_{x,y}(\psi_1 \wedge \psi_2 \wedge \psi_3). \end{aligned}$$

[2] We can not use $\forall u(\varphi(x,u) \rightarrow x \neq u)$ to instead of $(n - max_b \neq x)$, in $\underline{\mathbf{Odd}}_{trace}$, although they seem to realize the same effect at first sight. For in the former "$x \neq u$" fails to mean "$u \in Y$" under $P^{1,1}$ (Y is the partition subset relating to y) as we wish. However $\forall u(\varphi^2(x,u) \rightarrow y \neq u)$ doesn't contain this defect.

The above examples show that the partition quantifiers can characterize some basic recursive constructs through an indirect but succinct way, which is the starting point of many its applications in computer science, specifically the inductive or recurrent meaning of $\neg P^{1,1}$. H.Imhof found an interpretation of tree induction for $\neg P^{m,1}$[4]. For more general $\neg P^{m,n}$ we shall offer an interesting intuitive interpretation later.

**Example 5.** Undefinability of some partition property in $\mathcal{L}$(MP): The Dedekind cut property is not axiomizable in $\mathcal{L}(P^{n,n})$, and nor even in $\mathcal{L}$(MP) [9], but it is definable in $MSO$ [2]. So $\mathcal{L}$(MP) is a proper fragment of $MSO$, and a proper extension of $\mathcal{L}_{\omega\omega}$ (see example 2). This result was first proved by colored pebble game of $\mathcal{L}$(MP) in [9], we will give another simple proof using the Löwenheim-Skolen-Tarski theorem later in this paper.

The Compactness and Downward Löwenheim-Skolen theorems are two essential features of $\mathcal{L}_{\omega\omega}$. The Lindström theorem shows that in some sense they characterize $\mathcal{L}_{\omega\omega}$ completely [1]. The Compactness does not work in partition logics (see example 2), while the Löwenheim-Skolen theorem, even its stronger form, the Löwenheim-Skolen-Tarski theorem, does survive. The idea behind the argument we will offer is: via introducing new predicates to lower the problem down to $\mathcal{L}_{\omega\omega}$(even $\mathbf{\Sigma_1}$) level, therefore the Compactness is applicable; then try to overcome the difficulty of exponential blow-up of the number of partitions and their witnesses, which is going to occur at first glance, with the aid of some concepts and technique borrowed from graph theory. While the notions of generalized (Gaifman) graphs and pseudo-reachability themselves may deserve their own merit.

## §2. The Downward Löwenheim-Skolen-Tarski Theorem for Partition Logics.

**Definition 2.** The *partition quantifier rank* of a formula $\varphi$, $P-\mathrm{qr}(\varphi)$, is defined inductively:

$$\begin{aligned}
P-\mathrm{qr}(\varphi) &= 0, \text{where } \varphi \text{ is atomic;}\\
P-\mathrm{qr}(\neg\varphi) &= P-\mathrm{qr}(\varphi);\\
P-\mathrm{qr}(\varphi_0\wedge\varphi_1) &= \max(P-\mathrm{qr}(\varphi_0), P-\mathrm{qr}(\varphi_1));\\
P-\mathrm{qr}(P\varphi) &= P-\mathrm{qr}(\exists x\varphi) \;=\; P-\mathrm{qr}(\varphi)+1.
\end{aligned}$$

**Definition 3.** $\mathfrak{A}^*$ is an $\mathcal{L}$(MP) *elementary substructure* of $\mathfrak{A}$ iff $\mathfrak{A}^*$ is a substructure of $\mathfrak{A}$ and for all formulae $\varphi$ of $\mathcal{L}(MP)[\tau]$ and all elements $a_0,\ldots,a_{n-1}$ in $A^*$, we have $\mathfrak{A}\models\varphi[a_0,\ldots,a_{n-1}]$ iff $\mathfrak{A}^*\models\varphi[a_0,\ldots,a_{n-1}]$. Denote this situation as $\mathfrak{A}^*\prec_{L(MP)}\mathfrak{A}$. Sometimes we restrict our attention to some subset of all formulae (say by the partition quantifier rank of $\varphi$), the corresponding (hierarchical) concept of elementary substructure can also be defined ($\prec_P^n$).

Without loss of generality, we shall fix a countable vocabulary $\tau$ in the sequel. First we consider the simple case of $\mathcal{L}(P^{1,1})$, then generalize the result to $\mathcal{L}$(MP), etc.

**Theorem 1.** Let $\mathfrak{A}$ be an infinite $\tau$-structure, given any countable $A_0\subseteq A$, $\mathfrak{A}$ has an $\mathcal{L}(P^{1,1})$ elementary substructure $\mathfrak{A}^*$ such that $A_0\subseteq A^*$ and $|A^*|\leq\aleph_0$.

The proof is a simple application of chain construction combined with the graph interpretation of $\neg P^{1,1}$ in above example, see [7]. A related result "the Lövenheim number of $\mathcal{L}(P^{1,1})$ is $\aleph_0$" was proved by reducing technique in [11].

The graph intuition remains useful in general cases $\mathcal{L}$(MP),and even $\mathcal{L}_P$. Of course, some new light needs to be shedded on it. We first introduce two technical lemmas. They can be viewed as generalization of some properties of connectivity depicted by $P^{1,1}$. Consider $(m+n)-$ary relation symbol $R(\bar{x},\bar{y})$ and structure $\mathfrak{A}=(A,R^{\mathfrak{A}},\ldots)$, to be called *Gaifman graph.* If $\mathfrak{A}\models\neg P^{m,n}_{\bar{x};\bar{y}}\neg R(\bar{x},\bar{y})$, then we say that $\mathfrak{A}$ is *pseudo-connected* (with respect to $R$); for any two $e,f\in A, e\neq f$, if $\mathfrak{A}\models\neg P^{m,n}_{\bar{x};\bar{y}}[\neg R(\bar{x},\bar{y})\wedge y_0\neq u\wedge x_0\neq v][e,f]$, then we say that $e$ is *pseudo-reachable* to $f$ in $\mathfrak{A}$. Now that Lemma 1 means the coincidence of the local and global descriptions of pseudo-connectivity in Gaifman graph. (Compare to (1) in example 3.) While Lemma 2 shows that the pseudo-reachability between two points in generalized graph needs in fact only finitely many witnesses.

**Lemma 1.** For any formula $\varphi(\bar{x},\bar{y})$ with $\bar{x}=x_0,\ldots,x_{m-1}$ and $\bar{y}=y_0,\ldots,y_{n-1}$,

$$\mathfrak{A}\models\neg P^{m,n}_{\bar{x};\bar{y}}\varphi \iff \mathfrak{A}\models\forall uv\{u\neq v\to\neg P^{m,n}_{\bar{x};\bar{y}}[\varphi\wedge y_0\neq u\wedge x_0\neq v]\}.$$

Proof. Easily verified by the semantic meaning of $\neg P^{m,n}_{\bar{x};\bar{y}}$, which occurs in above examples and will be reused later again and again. □

**Lemma 2.** Suppose $R$ is $(m+n)-$ary relation symbol, $\mathfrak{A}=(A,R^{\mathfrak{A}})\models\neg P^{m,n}_{\bar{x};\bar{y}}[\neg R(\bar{x},\bar{y})\wedge y_0\neq u\wedge x_0\neq v][e,f]$, where $e,f\in A$, $e\neq f$,[e/u,f/v], and $\bar{x}=x_0,\ldots,x_{m-1}$, $\bar{y}=y_0,\ldots,y_{n-1}$. Denote $D$ to be the diagram of $\mathfrak{A}$. There exists a finite subset $D_{e,f}$ of $D$ satisfying $D_{e,f}\models\neg P^{m,n}_{\bar{x};\bar{y}}[\neg R(\bar{x},\bar{y})\wedge y_0\neq \underline{e}\wedge x_0\neq \underline{f}]$, where $\underline{e}$, $\underline{f}$ are new constant symbols interpreted by $e$ and $f$ respectively.

Proof. Let $\pi=\{R,S,S',\underline{e},\underline{f}\}$, where $S$ and $S'$ both are unary relation symbols.
**Claim.** Over the vocabulary $\pi$,

$$D\cup\{\forall w(Sw\vee S'w),\neg\exists z(Sz\wedge S'z),S\underline{e},S'\underline{f}\}\models\exists\bar{x}\bar{y}[\bar{x}\in S^m\wedge\bar{y}\in(S')^n\wedge R(\bar{x},\bar{y})],$$

where "$\bar{x}\in S^m$" is the abbreviation for $Sx_0\wedge\ldots\wedge Sx_{m-1}$, similarly for "$\bar{y}\in(S')^n$".

For any $\pi$-structure $\mathfrak{B}$, such that $\mathfrak{B}\models D\cup\{\forall w(Sw\vee S'w),\neg\exists z(Sz\wedge S'z),S\underline{e},S'\underline{f}\}$, the restriction $\mathfrak{B}\uparrow A$ is a substructure of $\mathfrak{B}$, as well as a $\pi$-expansion of $\mathfrak{A}$; $S^{\mathfrak{B}}|S'^{\mathfrak{B}}$ is a non-trivial 2-partition of $B$ with $e\in S^{\mathfrak{B}}$, $f\in S'^{\mathfrak{B}}$. From assumption, $(\mathfrak{A},e,f)\models\neg P^{m,n}_{\bar{x};\bar{y}}[\neg R(\bar{x},\bar{y})\wedge y_0\neq\underline{e}\wedge x_0\neq\underline{f}]$, which implies that for any $\mathfrak{A}$'s $\pi$-expansion $\overline{\mathfrak{A}}=(\mathfrak{A},e,f,S^{\overline{\mathfrak{A}}},S'^{\overline{\mathfrak{A}}})$, if $S^{\overline{\mathfrak{A}}}|S'^{\overline{\mathfrak{A}}}$ is a non-trivial partition of $A$ with $e\in S^{\overline{\mathfrak{A}}}$, $f\in S'^{\overline{\mathfrak{A}}}$, then $\overline{\mathfrak{A}}\models\exists\bar{x}\bar{y}[\bar{x}\in S^m\wedge\bar{y}\in(S')^n\wedge R(\bar{x},\bar{y})]$. Clearly above $\mathfrak{B}\uparrow A(\subseteq\mathfrak{B})$ is such an $\overline{\mathfrak{A}}$, Hence $\mathfrak{B}\models\exists\bar{x}\bar{y}[\bar{x}\in S^m\wedge\bar{y}\in(S')^n\wedge R(\bar{x},\bar{y})]$. (Claim is proved.)

By an application of the Compactness to the claim in $\mathcal{L}_{\omega\omega}[\pi]$, there is a finite subset $D_{e,f}\subseteq_{fin}D$, such that

$$\begin{aligned}D_{e,f}\ \models\ &[\text{"}S|S'\text{ is a non-trivial partition with }e\in S\text{ and }f\in S'\text{"}]\\&\to\exists\bar{x}\bar{y}[\bar{x}\in S^m\wedge\bar{y}\in(S')^n\wedge R(\bar{x},\bar{y})].\end{aligned}$$

Observe that symbols $S$ and $S'$ do not occur in $D_{e,f}$, we conclude $D_{e,f}\models\neg P^{m,n}_{\bar{x};\bar{y}}[\neg R(\bar{x},\bar{y})\wedge y_0\neq\underline{e}\wedge x_0\neq\underline{f}]$. □

It can be verified that the pseudo-reachability is transitive, as the ordinary reachability in graph.

**Proposition 1.** For any $\mathfrak{A}$, $\mathfrak{A}\models\neg P^{m,n}_{\bar{x},\bar{y}}[\varphi(\bar{x},\bar{y})\wedge y_0\neq u\wedge x_0\neq v][e,f]$ and $\mathfrak{A}\models\neg P^{m,n}_{\bar{x},\bar{y}}[\varphi(\bar{x},\bar{y})\wedge y_0\neq u\wedge x_0\neq v][f,g]$ imply $\mathfrak{A}\models\neg P^{m,n}_{\bar{x},\bar{y}}[\varphi(\bar{x},\bar{y})\wedge y_0\neq u\wedge x_0\neq v][e,g]$.

At first glance of Lemma 2, It is not clear whether the union of the finite witnesses of the pseudo-reachability from $e$ to $f$ and from $f$ to $g$ would be the witnesses of the pseudo-reachability from $e$ to $g$, as the case in ordinary graph. Now by above proposition, we can say that it is also true in Gaifman graph, which justifies our nomenclature for these notions.

**Corollary 1.** If both $D_{e,f}\models\neg P^{m,n}_{\bar{x},\bar{y}}[\neg R(\bar{x},\bar{y})\wedge y_0\neq\underline{e}\wedge x_0\neq\underline{f}]$ and $D_{f,g}\models\neg P^{m,n}_{\bar{x},\bar{y}}[\neg R(\bar{x},\bar{y})\wedge y_0\neq\underline{f}\wedge x_0\neq\underline{g}]$, then $D_{e,f}\cup D_{f,g}\models\neg P^{m,n}_{\bar{x},\bar{y}}[\neg R(\bar{x},\bar{y})\wedge y_0\neq\underline{e}\wedge x_0\neq\underline{g}]$, where $D_{e,f}$ and $D_{f,g}$ are defined as in Lemma 2.

Although the interpretation of Lemma 1,2 and their corollaries relies on graph background, their application need not be restricted only to 2-partition quantifiers. The arguments in above proofs also work for multi-partition and even non-monadic cases. For instance, in case of 3-partition quantifiers, Lemma 1 should read: $\mathfrak{A} \models \neg P^{m,n,r}_{\bar{x};\bar{y};\bar{z}}\varphi \iff \mathfrak{A} \models \forall uvw(\psi_1 \rightarrow \psi_2)$,where $\psi_1 := u \neq v \wedge u \neq w \wedge v \neq w$, and $\psi_2 := \neg P^{m,n,r}_{\bar{x};\bar{y};\bar{z}}[\varphi \wedge y_0 \neq u \neq z_0 \wedge x_0 \neq v \neq z_0 \wedge x_0 \neq w \neq y_0]$. The 3-partition case of Lemma 2 is as follows: if $\mathfrak{A} \models \neg P^{m,n,r}_{\bar{x};\bar{y};\bar{z}}[\neg R(\bar{x},\bar{y},\bar{z}) \wedge y_0 \neq u \neq z_0 \wedge x_0 \neq v \neq z_0 \wedge x_0 \neq w \neq y_0][e,f,g]$, then there exists a finite subset $D_{e,f,g}$ of the diagram of $\mathfrak{A}$, such that $D_{e,f,g} \models \neg P^{m,n,r}_{\bar{x};\bar{y};\bar{z}}[\neg R(\bar{x},\bar{y},\bar{z}) \wedge y_0 \neq \underline{e} \neq z_0 \wedge x_0 \neq \underline{f} \neq z_0 \wedge x_0 \neq \underline{g} \neq y_0]$.

**Theorem 2.** (**Downward Löwenheim-Skolen-Tarski Theorem of** $\mathcal{L}$(MP))
For any infinite $\tau-$structure $\mathfrak{A}$, and a countable subset $A_0$ of its universe $A$, there is an $\mathcal{L}$(MP) elementary substructure $\mathfrak{A}^*$ of $\mathfrak{A}$ with $A_0 \subseteq A^*$ and $|A^*| \leq \aleph_0$.

Proof. We shall build the required $\mathfrak{A}^*$ by chain construction, and observe only the non-trivial (inductive) cases relating to 2-partition quantifiers, say in $\mathcal{L}(P^{m,n})$, since the argument can be easily generalized to multi-partition case on the base of multi-partition extensions of Lemma 1 and 2.

Given the countable $A_k$, consider all the formulae of $\mathcal{L}(P^{m,n})[\tau]$ $\varphi(\bar{x},\bar{y},z_0,\ldots,z_{q-1})$ (countably many) and all $c_0,\ldots,c_{q-1} \in A_k$, where $\bar{x} = x_0,\ldots,x_{m-1}$, $\bar{y} = y_0,\ldots,y_{n-1}$.
(1) If $\mathfrak{A} \models P^{m,n}_{\bar{x};\bar{y}}\varphi[c_0,\ldots,c_{q-1}]$, choose two points $e,f \in A$ as the non-empty witnesses of the corresponding 2-partition, and put them into $A_{k+1}$.
(2) If $\mathfrak{A} \models \neg P^{m,n}_{\bar{x};\bar{y}}\varphi[c_0,\ldots,c_{q-1}]$, let $\pi_0 = \tau \cup \{R\}$, where $R$ is a new $(m+n)-$ary relation symbol. Consider the $\pi_0$-expansion $\mathfrak{A}_R$ of $\mathfrak{A}$, with $R^{\mathfrak{A}_R} = \neg\varphi^{\mathfrak{A}}(\bar{x},\bar{y},c_0,\ldots,c_{q-1})$,which is a non-empty relation. By the property of definability[3], such $\mathfrak{A}_R$ is unique. Now $\mathfrak{A} \models \neg P^{m,n}_{\bar{x};\bar{y}}\varphi[c_0,\ldots,c_{q-1}]$ implies $\mathfrak{A}_R \models \neg P^{m,n}_{\bar{x};\bar{y}}\neg R(\bar{x},\bar{y})$. By Lemma 1, for any $e,f \in A_k \subseteq A$, $e \neq f$, $\mathfrak{A}_R \models \neg P^{m,n}_{\bar{x};\bar{y}}[\neg R(\bar{x},\bar{y}) \wedge y_0 \neq u \wedge x_0 \neq v][e,f]$, i.e. $e$ is pseudo-reachable to $f$ in $\mathfrak{A}_R$ (with respect to $R$). Furthermore, observe the reduction structure of $(\mathfrak{A}_R,e,f)$ over the vocabulary $\{R,\underline{e},\underline{f}\}$, $\overline{\mathfrak{A}_R} = (A, R^{\mathfrak{A}_R}, e, f)$. By Lemma 2, there exists a finite subset $D_{\varphi,c_0,\ldots,c_{q-1},e,f}$ of the diagram of $\overline{\mathfrak{A}_R}$, such that

$$D_{\varphi,c_0,\ldots,c_{q-1},e,f} \models \neg P^{m,n}_{\bar{x};\bar{y}}[\neg R(\bar{x},\bar{y}) \wedge y_0 \neq \underline{e} \wedge x_0 \neq \underline{f}]. \tag{2}$$

Put all the elements in $D_{\varphi,c_0,\ldots,c_{q-1},e,f}$ (interpretations of finitely many constant symbols) into $A_{k+1}$, where $e,f$ range over all the distinct pairs of $A_k \times A_k$.

Since $A_k$ is countable, $A_{k+1}$ thus constructed is countable too.

Iterated the above construction $\omega$ times, we get an $\omega-$chain: $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$, let $A^* = \bigcup_{n<\omega} A_n$. Clearly $A^*$ is countable, and closed under $\tau-$terms, so $\mathfrak{A}^* := \mathfrak{A} \upharpoonright A^*$ is a substructure of $\mathfrak{A}$.

We claim $\mathfrak{A}^* \prec_{L(P^{m,n})} \mathfrak{A}$, which suffices to show by induction on the formula $\psi \in \mathcal{L}(P^{m,n}[\tau])$ and any $c_0,\ldots,c_{q-1} \in A^*$ that $\mathfrak{A}^* \models \psi[c_0,\ldots,c_{q-1}]$ iff $\mathfrak{A} \models \psi[c_0,\ldots,c_{q-1}]$. Since $\exists$ is definable from partition quantifier, only the $P$-quantified formulae are discussed in detail.
($\alpha$) If $\mathfrak{A} \models P^{m,n}_{\bar{x};\bar{y}}\varphi[c_0,\ldots,c_{q-1}]$, by the construction(1), we have a non-trivial partition $A'|A''$ of $A$, where $\varphi$ is satisfied homogeneously (under $\mathfrak{A}$), and there are two elements of $e,f \in A^* \subseteq A$, which witness this partition, i.e. $e \in A'$ and $f \in A''$. This fact ensures that the restriction of $A'|A''$ on $A$,i.e. $(A' \cap A^*)|(A'' \cap A^*)$, is non-trivial. By the induction hypotheses, $\varphi$ is satisfied homogeneously on this partition (under $\mathfrak{A}^*$). So $\mathfrak{A}^* \models P^{m,n}_{\bar{x};\bar{y}}\varphi[c_0,\ldots,c_{q-1}]$.
($\beta$) if $\mathfrak{A} \models \neg P^{m,n}_{\bar{x};\bar{y}}\varphi[c_0,\ldots,c_{q-1}]$, take any $e,f \in A^*$, $e \neq f$. By the construction of $A^*$, for some $k$, $c_0,\ldots,c_{q-1},e,f \in A_k$. Consider the $\pi_0$-expansion $\mathfrak{A}^*_R$ of $\mathfrak{A}^*$, where $R^{\mathfrak{A}^*_R} = \neg\varphi^{\mathfrak{A}^*}(\bar{x},\bar{y},c_0,\ldots,c_{q-1})$. By the induction hypotheses, for any $\bar{a} \in (A^*)^m$ and $\bar{b} \in (A^*)^n$, we have $\mathfrak{A}^* \models \neg\varphi[\bar{a},\bar{b},c_0,\ldots,c_{q-1}]$ iff

[3]This first-order feature is obviously true in extended logics such as $\mathcal{L}$(MP), *MSO* etc.

$\mathfrak{A} \models \neg\varphi[\bar{a}, \bar{b}, c_0, \ldots, c_{q-1}]$, i.e. $R^{\mathfrak{A}^*_R} = R^{\mathfrak{A}_R} \cap ((A^*)^m \times (A^*)^n)$, meanwhile $A_{k+1} \subseteq A^*$. So by the construction(2), the expansion-reduction $\overline{\mathfrak{A}^*_R}$ of $\mathfrak{A}^*_R$ over the vocabulary $\{R, \underline{e}, \underline{f}\}$, $\overline{\mathfrak{A}^*_R} \models D_{\varphi, c_0, \ldots, c_{q-1}, e, f}$. It follows that $\overline{\mathfrak{A}^*_R} \models \neg P^{m,n}_{\bar{x};\bar{y}}[\neg R(\bar{x}, \bar{y}) \wedge y_0 \neq \underline{e} \wedge x_0 \neq \underline{f}]$, by expression(2). So $\mathfrak{A}^*_R \models \neg P^{m,n}_{\bar{x};\bar{y}}[\neg R(\bar{x}, \bar{y}) \wedge y_0 \neq u \wedge x_0 \neq v][e, f]$, i.e. $e$ is also pseudo-reachable to $f$ in $\mathfrak{A}^*_R$ (with respect to $R$). Since $e, f$ are arbitrary, $\mathfrak{A}^*_R \models \forall uv\{u \neq v \to \neg P^{m,n}_{\bar{x};\bar{y}}[\neg R(\bar{x}, \bar{y}) \wedge y_0 \neq u \wedge x_0 \neq v]\}$. Finally by Lemma 1, it is equivalent to $\mathfrak{A}^*_R \models \neg P^{m,n}_{\bar{x};\bar{y}} \neg R(\bar{x}, \bar{y})$, so $\mathfrak{A}^* \models \neg P^{m,n}_{\bar{x};\bar{y}} \varphi[c_0, \ldots, c_{q-1}]$. □

**Remark.** In the above proof of ($\beta$), instead of starting from the definition of $\mathfrak{A}^* \models \neg P^{m,n}_{\bar{x};\bar{y}} \varphi$ and observing all the ($2^{\aleph_0}$ many) non-trivial partition of $A^*$, we transform, by Lemma 1, the problem into the pseudo-reachability from $e$ to $f$ for all $(e, f) \in A^* \times A^*$ with $e \neq f$, each of which needs only finite many witnesses by Lemma 2. In this way, we are able to avoid the direct involving in partitions which may lead to the awkward predicament of the possible exponential blowup of partition witnesses. However there is indeed a way out of this difficult situation of exponential explosion directly. The argument along this approach will take a bit of more space and less intuition, see [4].

The above argument also works for non-monadic partition quantifiers, so we have

**Corollary 2.** $\mathcal{L}_P$ possesses the Downward Löwenheim-Skolen-Tarski property.

The combination of Lemma 1 and 2 (as applied in above argument) is a useful tool in proving results about partition logics. As another example, we will prove Tarski Chain Theorem in $\mathcal{L}$(MP).

**Definition 4.** A set of $\tau$-structures $\{\mathfrak{A}_i\}_{i<\lambda}$, where $\lambda \in \mathbf{On}$, is called $\mathcal{L}$(MP) elementary chain, if for any $\alpha < \beta < \lambda$ , we have $\mathfrak{A}_\alpha \prec_{L(MP)} \mathfrak{A}_\beta$. Usually $\lambda$ is a limit ordinal.

**Theorem 3.** **(Elementary Chain Theorem of $\mathcal{L}$(MP))**
For an $\mathcal{L}$(MP) chain $\{\mathfrak{A}_i\}_{i<\lambda}$, we have $\mathfrak{A}_\alpha \prec_{L(MP)} \bigcup_{i<\lambda} \mathfrak{A}_i$ for all the $\alpha < \lambda$, where $\bigcup_{i<\lambda} \mathfrak{A}_i$ is the (first order) union of ascending elementary chain $\{\mathfrak{A}_i\}_{i<\lambda}$.

Proof. Let $\mathfrak{A} = \bigcup_{i<\lambda} \mathfrak{A}_i$, and $A = \bigcup_{i<\lambda} A_i$. By the induction on the ordinals and the partition quantifier rank of $\varphi(x_0, \ldots, x_{n-1}) \in L(MP)[\tau]$, we shall verify: for any $\alpha < \lambda$ and $a_0, \ldots, a_{n-1} \in A_\alpha$, $\mathfrak{A}_\alpha \models \varphi[a_0, \ldots, a_{n-1}]$ iff $\mathfrak{A} \models \varphi[a_0, \ldots, a_{n-1}]$.

The cases for atomic formulae and boolean connectives are too simple to be worth concern, so we will only check the partition-quantified formula, say $\varphi(x_0, \ldots, x_{n-1}) = P^{m,r}_{\bar{y};\bar{z}} \psi(\bar{y}, \bar{z}, x_0, \ldots, x_{n-1})$, where $\bar{y} = y_0, \ldots, y_{m-1}$ and $\bar{z} = z_0, \ldots, z_{r-1}$.
($\alpha$) If $\mathfrak{A} \models P^{m,r}_{\bar{y};\bar{z}} \psi[a_0, \ldots, a_{n-1}]$, there exists a non-trivial partition of $A$: $A' \neq \emptyset \neq A''$,$A = A' \dot{\cup} A''$, such that for any $\bar{b} \in (A')^m$ and $\bar{c} \in (A'')^r$, we have $\mathfrak{A} \models \psi[\bar{b}, \bar{c}, a_0, \ldots, a_{n-1}]$. Let $e$ and $f$ be any nonempty witnesses of $A'$ and $A''$ respectively. By the construction of $A$, for some sufficiently large $\beta \geq \alpha (\lambda > \beta)$, $e, f, a_0, .., a_{n-1} \in A_\beta$. So $A' \cap A_\beta \neq \emptyset \neq A'' \cap A_\beta$, $A_\beta = (A' \cap A_\beta) \dot{\cup} (A'' \cap A_\beta)$, namely, $(A' \cap A_\beta) | (A'' \cap A_\beta)$ is a non-trivial partition of $A_\beta$. By the induction hypotheses, for any $\bar{b} \in (A' \cap A_\beta)^m$ and $\bar{c} \in (A'' \cap A_\beta)^r$, we have $\mathfrak{A}_\beta \models \psi[\bar{b}, \bar{c}, a_0, \ldots, a_{n-1}]$. It follows $\mathfrak{A}_\beta \models P^{m,r}_{\bar{y};\bar{z}} \psi[a_0, \ldots, a_{n-1}]$. Finally since $\mathfrak{A}_\alpha \prec_{\mathcal{L}(MP)} \mathfrak{A}_\beta$, $\mathfrak{A}_\alpha \models P^{m,r}_{\bar{y};\bar{z}} \psi[a_0, \ldots, a_{n-1}]$.
($\beta$) If $\mathfrak{A}_\alpha \models P^{m,r}_{\bar{y};\bar{z}} \psi[a_0, \ldots, a_{n-1}]$, there exists a non-trivial partition $A_{\alpha 0} | A_{\alpha 1}$ of $A_\alpha$, over which $\mathfrak{A}_\alpha \models \psi[\cdot, \cdot, \bar{a}]$ is homogeneously true. Let $e$ and $f$ be some nonempty witnesses of $A_{\alpha 0}$ and $A_{\alpha 1}$ respectively, obviously $(\mathfrak{A}_\alpha, e, f) \models P^{m,r}_{\bar{y};\bar{z}}[\psi \wedge z_0 \neq \underline{e} \wedge y_0 \neq \underline{f}]$, in other words, e is unable to pseudo-reach to f in the Gaifman graph $(A_\alpha, R^{\mathfrak{A}_\alpha}, \ldots)$, where new $R := \neg\psi$.

For contradiction, suppose $\mathfrak{A} \models \neg P^{m,r}_{\bar{y};\bar{z}} \psi[\bar{a}]$, i.e. Gaifman graph $(A, R^{\mathfrak{A}}, \ldots)$ is pseudo-connected, particularly e is pseudo-reachable to f. By Lemma 2, finitely many elements are enough to witness this reachability. According to the construction of $A$ there exists large enough $\beta$ ($\lambda > \beta \geq \alpha$), such that $A_\beta$ contains all these reachability witnesses. By the induction hypotheses, $(\mathfrak{A}_\alpha \prec_P)\mathfrak{A}_\beta \prec_P^k \mathfrak{A}$, where $k = P - \mathrm{qr}(\neg\psi) = P - \mathrm{qr}(\psi) < P - \mathrm{qr}(P^{m,r}_{\bar{y},\bar{z}}\psi)$, hence e is pseudo-reachable to f in generalized subgraph $(A_\beta, R^{\mathfrak{A}_\beta}, \ldots)$; and furthermore, so does in $(A_\alpha, R^{\mathfrak{A}_\alpha}, \ldots)$ as well, for $\mathfrak{A}_\alpha \prec_{\mathcal{L}(MP)} \mathfrak{A}_\beta$ and pseudo-reachability is an $\mathcal{L}(MP)$-property. A contradiction to the assumption. So $\mathfrak{A} \models P^{m,r}_{\bar{y};\bar{z}}\psi[\bar{a}]$.

It is not difficult to fill up the technical details, such as expansion-reduction operations etc., to turn the above intuitive sketch into a complete proof. □

**Remark.** There is another (standard) way to accomplish part($\beta$) of the above proof, similar to that in case of $\mathcal{L}_{\omega\omega}$, $\mathcal{L}(Q)$ etc. The argument we adopted here reflects our taste of sticking to graph intuition.

**Corollary 3.** Dedekind cut property is not definable in $\mathcal{L}_P$.

Proof. Let $D$ denote the Dedekind cut property for linear order. Then, $(\mathbb{R}, <) \models D$, $(\mathbb{Q}, <) \models \neg D$, where $\mathbb{R}$ is the set of all reals, $\mathbb{Q}$, the set of all relational numbers; and $\neg D$ is a monadic $\Sigma^1_1$-sentence,see [2]. Let $(\mathbb{R}_0, <)$ be a countable $\mathcal{L}_P$ elementary substructure of $(\mathbb{R}, <)$. $(\mathbb{R}_0, <)$ is dense linear order without ends (this property is $\mathcal{L}_{\omega\omega}$ definable), hence $(\mathbb{Q}, <) \cong (\mathbb{R}_0, <)$. If $D$ were $\mathcal{L}_P$ definable, then $(\mathbb{R}_0, <) \models D$, a contradiction. □

Using automata-theoretic technique, Büchi proved that the monadic theories of finite and $\omega$ words ($S1S$) are decidable. Rabin extended these results and method to $S2S$, and further by reducing technique to obtain the decidability of the monadic theories of countable chains. A natural thought is: via introducing the notion of $\alpha-$automata ($\alpha$ is an infinite ordinal), study the decision problem of monadic theory of infinite ordinals (even arbitrary chains) . But all the endeavor along this line by Büchi and his followers stopped by the wall of $\omega_2$, i.e. the extensions only work for all those ordinals whose cardinality are less or equal to $\aleph_1$. Later Gurevich-Magidor-Shelah clarified the intrinsic and deep contents under this phenomenon: decidability of the monadic theory of $\omega_2$ depends on the existence of weakly compact cardinal (a set-theoretic hypotheses), and hence it is independent of **ZFC**. ([3] is a good survey in this respect.) For a decision problem, this situation seems to be a little uneasy, and also hints that maybe the expressive power of $MSO$ is a little bit too strong (but not strong enough, like $SO$, to settle down the problem again, negatively). Indeed, if we live in the world of partition logics, all these uncertainty (independency) will soon vanish, because of the Downward Löwenheim-Skolen-Tarski property of partition logics and $\mathcal{L}$(MP) $\leq$ MSO.

**Corollary 4.** The $\mathcal{L}$(MP) theory of any ordinal (even any chain) is decidable.

The following table summerizes the decidability of theories of infinite ordinals in three logic frameworks.

| Range of ordinal $\alpha$ | $\mathcal{L}$(MP) | $MSO$ | $SO$ |
|---|---|---|---|
| $\alpha \geq \omega_2$ | decidable | independent | undecidable |
| $\omega_2 > \alpha \geq \omega$ | decidable | decidable | undecidable |

**§3. Extensions with Malitz Quantifiers.**

Malitz quantifiers are the monadic partition quantifiers with cardinality restriction. $\mathcal{L}$(MQ) will be used to denote the extended logic by adding all such quantifiers to $\mathcal{L}_{\omega\omega}$.

**Definition 5.** $\mathfrak{A} \models Q^{2,1}_{x,x';y}\varphi(x, x', y)$ iff there is a non-trivial partition of universe $A(|A| \geq \kappa \geq \aleph_0)$ with $A = A_0 \dot{\cup} A_1$, $|A_0| \geq \kappa$ ($\kappa$-interpretation) where $\kappa$ is an infinite cardinal, and $A_1 \neq \emptyset$, such that for all $a, a' \in A_0, b \in A_1$, $\mathfrak{A} \models \varphi[a, a', b]$.

Note that if the cardinality of the structure is at least $\kappa$, then for any non-trivial partition of its universe, $A_0|A_1$, $\max(|A_0|, |A_1|) \geq \kappa$. So we have the following simple lemma.

**Lemma 3.** For any $\mathfrak{A}$, $|A| \geq \kappa$, $\mathfrak{A} \models P^{m,n}_{\bar{x};\bar{y}}\varphi$, iff $\mathfrak{A} \models Q^{m,n}_{\bar{x};\bar{y}}\varphi \vee Q^{n,m}_{\bar{y};\bar{x}}\varphi$.

The next lemma will establish a substantial relation between Ebbinghaus quantifiers and Malitz quantifiers, in reverse direction. The Malitz quantifier may be looked as, concerning witnesses, a (locally finite) direct limit of a series of Ebbinghaus quantifiers. First we need some extra notions.

**Definition 6.** A *directed system* $\mathcal{M}$ of set $A$ is a subset of $\mathcal{P}(A)$, such that for all $M_0, M_1 \in \mathcal{M}$ there exists a $M \in \mathcal{M}$ with both $M_0 \subseteq M$ and $M_1 \subseteq M$. Further if for all the $M \in \mathcal{M}$, $|M| < \aleph_0$, $\mathcal{M}$ is a *finite directed system* of $A$.

**Lemma 4.** For any $\mathfrak{A}$, $\mathfrak{A} \models Q^{m,n}_{\bar{x};\bar{y}}\varphi$, where the Malitz quantifiers are under $\kappa$ interpretation, iff there exist a finite directed system $\mathcal{M}$ of $A$ with $|\bigcup\mathcal{M}| \geq \kappa$, and an element $f$ of $A$, such that for any $M \in \mathcal{M}$, $\mathfrak{A} \models P^{m,n}_{\bar{x};\bar{y}}[\varphi \wedge \bigwedge_{e\in M} y_0 \neq u_e \wedge x_0 \neq v][M,f]$, where $[M,f]$ means the substitution $[e/u_e, f/v : e \in M]$.

Proof.
(1) $\mathfrak{A} \models Q^{m,n}_{\bar{x};\bar{y}}\varphi$, hence there exists a non-trivial partition $A_0|A_1$ of $A$ with $|A_0| \geq \kappa$, over which $\varphi$ is satisfied homogeneously. The result follows immediately by taking $\mathcal{M} = \{M | M \subseteq_{fin} A_0\}$ and $f$ any element of $A_1$.
(2) Conversely, assume $\mathfrak{A} \models \neg Q^{m,n}_{\bar{x};\bar{y}}\varphi$. Define the $\tau \cup \{R\}$-expansion $\mathfrak{A}_R$ of $\mathfrak{A}$, where $R^{\mathfrak{A}_R}(\bar{x},\bar{y}) = \neg\varphi^{\mathfrak{A}}(\bar{x},\bar{y})$, then we reduce $\mathfrak{A}_R$ to $\overline{\mathfrak{A}_R}$ over the vocabulary $\{R\}$. Similar to the proof of lemma 2, let $\pi = \{R, S, S', \underline{f}\} \cup \{\underline{e} | e \in \bigcup\mathcal{M}\}$, where $S$ and $S'$ both are unary relation symbol, suppose $D$ is the diagram of $\overline{\mathfrak{A}_R}$, we can prove the following claim:

$$D \cup \{\forall w(Sw \vee S'w), \neg\exists z(Sz \wedge S'z)\} \cup \{S\underline{e} | e \in \bigcup\mathcal{M}\} \cup \{S'\underline{f}\} \models \exists\bar{x}\bar{y}[\bar{x} \in S^m \wedge \bar{y} \in (S')^n \wedge R(\bar{x},\bar{y})]$$

Then by the Compactness of $\mathcal{L}_{\omega\omega}$, we can pick out a finite $M_0 \subset \bigcup\mathcal{M}$, such that,

$$D \cup \{\forall w(Sw \vee S'w), \neg\exists z(Sz \wedge S'z)\} \cup \{S\underline{e} | e \in M_0\} \cup \{S'\underline{f}\} \models \exists\bar{x}\bar{y}[\bar{x} \in S^m \wedge \bar{y} \in (S')^n \wedge R(\bar{x},\bar{y})]. \quad (3)$$

For $\mathcal{M}$ is directed, without the loss of generality we can suppose $M_0 \in \mathcal{M}$. By the assumption, $\mathfrak{A} \models P^{m,n}_{\bar{x};\bar{y}}[\varphi \wedge \bigwedge_{e\in M_0} y_0 \neq u_e \wedge x_0 \neq v][M_0, f]$, which can be reduced to $\overline{\mathfrak{A}_R} \models P^{m,n}_{\bar{x};\bar{y}}[\neg R(\bar{x},\bar{y}) \wedge \bigwedge_{e\in M_0} y_0 \neq u_e \wedge x_0 \neq v][M_0, f]$. Hence there is an appropriate partition $A_0|A_1$ of A with $M_0 \subseteq A_0$ and $f \in A_1$, such that for the $\{R, S, S'\}$-expansion $\overline{\mathfrak{A}_R}^*$ of $\overline{\mathfrak{A}_R}$, with $S^{\overline{\mathfrak{A}_R}^*} = A_0$ and $S'^{\overline{\mathfrak{A}_R}^*} = A_1$,

$$\overline{\mathfrak{A}_R}^* \models \forall\bar{x}\bar{y}[(\bar{x} \in S^m \wedge \bar{y} \in (S')^n) \to \neg R(\bar{x},\bar{y})]. \quad (4)$$

Note $\overline{\mathfrak{A}_R}^* \models \{\forall w(Sw \vee S'w), \neg\exists z(Sz \wedge S'z)\} \cup \{S\underline{e} | e \in M_0\} \cup \{S'\underline{f}\}$, so by (3)

$$\overline{\mathfrak{A}_R}^* \models \exists\bar{x}\bar{y}[\bar{x} \in S^m \wedge \bar{y} \in (S')^n \wedge R(\bar{x},\bar{y})],$$

which surely contradicts to (4). So $\mathfrak{A} \models Q^{m,n}_{\bar{x};\bar{y}}\varphi$. □

Observe that we can apply a weaker assumption "$\mathcal{M}$ is a finite-covered family" (i.e. for each finite subset $M_0$ of A there is an $M \in \mathcal{M}$ with $M_0 \subseteq M$ ), to instead of $\mathcal{M}$'s directedness. In fact when $\mathcal{M} \subseteq \mathcal{P}^\omega(A)$, these two notions are equivalent.

Using above two lemmas, we can prove the Downward Löwenheim-Skolen-Tarski property of $\mathcal{L}(\mathrm{MP} + \mathrm{MQ})$, the logic extending $\mathcal{L}_{\omega\omega}$ by adding all the Ebbinghaus and Malitz quantifiers.

**Theorem 4.** Assume $|\tau| \leq \kappa$, let $\mathfrak{A}$ be a infinite $\tau$-structure, given any $\kappa$-powered $A_0 \subseteq A$, $\mathfrak{A}$ has an $\mathcal{L}(\mathrm{MP} + \mathrm{MQ})$ elementary substructure $\mathfrak{A}^*$ with $A_0 \subseteq A^*$ and $|A^*| = \kappa$, where the Malitz quantifiers are under $\kappa$ interpretation.

Proof. Without loss of generality, we consider a typical fragment of $\mathcal{L}(\mathrm{MP}+\mathrm{MQ})$, say $\mathcal{L}(P^{m,m}+Q^{m,m})$ (note, similarly to example 1, if $m \geq n$, $\mathcal{L}(P^{m,n}+Q^{m,n}) \leq \mathcal{L}(P^{m,m}+Q^{m,m})$,) and assume $|A| > \kappa$. The construction proceeds similarly to that of Theorem 2. Given the $\kappa$-powered $A_k$, consider all the ($\kappa$ many) formulae of $\mathcal{L}(P^{m,m}+Q^{m,m})[\tau]$ $\varphi(\bar{x}, \bar{y}, z_0, \ldots, z_{q-1})$ and all $c_0, \ldots, c_{q-1} \in A_k$, where $\bar{x} = x_0, \ldots, x_{m-1}$, $\bar{y} = y_0, \ldots, y_{m-1}$.
(1) if $\mathfrak{A} \models \neg P^{m,m}_{\bar{x};\bar{y}}\varphi[c_0, \ldots, c_{q-1}]$, let $\pi = \tau \cup \{R\}$, where $R$ is a new $(m+m)-$ary relation symbol. Consider the $\pi$-expansion $\mathfrak{A}_R$ of $\mathfrak{A}$, with $R^{\mathfrak{A}_R} = \neg\varphi^{\mathfrak{A}}(\bar{x}, \bar{y}, c_0, \ldots, c_{q-1})$. So the assumption implies $\mathfrak{A}_R \models \neg P^{m,n}_{\bar{x};\bar{y}}\neg R(\bar{x}, \bar{y})$. Just like the construction (2) in the proof of theorem 2, we put all the witnesses of the pseudo-reachability from $e$ to $f$ into $A_{k+1}$, for all the $e, f \in A_k$ with $e \neq f$.
(2) if $\mathfrak{A} \models Q^{m,m}_{\bar{x};\bar{y}}\varphi[c_0, \ldots, c_{q-1}]$, there is a partition $A'|A''$ of $A$ with $|A'| \geq \kappa$ and $A'' \neq \emptyset$, where $\varphi$ is satisfied homogeneously. Then we can choose $\kappa$ many elements in $A'$ and one element in $A''$ as the witnesses of the partition, and put them into $A_{n+1}$.

It is also easy to verify that $A_{k+1}$ is of power $\kappa$. Observe that in above construction, we take no notice of the cases: $\mathfrak{A} \models P^{m,m}_{\bar{x};\bar{y}}\varphi[c_0, \ldots, c_{q-1}]$ and $\mathfrak{A} \models \neg Q^{m,m}_{\bar{x};\bar{y}}\varphi[c_0, \ldots, c_{q-1}]$.

Iterated the above construction $\omega$ times, we get $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$, let $A^* = \bigcup_{n<\omega} A_n$. Surely $\mathfrak{A}^* := \mathfrak{A} \upharpoonright A^*$ is a $\kappa$-powered substructure of $\mathfrak{A}$.

**Claim** $\mathfrak{A}^* \prec_{L(P^{m,m}+Q^{m,m})} \mathfrak{A}$.

By induction, for any formula $\psi \in \mathcal{L}(P^{m,m}+Q^{m,m})[\tau]$ and $c_0, \ldots, c_{q-1} \in A^*$:
($\alpha$) If $\mathfrak{A} \models Q^{m,m}_{\bar{x};\bar{y}}\varphi[c_0, \ldots, c_{q-1}]$, the construction(2) guarantees $\mathfrak{A}^* \models Q^{m,m}_{\bar{x};\bar{y}}\varphi[c_0, \ldots, c_{q-1}]$, in a way similar to ($\alpha$) in the proof of Theorem 2, with additional consideration about cardinal constraint.
($\beta$) If $\mathfrak{A} \models P^{m,m}_{\bar{x};\bar{y}}\varphi[c_0, \ldots, c_{q-1}]$, note $|A| \geq \kappa$, so by lemma 3, $\mathfrak{A} \models (Q^{m,m}_{\bar{x};\bar{y}}\varphi \vee Q^{m,m}_{\bar{y};\bar{x}}\varphi)[c_0, \ldots, c_{q-1}]$.[4] Similarly to the discussion in ($\alpha$), $\mathfrak{A}^* \models (Q^{m,m}_{\bar{x};\bar{y}}\varphi \vee Q^{m,m}_{\bar{y};\bar{x}}\varphi)[c_0, \ldots, c_{q-1}]$. Again by lemma 3 $\mathfrak{A}^* \models P^{m,m}_{\bar{x};\bar{y}}\varphi[c_0, \ldots, c_{q-1}]$.
($\gamma$) If $\mathfrak{A} \models \neg P^{m,m}_{\bar{x};\bar{y}}\varphi[c_0, \ldots, c_{q-1}]$,then $\mathfrak{A}^* \models \neg P^{m,m}_{\bar{x};\bar{y}}\varphi[c_0, \ldots, c_{q-1}]$. this case is exactly the same as the ($\beta$) in the proof of theorem 2,by construction (1).
($\delta$) If $\mathfrak{A} \models \neg Q^{m,m}_{\bar{x};\bar{y}}\varphi[c_0, \ldots, c_{q-1}]$, by contradiction, assume $\mathfrak{A}^* \models Q^{m,m}_{\bar{x};\bar{y}}\varphi[c_0, \ldots, c_{q-1}]$. So by Lemma 4, there exist a finite directed system $\mathcal{M}$ of $A^*$ with $|\bigcup\mathcal{M}| \geq \kappa$, and an element $f$ of $A^*$, such that for any $M \in \mathcal{M}$, $\mathfrak{A}^* \models P^{m,m}_{\bar{x};\bar{y}}[\varphi \wedge \bigwedge_{e\in M} y_0 \neq u_e \wedge x_0 \neq v][M, f, c_0, \ldots, c_{q-1}]$. From construction(1) which guarantees that $\mathfrak{A} \models \neg P^{m,m}_{\bar{x};\bar{y}}[\varphi \wedge \bigwedge_{e\in M} y_0 \neq u_e \wedge x_0 \neq v][M, f, c_0, \ldots, c_{q-1}]$ would imply $\mathfrak{A}^* \models \neg P^{m,m}_{\bar{x};\bar{y}}[\varphi \wedge \bigwedge_{e\in M} y_0 \neq u_e \wedge x_0 \neq v][M, f, c_0, \ldots, c_{q-1}]$ (see ($\gamma$)), we get $\mathfrak{A} \models P^{m,m}_{\bar{x};\bar{y}}[\varphi \wedge \bigwedge_{e\in M} y_0 \neq u_e \wedge x_0 \neq v][M, f, c_0, \ldots, c_{q-1}]$, for any $M \in \mathcal{M}$. Applying Lemma 4 again, $\mathfrak{A} \models Q^{m,m}_{\bar{x};\bar{y}}\varphi[c_0, \ldots, c_{q-1}]$, which contradicts to the assumption. So $\mathfrak{A}^* \models \neg Q^{m,m}_{\bar{x};\bar{y}}\varphi[c_0, \ldots, c_{q-1}]$. □

Note $\mathcal{L}(\mathrm{MQ})$ is a sublogic of $\mathcal{L}(\mathrm{MP}+\mathrm{MQ})$, hence the above theorem implies,

**Corollary 5.** $\mathcal{L}(\mathrm{MQ})$ has the Downward Löwenheim-Skolen-Tarski property.

The downward Löwenheim-Skolen-Tarski property of $\mathcal{L}(\mathrm{MQ})$ can be reached by some other ways, for instance, by an approach similar to that of Theorem 2, or by one like Imhof [4].

**Historical Notes.** The main result of section 2 is a part of the manuscript [7]. While H.Imhof had obtained independently some similar results in a different way, which is focused on Malitz quantifiers first [4].

[4]This is why we prefer $\mathcal{L}(P^{m,m}+Q^{m,m})$ to $\mathcal{L}(P^{m,n}+Q^{m,n})$, otherwise we should consider $Q^{n,m}$ which would make the argument more tedious.

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