An analysis of the W*-hierarchy

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January 24, 2007

Abstract

We observe that the W*-hierarchy, a variant (introduced by Downey, Fellows, and Taylor [8]) of the better known W-hierarchy, coincides with the W-hierarchy, though not level wise, but just as a whole hierarchy. More precisely, we prove that $\mathrm{W}[t] \subseteq \mathrm{W}^*[t] \subseteq \mathrm{W}[2t-2]$ for each $t \geq 2$. It was known before that $\mathrm{W}[1] = \mathrm{W}^*[1]$ and $\mathrm{W}[2] = \mathrm{W}^*[2]$.

Our second main result is a new logical characterization of the W*-hierarchy in terms of "Fagin-definable problems." As a by-product, we also obtain an improvement of our earlier characterization of the hierarchy in terms of model-checking problems. Furthermore, we obtain new complete problems for the classes W[3] and W*[3].

1. Introduction

It is well-known that the landscape of fixed-parameter intractable problems is not as nice and simple as the landscape of classical intractable problems provided by the theory of NP-completeness. Instead, there is a huge variety of seemingly different classes of fixed-parameter intractable problems (see [5, 11]). Various hierarchies of such problems have been considered. The most important of these is the *W-hierarchy*, introduced by Downey and Fellows [2, 3] in their ground breaking series of papers on parameterized (in)tractability. The W-hierarchy is defined "syntactically" by means of formulas of propositional logic or Boolean circuits. An innocent looking syntactical variation of the definition of the W-hierarchy leads to the *W*-hierarchy* [8, 4]. Besides the "foundational" interest in the W*-hierarchy from a structural complexity theoretic point of view, the hierarchy has also received attention because some parameterized problems can naturally placed into the lower levels of this hierarchy (see [1, 4, 11]). It is still not known whether the W-hierarchy and the W*-hierarchy coincide.

In this paper, we show that at least the two hierarchies coincide as a whole, that is, a problem is contained in some level of the W-hierarchy if and only if it is contained in some level of the W*-hierarchy. Actually, this is an easy consequence of a normal form result for the W*-hierarchy due to Downey, Fellows, and Taylor (Lemma 20 of [8]). Our first main result is the following tighter relationship between the hierarchies:

Theorem 1. *For every $t \geq 2$,*

$$\mathrm{W}[t] \subseteq \mathrm{W}^*[t] \subseteq \mathrm{W}[2t-2].$$

This generalizes a result due to Downey and Fellows [4] that $\mathrm{W}[2] = \mathrm{W}^*[2]$. Downey et al. [8] had already proved earlier that $\mathrm{W}[1] = \mathrm{W}^*[1]$. The proof of Theorem 1 heavily builds on our approach to parameterized complexity theory based on first-order logic and specifically on techniques that were developed in [10].

As mentioned above, the W-hierarchy and the W*-hierarchy were originally defined in terms of propositional logic (or, equivalently, Boolean circuits). However, it has been shown that they can also be characterized in terms of first-order logic. In fact, in Section 2 we define them via first-order logic and based on these definitions, we prove Theorem 1 in Section 3. We exemplify the difference between the W-hierarchy and the W*-hierarchy with help of the parameterized maximal irredundant set problem.

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The first-order characterizations have given new insights into the structure of the hierarchies and also led to simpler proofs of the core structural theorems (see [10, 11]). Characterizations of the W-hierarchy in terms of first-order logic were first obtained by Downey, Fellows, and Regan [6] and later improved by Flum and Grohe [9, 10]. In the terminology of [9], we distinguish between two types of such characterizations, one based on *model-checking problems* and one based on *Fagin-definable problems*. As a matter of fact, in [11] we decided to take the characterization by Fagin-definable problems as the definition of the W-hierarchy. A characterization of the W*-hierarchy in terms of model-checking problems was given in [10]. The second main result of this paper is a characterization of the W*-hierarchy in terms of Fagin-definable problems (see Theorem 17 for the precise statement). As already seen in [11], when translating characterizations based on model-checking problems into characterizations in terms of Fagin-definable problems, propositional logic serves as a useful bridge. While using this bridge we slightly improve a propositional normal form of [8] for the W*-hierarchy (see Section 4). Moreover, the analysis of the structure of the W*-hierarchy underlying the proof of the Fagin-type characterization of this hierarchy yields a strengthening of the model-checking characterization in that it allows a restriction to graphs (see Theorem 18 for the precise statement).

To give some more intuition for the higher levels of the W-hierarchy and the W*-hierarchy, in the last section of the paper we define two algorithmic problems on hypergraphs that are complete for the third levels of the respective hierarchies. These problems can be viewed as generalizations of the well-known dominating set problem to families of graphs. Though admittedly somewhat artificial, the problems nevertheless illustrate the difference between the two hierarchies (on the lowest level where the hierarchies are not known to coincide).

2. The W-hierarchy and the W*-hierarchy

A (relational) *vocabulary* τ is a finite set of relation symbols. Each relation symbol has an *arity*. A τ-*structure* (or, simply structure), consists of a set A called the *universe*, which we assume to be finite, and an interpretation $R^{\mathcal{A}} \subseteq A^r$ of each r-ary relation symbol $R \in \tau$. For example, we view a *graph* as a structure $\mathcal{G} = (G, E^{\mathcal{G}})$, where E is a binary relation symbol and $E^{\mathcal{G}}$ is a irreflexive and symmetric binary relation on the set of vertices G. Nevertheless, sometimes we denote the vertex set of a graph $\mathcal{G}$ by V and the edge set by E (instead of G and $E^{\mathcal{G}}$) and use the set notation $\{v, w\}$ for edges.

Formulas of first-order logic are built up from atomic formulas using the usual boolean connectives and existential and universal quantification. Recall that *atomic formulas* are formulas of the form $x = y$ or $Rx_1 \dots x_r$, where $x, y, x_1, \dots, x_r$ are variables and R is an r-ary relation symbol. *Literals* are atomic or negated atomic formulas. For $t \geq 1$, let Σ_t denote the class of all first-order formulas of the form

$$\exists x_{11} \dots \exists x_{1k_1} \forall x_{21} \dots \forall x_{2k_2} \ \dots \ Qx_{t1} \dots Qx_{tk_t} \ \psi,$$

where $Q = \forall$ if t is even and $Q = \exists$ otherwise, and where ψ is quantifier-free. Π_t-formulas are defined analogously starting with a block of universal quantifiers. Let $t, u \geq 1$. A formula φ is $\Sigma_{t,u}$, if it is Σ_t and all quantifier blocks after the leading existential block have length $\leq u$.

While conjunctions and disjunctions of Σ_t-formulas are logically equivalent to a Σ_t-formula, the classes $\Sigma_{t,u}$ for $u \geq 1$ do not share this property. We define the classes $\Sigma^*_{t,u}$ that have this closure property.

If Φ is a class of first-order formulas and $u \geq 1$, let

- $\exists^u(\Phi) := \{\exists x_1 \dots \exists x_m \varphi \mid m \leq u, \, \varphi \in \Phi\}$; $\quad \forall^u(\Phi) := \{\forall x_1 \dots \forall x_m \varphi \mid m \leq u, \, \varphi \in \Phi\}$;
- $\exists^*(\Phi) := \{\exists x_1 \dots \exists x_m \varphi \mid m \geq 0, \, \varphi \in \Phi\}$; $\quad \forall^*(\Phi) := \{\forall x_1 \dots \forall x_m \varphi \mid m \geq 0, \, \varphi \in \Phi\}$;
- $\mathrm{BC}(\Phi)$:= the class of boolean combinations of formulas in Φ.

Let ATOM be the class of all atomic formulas.

Definition 2. For $m \geq 0$ and $u \geq 1$ let the set $Q_{m,u}$ of first-order formulas be defined by induction on m:

- $Q_{0,u} :=$ ATOM.
- $Q_{m+1,u} := \forall^u(\mathrm{BC}(Q_{m,u}))$.

For $t, u \geq 1$, we set

$$\Sigma^*_{t,u} := \exists^*(\mathrm{BC}(Q_{t-1,u})).$$

Note that $\Sigma_{t,u} \subseteq \Sigma^*_{t,u}$. For $\Phi \subseteq \Sigma^*_{t,u}$ we denote by strict-Φ the class of formulas φ in Φ with the property that every atomic subformula in φ contains at most one variable of the leading existential (the unbounded) block.

We assume that the reader is familiar with the basic notions of parameterized complexity theory (cf. [5, 11]). In particular, a parameterized problem is *fixed-parameter tractable* if it is solvable in time $f(k) \cdot p(n)$ for some computable function f and some polynomial p; here n is the length of the instance and k denotes its parameter. For parameterized problems P and P' we write $P \leq^{\text{fpt}} P'$ if there is a (many-one) fpt-reduction from P to P'. By $[P]^{\text{fpt}}$ we denote the class of problems fpt-reducible to P. Similarly, if C is a class of parameterized problems, $[\mathrm{C}]^{\text{fpt}}$ is the class of problems fpt-reducible to some problem in C.

Let Φ be a class of first-order formulas. By p-$\mathrm{MC}(\Phi)$ we denote the *parameterized model-checking problem for* Φ:

p-$\mathrm{MC}(\Phi)$	
Input:	A structure $\mathcal{A}$ and a sentence $\varphi \in \Phi$.
Parameter:	$\|\varphi\|$.
Problem:	Decide whether $\mathcal{A}$ is a model of φ.

If S is a class of structures, we let p-$\mathrm{MC}(\mathrm{S}, \Phi)$ be the restriction of p-$\mathrm{MC}(\Phi)$ to input structures from S.

The classes of the W-hierarchy and the classes of the W*-hierarchy can be defined in the following way:

Definition 3. For $t \geq 1$ let

$$\mathrm{W}[t] := [p\text{-}\mathrm{MC}(\Sigma_{t,1})]^{\text{fpt}} \qquad \text{and} \qquad \mathrm{W}^*[t] := [p\text{-}\mathrm{MC}(\Sigma^*_{t,1})]^{\text{fpt}}.$$

For the sake of completeness let us mention that $\mathrm{A}[t] := [p\text{-}\mathrm{MC}(\Sigma_t)]^{\text{fpt}}$. Let GRAPH be the class of graphs. The following results are known (cf. [6, 10]):

Theorem 4. *Let* $t, u \geq 1$.

(1) p-$\mathrm{MC}(\Sigma_{t,u})$ *and, for* $t \geq 2$*, the problem* p-$\mathrm{MC}(\text{strict-}\Sigma_{t,u})$ *are* $\mathrm{W}[t]$*-complete under fpt-reductions.*

(2) p-$\mathrm{MC}(\mathrm{GRAPH}, \Sigma_{t,1})$ *is* $\mathrm{W}[t]$*-complete under fpt-reductions.*

(3) p-$\mathrm{MC}(\Sigma^*_{t,u})$ *and, for* $t \geq 2$*, the problem* p-$\mathrm{MC}(\text{strict-}\Sigma^*_{t,u})$ *are* $\mathrm{W}^*[t]$*-complete under fpt-reductions.*

Proposition 8 will show that already p-$\mathrm{MC}(\text{strict-}\exists^*(Q_{t-1,1}))$ is $\mathrm{W}^*[t]$-complete under fpt-reductions.

We close this section by introducing a parameterized problem (cf. [1]), which exemplifies the differences between the W-hierarchy and the W*-hierarchy (and the A-hierarchy).

Let I be a set of vertices of the graph $\mathcal{G} = (V, E)$ and $u \in I$. A vertex $v \in V$ is a *private neighbor of* u *with respect to* I if $v \in N(u)$ and $v \notin N(u')$ for all $u' \in I$ with $u' \neq u$. Here $N(w) := \{w' \mid w' = w \text{ or } \{w, w'\} \in E\}$. The set I is *irredundant* if each vertex in I has a private neighbor with respect to I, and I is a *maximal irredundant set* if it is irredundant and there is no irredundant set I' with $I \subset I' \subseteq V$. We consider the problem

p-MAXIMAL-IRREDUNDANT-SET	
Input:	A graph $\mathcal{G}$ and $k \in \mathbb{N}$.
Parameter:	k.
Problem:	Decide whether $\mathcal{G}$ has a maximal irredundant set of cardinality k.

Let $priv_{k,i}(y, x_1, \dots, x_k)$ be a quantifier-free first-order formula expressing that "y is a private neighbor of x_i with respect to $\{x_1, \dots, x_k\}$" (clearly the formula depends on k). Then $(\mathcal{G}, k)$ is a positive instance of p-MAXIMAL-IRREDUNDANT-SET if and only if

$$\mathcal{G} \models \exists x_1 \dots \exists x_k \exists u_1 \dots \exists u_k \forall x_{k+1} \forall v_1 \dots \forall v_{k+1} \Big(\bigwedge_{i\in[k]} priv_{k,i}(u_i, x_1, \dots, x_k) \wedge \bigvee_{j\in[k+1]} \neg priv_{k+1,j}(v_j, x_1, \dots, x_k, x_{k+1})\Big).$$

(For a natural number s let $[s] := \{1, \dots, s\}$.) Hence, p-MAXIMAL-IRREDUNDANT-SET $\leq^{\text{fpt}}$ p-MC(Σ_2) and thus, p-MAXIMAL-IRREDUNDANT-SET $\in$ A[2]. To place the problem in a level of the W-hierarchy, we cannot use the preceding formula, since the length of the universal quantifier block depends on k. But we can express that $(\mathcal{G}, k)$ is a positive instance also by the formula

$$\exists x_1 \dots \exists x_k \exists u_1 \dots \exists u_k \forall x_{k+1} \bigvee_{j\in[k+1]} \forall v \Big(\bigwedge_{i\in[k]} priv_{k,i}(u_i, x_1, \dots, x_k) \wedge \neg priv_{k+1,j}(v, x_1, \dots, x_k, x_{k+1})\Big).$$

By standard techniques (see Lemma 6) one can replace the disjunction $\bigvee_{j\in[k+1]}$ by an existential quantifier, thus obtaining a reduction to p-MC$(\Sigma_{4,1})$. Hence, p-MAXIMAL-IRREDUNDANT-SET $\in$ W[4]. Otherwise the previous formula shows at the same time that we can reduce the problem to p-MC$(\Sigma^*_{3,1})$ and thus, p-MAXIMAL-IRREDUNDANT-SET $\in$ W*[3].

3. The equivalence between the W-hierarchy and the W*-hierarchy

In this section we show the main result of this paper:

Theorem 5. *For any $t \geq 2$,*

$$\text{W}[t] \subseteq \text{W}^*[t] \subseteq \text{W}[2t-2].$$

Essentially this theorem will be proven by Proposition 8, Lemma 9, and Lemma 10. Before turning to them we list two simple results which will be used again and again. The first one shows how a conjunction can be replaced by a universal quantifier followed by a disjunction. How in the formula obtained thereby further quantifiers can be brought in front of this additional disjunction in an "economical way" is shown by the second result.

Lemma 6. *Let $R_1, \dots, R_s$ be unary relation symbols and $\varphi_1(\bar{x}), \dots, \varphi_s(\bar{x})$ formulas of first-order logic. If $\mathcal{A}$ is a structure such that $R_1^{\mathcal{A}}, \dots, R_s^{\mathcal{A}}$ is a partition of A into nonempty sets , then*

- $\mathcal{A} \models \forall \bar{x}\Big(\bigwedge_{i\in[s]} \varphi_i \longleftrightarrow \forall y \bigvee_{i\in[s]}(R_i y \wedge \varphi_i)\Big)$;
- $\mathcal{A} \models \forall \bar{x}\Big(\bigvee_{i\in[s]} \varphi_i \longleftrightarrow \exists y \bigwedge_{i\in[s]}(R_i y \to \varphi_i)\Big)$.

Lemma 7. *Assume that $\varphi_1(\bar{x}), \dots, \varphi_s(\bar{x})$ and $\psi_1(\bar{x}, y), \dots, \psi_s(\bar{x}, y)$ with y distinct from the variables in $\bar{x}$ are formulas of first-order logic and that $Q \in \{\forall, \exists\}$. If $\mathcal{A}$ is a structure such that $\mathcal{A} \models \forall \bar{x} \neg(\varphi_i \wedge \varphi_j)$ for $i \neq j$, then*

- $\mathcal{A} \models \forall \bar{x}\Big(\bigwedge_{i\in[s]}(\varphi_i \to Qy\psi_i) \longleftrightarrow Qy \bigwedge_{i\in[s]}(\varphi_i \to \psi_i)\Big)$;
- $\mathcal{A} \models \forall \bar{x}\Big(\bigvee_{i\in[s]}(\varphi_i \wedge Qy\psi_i) \longleftrightarrow Qy \bigvee_{i\in[s]}(\varphi_i \wedge \psi_i)\Big)$.

The following result can be viewed as a first-order normal form for the W*-hierarchy.

Proposition 8. *For $t \geq 2$ the problem p-MC(strict-$\exists^*(Q_{t-1,1})$) is W*$[t]$-complete under fpt-reductions.*

Proof: By Theorem 4(3) it suffices to show that p-MC(strict-$\Sigma^*_{t,1}$) $\leq^{\text{fpt}}$ p-MC(strict-$\exists^*(Q_{t-1,1})$), i.e.

$$p\text{-MC(strict-}\exists^*(\text{BC}(Q_{t-1,1}))) \leq^{\text{fpt}} p\text{-MC(strict-}\exists^*(Q_{t-1,1})).$$

Let $(\mathcal{A}, \varphi)$ be an instance of p-MC(strict-$\exists^*(\text{BC}(Q_{t-1,1}))$). We may assume that

$$\varphi = \exists x_1 \ldots \exists x_k \bigvee_{i \in [s]} \big(\bigwedge_{j \in [\ell]} \psi_{ij} \wedge \bigwedge_{j \in [m]} \psi'_{ij} \big)$$

where $\psi_{ij} = \forall z \chi_{ij}$ for $j \in [\ell]$ and $\psi'_{ij} = \exists y_j \chi'_{ij}$ for $j \in [m]$ and where all χ_{ij} and χ'_{ij} are in BC$(Q_{t-2,1})$. Furthermore we may assume that $z, y_1, \ldots, y_m$ are pairwise distinct. If $|A| < s$ we check whether $\mathcal{A} \models \varphi$ and choose an equivalent instance of p-MC(strict-$\exists^*(Q_{t-1,1})$). Let $|A| \geq s$. We expand $\mathcal{A}$ to a structure $\mathcal{A}^* := (\mathcal{A}, (R_i^{\mathcal{A}})_{i \in [s]})$, where $R_1, \ldots, R_s$ are new unary relation symbols and $(R_i^{\mathcal{A}})_{i \in [s]}$ is a partition of A into nonempty sets. By Lemma 6

$$\mathcal{A}^* \models \Big(\varphi \longleftrightarrow \exists x_1 \ldots \exists x_k \exists u \bigwedge_{i \in [s]} \Big(R_i u \rightarrow \big(\bigwedge_{j \in [\ell]} \psi_{ij} \wedge \bigwedge_{j \in [m]} \psi'_{ij} \big) \Big) \Big)$$

and hence

$$\mathcal{A}^* \models \Big(\varphi \longleftrightarrow \exists x_1 \ldots \exists x_k \exists u \Big(\bigwedge_{i \in [s]} \big(R_i u \rightarrow \exists y_1 \ldots \exists y_m \forall z \big(\bigwedge_{j \in [\ell]} \chi_{ij} \wedge \bigwedge_{j \in [m]} \chi'_{ij} \big) \big) \Big)$$

and thus by Lemma 7

$$\mathcal{A}^* \models \Big(\varphi \longleftrightarrow \exists x_1 \ldots \exists x_k \exists u \exists y_1 \ldots \exists y_m \forall z \underbrace{\Big(\bigwedge_{i \in [s]} \big(R_i u \rightarrow \big(\bigwedge_{j \in [\ell]} \chi_{ij} \wedge \bigwedge_{j \in [m]} \chi'_{ij} \big) \big) \Big)}_{\rho} \Big).$$

Since ρ is a boolean combination of formulas in BC$(Q_{t-2,1})$, it is itself in BC$(Q_{t-2,1})$. Hence the formula on the right hand side, that is $\exists \bar{x} \exists u \exists \bar{y} \forall z \rho$, is in $\exists^*(Q_{t-1,1})$. Note that in general the preceding formula is not strict, as an atomic subformula may contain a variable from $\bar{x}$ and a variable from $\bar{y}$. So, it remains to show that one can pass to a formula in strict-$\exists^*(Q_{t-1,1})$. This can be done as in [10] for the class $\Sigma_{t,1}$:

Let $\bar{v} = v_1 \ldots v_r$ consist of all variables occurring in ρ and distinct from u and the variables in $\bar{x}$ and $\bar{y}$. We may assume that $r \leq t-1$, because there are at most $t-1$ nested quantifiers in $\forall z \rho$. We define a structure $\mathcal{A}'$ and a strict-$\exists^*(Q_{t-1,1})$-sentence φ' such that $(\mathcal{A}^* \models \varphi \iff \mathcal{A}' \models \varphi')$. The universe of $\mathcal{A}'$ is $A' := A \cup A^2$. In addition to the relations of $\mathcal{A}^*$, which keep their interpretations in $\mathcal{A}'$, the structure $\mathcal{A}'$ contains the interpretations of new relation symbols E (unary), T_1, T_2 (binary), and for every atomic subformula $\lambda(x_i, y_j, \bar{v})$ of ρ containing the variables x_i and y_j (that is, two variables from the leading existential block) the interpretation of a new $(r+1)$-ary relation symbol S_λ:

$$\begin{aligned}
E^{\mathcal{A}'} &:= A; \\
T_1^{\mathcal{A}'} &:= \{(a,b) \mid a \in A,\ b \in A^2,\ \text{and there is an } a' \in A \text{ with } b = (a, a')\}; \\
T_2^{\mathcal{A}'} &:= \{(a,b) \mid a \in A,\ b \in A^2,\ \text{and there is an } a' \in A \text{ with } b = (a', a)\}; \\
S_\lambda^{\mathcal{A}'} &:= \big\{(b, a_1, \ldots, a_r) \bigm| b \in A^2,\ a_1, \ldots, a_r \in A,\ \text{and } \mathcal{A} \models \lambda(b, a_1, \ldots, a_r)\big\}.
\end{aligned}$$

Recall that $\bar{x} = x_1 \ldots x_k$ and $\bar{y} = y_1 \ldots y_m$. For every $i \in [k]$ and $j \in [m]$ we let $w_{i,j}$ be a new variable and we denote by $\bar{w}$ the sequence of these $w_{i,j}$. We set

$$\mu(\bar{x}, \bar{y}, \bar{w}, z) := \bigwedge_{i \in [k]} \bigwedge_{j \in [m]} \Big((T_1 x_i z \wedge T_2 y_j z) \rightarrow z = w_{i,j} \Big).$$

Then we can set

$$\varphi' := \exists \bar{x} \exists u \exists \bar{y} \exists \bar{w} \forall z \Big(Eu \wedge \bigwedge_{i \in [k],\ j \in [m]} (E x_i \wedge E y_j \wedge \neg E w_{i,j}) \wedge \mu \wedge (Ez \rightarrow \rho') \Big),$$

where ρ' is the formula obtained from ρ by replacing each atomic subformula $\lambda(x_i, y_j, \bar{v})$ by $S_\lambda w_{i,j}\bar{v}$ and by relativizing each quantifier to E. $\square$

The next lemma contains the main step in a proof of Theorem 5.

Lemma 9. *Let $m \geq 0$ and $u \geq 1$. There are a computable function h and an fpt-algorithm that assigns to every $(\mathcal{A}, \varphi(\bar{x}))$ with $\varphi \in \mathrm{BC}(Q_{m,u})$ and $|A| \geq h(|\varphi|)$ a pair $(\mathcal{A}^*, \varphi^*(\bar{x}))$ such that*

- *$\mathcal{A}^*$ is an expansion of $\mathcal{A}$;*
- *$\varphi^* \in \Pi_{2m+1}$, the length of each quantifier block in φ^* is bounded by $u+1$, and the first quantifier block consists of a single $\forall$-quantifier;*
- *$\mathcal{A}^* \models \forall\bar{x}\big(\varphi(\bar{x}) \leftrightarrow \varphi^*(\bar{x})\big)$.*

Proof: We proceed by induction on m. The claim is trivial for $m = 0$. Let $m \geq 1$ and $\varphi(\bar{x}) \in \mathrm{BC}(Q_{m,u})$. We may assume that

$$\varphi(\bar{x}) = \bigwedge_{i \in I} \varphi_i(\bar{x}), \tag{1}$$

where for each $i \in I$

$$\varphi_i(\bar{x}) = \bigvee_{j \in I} \varphi_{ij}$$

and the φ_{ij} are of the form

$$\exists y_1 \ldots \exists y_u\ \chi_{ij} \qquad \text{or} \qquad \forall z_1 \ldots \forall z_u\ \chi_{ij}$$

for some $\chi_{ij} \in \mathrm{BC}(Q_{m-1,u})$. Here, but also in the formulas to which the induction hypothesis will be applied, we always assume that the index set is a fixed set I. This can be achieved by repeating conjuncts or disjuncts if necessary. The cardinality of I depends only on $|\varphi|$.

Let $\mathcal{A}$ be a structure with $|A| \geq |I|$. We expand $\mathcal{A}$ to a structure $\mathcal{A}^* := (\mathcal{A}, (R_i^{\mathcal{A}})_{i \in I})$, where the R_i are new unary relation symbols and $(R_i^{\mathcal{A}})_{i \in I}$ is a partition of A into nonempty sets.

Fix $i \in I$. By Lemma 6 we get

$$\mathcal{A}^* \models \forall\bar{x}\Big(\varphi_i(\bar{x}) \longleftrightarrow \exists y \bigwedge_{j \in I}(R_j y \to \varphi_{ij})\Big)$$

and by repeatedly applying Lemma 7

$$\mathcal{A}^* \models \forall\bar{x}\Big(\varphi_i(\bar{x}) \longleftrightarrow \exists y \exists y_1 \ldots \exists y_u \forall z_1 \ldots \forall z_u \bigwedge_{j \in I}(R_j y \to \chi_{ij})\Big).$$

Since $\bigwedge_{j \in I}\big(R_j y \to \chi_{ij}\big)$ is a boolean combination of formulas in $\mathrm{BC}(Q_{m-1,u})$, it is itself in $\mathrm{BC}(Q_{m-1,u})$. By induction hypothesis, we obtain a formula $\chi_i \in \Pi_{2(m-1)+1}$ with all quantifier blocks of length $\leq u+1$ and with leading block consisting of a single universal quantifier such that

$$\mathcal{A}^* \models \forall\bar{x}\Big(\varphi_i(\bar{x}) \longleftrightarrow \exists y \exists y_1 \ldots \exists y_u \forall z_1 \ldots \forall z_u \chi_i\Big).$$

Again applying Lemma 6 we get, by (1),

$$\mathcal{A}^* \models \forall\bar{x}\Big(\varphi(\bar{x}) \longleftrightarrow \forall z \bigvee_{i \in I}(R_i z \wedge \exists y \exists y_1 \ldots \exists y_u \forall z_1 \ldots \forall z_u \chi_i)\Big),$$

which in view of Lemma 7 yields

$$\mathcal{A}^* \models \forall\bar{x}\Big(\varphi(\bar{x}) \longleftrightarrow \forall z \exists y \exists y_1 \ldots \exists y_u \forall z_1 \ldots \forall z_u \bigvee_{i \in I}(R_i z \wedge \chi_i)\Big).$$

Applying Lemma 7 to the quantifier blocks of the χ_is one obtains a formula $\varphi^*(\bar{x})$ of the desired form such that

$$\mathcal{A}^* \models \forall\bar{x}(\varphi(\bar{x}) \leftrightarrow \varphi^*(\bar{x})).$$ $\square$

As an easy consequence of the preceding lemma we get:

Lemma 10. *For all $t \geq 2$ and $u \geq 1$*

$$p\text{-MC}(\exists^*(Q_{t-1,u})) \leq^{\text{fpt}} p\text{-MC}(\Sigma_{2t-2,u+1}).$$

Proof: Fix $t \geq 2$ and $u \geq 1$. Choose h according to Lemma 6. Let $(\mathcal{A}, \psi)$ be an instance of p-MC$(\exists^*(Q_{t-1,u}))$ say,

$$\psi = \exists x_1 \ldots \exists x_k \forall x_{k+1} \ldots \forall x_{k+u}\, \varphi(\bar{x}),$$

with $\varphi(\bar{x}) \in \text{BC}(Q_{t-2,u})$. If $|A| < h(|\varphi|)$ we check whether $\mathcal{A} \models \psi$ and choose an equivalent instance of p-MC$(\Sigma_{2t-2,u+1})$. Now assume $|A| \geq h(|\varphi|)$. We apply Lemma 9 to $(\mathcal{A}, \varphi(\bar{x}))$ in order to obtain $(\mathcal{A}^*, \varphi^*(\bar{x}))$, where $\mathcal{A}^*$ is an expansion of $\mathcal{A}$, the formula φ^* is in $\Pi_{2(t-2)+1}$,

$$\mathcal{A}^* \models \forall \bar{x}\big(\varphi(\bar{x}) \leftrightarrow \varphi^*(\bar{x})\big),$$

the length of each quantifier block in φ^* is bounded by $u+1$, and the first quantifier block consists of a single $\forall$-quantifier. Hence

$$\mathcal{A} \models \psi \quad \text{iff} \quad \mathcal{A}^* \models \exists x_1 \ldots \exists x_k \forall x_{k+1} \ldots \forall x_{k+u}\, \varphi^*.$$

Since $\exists x_1 \ldots \exists x_k \forall x_{k+1} \ldots \forall x_{k+u}\, \varphi^* \in \Sigma_{2t-2,u+1}$, we get the desired reduction. $\square$

Proof of Theorem 5: By Theorem 4(3) the problem p-MC$(\Sigma^*_{t,1})$ is W$^*[t]$-complete. Therefore, we have

$$p\text{-MC}(\Sigma^*_{t,1}) \leq p\text{-MC}(\exists^*(Q_{t-1,1})) \leq^{\text{fpt}} p\text{-MC}(\Sigma_{2t-2,2})$$

by Proposition 8 and Lemma 10, respectively. In view of Theorem 4(1) this yields W$^*[t] \subseteq$ W$[2t-2]$.

4. A propositional normal form for the W*-hierarchy

It is well-known that the parameterized model-checking problem for formulas of first-order logic can be reduced to the weighted satisfiability problem for propositional formulas. In Proposition 12 of this section we analyze what type of propositional formulas we obtain if we start with instances of the W$^*[t]$-complete problem p-MC$(\exists^*(Q_{t-1,1}))$. Since originally the classes of the W*-hierarchy were defined in terms of the weighted satisfiability problem for classes of propositional formulas, we view Proposition 12 as yielding a propositional normal form for the W*-hierarchy. It is more or less implicit in [4].

Formulas of propositional logic are built up from *propositional variables* $X_1, X_2, \ldots$ by taking conjunctions, disjunctions, and negations. We distinguish between *small conjunctions*, denoted by $\wedge$, which are just conjunctions of two formulas, and *big conjunctions*, denoted by $\bigwedge$, which are conjunctions of arbitrary finite nonempty sets of formulas. Analogously, we distinguish between *small disjunctions*, denoted by $\vee$, and *big disjunctions*, denoted by $\bigvee$.

Let V be a set of propositional variables. We identify each assignment $S : V \to \{\text{TRUE}, \text{FALSE}\}$ with the set $\{X \in V \mid S(X) = \text{TRUE}\} \in 2^V$. The *weight* of an assignment $S \in 2^V$ is the number of variables set to TRUE. A propositional formula α is *k-satisfiable* (where $k \in \mathbb{N}$), if there is an assignment for the set of variables of α of weight k satisfying α.

For a set A of propositional formulas, the *weighted satisfiability problem* p-WSAT(A) *for formulas in* A is the following parameterized problem:

p-WSAT(A)	
Input:	A propositional formula $\alpha \in A$ and $k \in \mathbb{N}$.
Parameter:	k.
Problem:	Decide whether α is k-satisfiable.

Let A be a set and $k \geq 1$. For all $a \in A$ and $i \in [k]$, let $X_{i,a}$ be a propositional variable with $X_{i,a} \neq X_{j,b}$ for $(i,a) \neq (j,b)$. Let $V = V(A,k)$ be the set of all these propositional variables. Let us call an assignment $S \in 2^V$ *functional* if for each i there is exactly one a such that $S(X_{i,a}) = \text{TRUE}$. The proof of the following lemma from [10] is straightforward. Its part (2) will be applied in Lemma 13 to translate literals ψ into propositional formulas.

Lemma 11. *Let $V = V(A,k)$.*

(1) For

$$\chi^+ := \bigwedge_{i\in[k]} \bigvee_{b\in A} X_{i,b} \quad \textit{and} \quad \chi^- := \bigwedge_{\substack{i\in[k]\\ a,b\in A, a\neq b}} (\neg X_{i,a} \vee \neg X_{i,b})$$

and for every assignment $S \subseteq V$ of weight $|S| = k$ we have

$$\begin{aligned} S \textit{ satisfies } \chi^+ \quad &\iff \quad S \textit{ is functional} \\ &\iff \quad S \textit{ satisfies } \chi^-. \end{aligned}$$

Moreover, for every assignment $S \subseteq V$

$$S \textit{ satisfies } (\chi^+ \wedge \chi^-) \quad \iff \quad (|S| = k \textit{ and } S \textit{ is functional}).$$

(2) Let $\mathcal{A}$ be a structure with universe A, $\bar{b} \in A^s$, $i \in [k]$, and $\psi(x_i, \bar{y})$ a formula in the vocabulary of $\mathcal{A}$ with $\bar{y} = y_1 \dots y_s$. For

$$\eta^{\bigvee}(\mathcal{A}, \psi, \bar{b}) := \bigvee_{\substack{a\in A\\ \mathcal{A}\models\psi(a,\bar{b})}} X_{i,a} \quad \textit{and} \quad \eta^{\bigwedge}(\mathcal{A}, \psi, \bar{b}) := \bigwedge_{\substack{a\in A\\ \mathcal{A}\not\models\psi(a,\bar{b})}} \neg X_{i,a}$$

and for every functional assignment $S \subseteq V$ with, say, $S(X_{i,a_0}) = \text{TRUE}$ we have

$$\begin{aligned} S \textit{ satisfies } \eta^{\bigvee}(\mathcal{A}, \psi, \bar{b}) \quad &\iff \quad \mathcal{A} \models \psi(a_0, \bar{b}) \\ &\iff \quad S \textit{ satisfies } \eta^{\bigwedge}(\mathcal{A}, \psi, \bar{b}). \end{aligned}$$

Essentially, W$[t]$ was first defined as the class of parameterized problems reducible to p-WSAT(Γ_t), where Γ_t contains the propositional formulas that are big conjunctions of big disjunctions of big conjunctions ... with t alternations. It was shown by Downey and Fellows that additionally one could allow a constant depth amount of small conjunctions and disjunctions. In the original definition of W$^*[t]$ the depth amount of these small gates was allowed to depend on the parameter k (by the way, in the definition of $\Sigma^*_{t,u}$ these small gates correspond to the boolean combinations allowed in the transition of $Q_{m,u}$ to $Q_{m+1,u}$, see Definition 2). We first strengthen the normal form for the W*-hierarchy of [8] by showing, among others, that we can omit the small gates on the two lowest levels. We introduce the class strict-$\Gamma^*_{t,k}$ of formulas in the corresponding normal form.

We use $\wedge_{i\in[k]}\alpha_i$ as an abbreviation for the formula $(\cdots((\alpha_1 \wedge \alpha_2) \wedge \alpha_3) \cdots \wedge \alpha_k)$. Similarly, we use $\vee_{i\in[k]}\alpha_i$. For $t \geq 2$ and $k \geq 1$ we define the sets strict-$\Gamma^*_{t,k}$ and strict-$\Delta^*_{t,k}$ of formulas of propositional logic by induction on t:

- We let strict-$\Gamma^*_{2,k}$ be the class of formulas of the form

$$\bigwedge_{i\in I} \bigvee_{j\in J} X_{ij}$$

where I and J are arbitrary (finite) nonempty sets and each X_{ij} is a propositional variable. And let strict-$\Delta^*_{2,k}$ be the class of formulas of the form

$$\bigvee_{i\in I} \bigwedge_{j\in J} \neg X_{ij}$$

where I and J are arbitrary (finite) nonempty sets and each X_{ij} is a propositional variable. (Hence strict-$\Gamma^*_{2,k}$ and strict-$\Delta^*_{2,k}$ do not depend on k.)

- For $t \geq 2$, we let strict-$\Gamma^*_{t+1,k}$ be the class of all formulas of the form

$$\bigwedge_{i\in I} \vee_{j\in[k]} \alpha_{ij}$$

where I is an arbitrary (finite) nonempty set and $\alpha_{ij} \in \text{strict-}\Gamma^*_{t,k} \cup \text{strict-}\Delta^*_{t,k}$ for all $i \in I,\ j \in [k]$. Similarly, we let strict-$\Delta^*_{t+1,k}$ be the class of all formulas of the form

$$\bigvee_{i\in I} \wedge_{j\in[k]}\alpha_{ij}$$

where I and the α_{ij} are as above.

The propositional normalization we aim at reads as follows:

Proposition 12. *Let $t \geq 2$ and $u \geq 1$. There is an fpt-algorithm that assigns to every instance $(\mathcal{A}, \varphi)$ of p-MC$(\Sigma^*_{t,u})$ an equivalent instance (α, k) of p-WSAT(strict-$\Gamma^*_{t,k}$).*

To obtain Proposition 12 we first show:

Lemma 13. *Let $t \geq 2$. There is an fpt-algorithm associating with $((\mathcal{A}, \bar{b}), \psi)$, where*

- *$\psi = \psi(\bar{x}, \bar{y})$ is a subformula of a strict-$\Sigma^*_{t,1}[2]$-formula $\exists\bar{x}\psi_0$ and, say, $\bar{x} = x_1 \ldots x_k$, and $\bar{y} = y_1 \ldots y_p$,*
- *$\mathcal{A}$ is a structure of the corresponding vocabulary and $\bar{b} \in A^p$,*

a propositional formulas $\xi(\psi, \mathcal{A}, \bar{b})$ with variables in $V(A, k) := \{X_{i,a} \mid i \in [k] \text{ and } a \in A\}$ such that for $m \geq 1$

(1) if $\psi \in Q_{m,1}$, then $\xi(\psi, \mathcal{A}, \bar{b}) \in$ strict-$\Gamma^*_{m+1,k}$; [1]

(2) if $\psi = \neg\psi'$ with $\psi' \in Q_{m,1}$, then $\xi(\psi, \mathcal{A}, \bar{b}) \in$ strict-$\Delta^*_{m+1,k}$;

(3) for every $a_1, \ldots, a_k \in A$

$$\mathcal{A} \models \psi(\bar{a}, \bar{b}) \quad\Longleftrightarrow\quad \{X_{1,a_1}, \ldots, X_{k,a_k}\} \text{ satisfies } \xi(\psi, \mathcal{A}, \bar{b}).$$

Proof: Let $t \geq 2$. The proof proceeds by induction on m. Let $m = 1$. First assume that $\psi \in Q_{1,1}$, that is,

$$\psi(\bar{x}, \bar{y}) = \forall z\rho$$

with quantifier-free ρ, which we can assume to be in conjunctive normal form; hence (compare the last footnote)

$$\psi(\bar{x}, \bar{y}) = \forall z \bigwedge_{r\in[k]} \bigvee_{s\in[k]} \lambda_{rs}$$

with literals λ_{rs}. By strictness every λ_{rs} can be written as $\lambda_{rs}(x_i, \bar{y}, z)$ with $i \in [k]$. We set

$$\xi(\psi, \mathcal{A}, \bar{b}) := \bigwedge_{\substack{b\in A,\\ r\in[k]}} \bigvee_{s\in[k]} \eta^{\bigvee}(\mathcal{A}, \lambda_{rs}, \bar{b}b).$$

Then, by Lemma 11(2), the equivalence claimed in (3) holds. In $\xi(\psi, \mathcal{A}, \bar{b})$ we merge the disjunction $\bigvee_{s\in[k]}$ with the leading disjunctions in the $\eta^{\bigvee}(\ldots)$s in order to obtain a formula in strict-$\Gamma^*_{2,1}$.

If $\psi = \neg\forall z\rho$ with quantifier-free ρ, then (up to logical equivalence) we can assume that

$$\psi(\bar{x}, \bar{y}) = \exists z \bigvee_{r\in[k]} \bigwedge_{s\in[k]} \lambda_{rs}$$

with literals λ_{rs}. We argue as above, now using the formula $\eta^{\bigwedge}(\ldots)$ in order to obtain $\xi(\psi, \mathcal{A}, \bar{b}) \in$ strict-$\Delta^*_{2,1}$.

[1] To be precise we should remark that we get $\xi(\psi, \mathcal{A}, \bar{b}) \in$ strict-$\Gamma^*_{m+1,\ell}$, where ℓ only depends on $\exists\bar{x}\psi_0$; hence by adding dummy variables to $\bar{x}$ if necessary, we can assume that $\ell = k$.

In the induction step assume that $\psi(\bar{x}, \bar{y}) \in Q_{m+1,1}$ for some $m \geq 1$, say,

$$\psi(\bar{x}, \bar{y}) = \forall z \rho,$$

where ρ is a boolean combination of formulas in $Q_{m,1}$. Then we can assume that

$$\psi(\bar{x}, \bar{y}) = \forall z \bigwedge_{r \in [k]} \bigvee_{s \in [k]} \rho_{rs}$$

with $\rho_{rs} \in Q_{m,1} \cup \{\neg\varphi \mid \varphi \in Q_{m,1}\}$. But then using the induction hypothesis we see that we can set

$$\xi(\psi, \mathcal{A}, \bar{b}) := \bigwedge_{\substack{b \in A, \\ r \in [k]}} \vee_{s \in [k]} \xi(\rho_{rs}, \mathcal{A}, \bar{b}b).$$

The case $\psi(\bar{x}, \bar{y}) = \neg\psi'$ with $\psi' \in Q_{m+1,1}$ is treated similarly. $\square$

Proof of Proposition 12: Let $t \geq 2$ and $u \geq 1$. Let $(\mathcal{A}, \varphi)$ be an instance of p-MC$(\Sigma^*_{t,u})$. By Proposition 8 we can assume that φ is a strict-$\exists^*(Q_{t-1,1})$-formula, say

$$\varphi = \exists x_1 \dots \exists x_k \psi$$

where $\psi(\bar{x}) \in Q_{t-1,1}$. We use the formulas $\xi(\psi, \mathcal{A}, \emptyset)$ of Lemma 13 and χ^+ of Lemma 11(1) and set

$$\alpha := (\chi^+ \wedge \xi(\psi, \mathcal{A}, \emptyset)). \tag{2}$$

One easily verifies that α is (equivalent to) a formula in strict-$\Gamma^*_{t,k}$ and that the instances $(\mathcal{A}, \varphi)$ and (α, k) are equivalent. $\square$

Define α in (2) by $\alpha := (\chi^+ \wedge \chi^- \wedge \xi(\psi, \mathcal{A}, \emptyset))$. If $t \geq 3$, then α is still equivalent to a formula in strict-$\Gamma^*_{t,k}$. Hence, by Lemma 11(1), we obtain the following stronger version of Proposition 12.

Corollary 14. *Let $t \geq 3$ and $u \geq 1$. There is an fpt-algorithm that assigns to every instance $(\mathcal{A}, \varphi)$ of p-MC$(\Sigma^*_{t,u})$ an equivalent instance (α, k) of p-WSAT(strict-$\Gamma^*_{t,k}$) with the property that every satisfying assignment of α has weight k.*

4.1. The parse structure. Before we apply the propositional normal form obtained in Proposition 12 in the next sections, we recall some notions and constructions introduced in [10]. More precisely, for every "strict propositional formula" α we introduce a parse structure and show how then we can express in first-order logic that an assignment satisfies α.

Let τ be the vocabulary $\{E, \mathit{Root}, S_1, \dots, S_k, C_1, \dots, C_k\}$ with binary E and unary $\mathit{Root}, S_1, \dots, S_k$, $C_1, \dots, C_k$ (read Exy as "y is a son of x," $\mathit{Root}\, x$ as "x is the root," $S_i x$ as "x is the ith son of its parent," and $C_i x$ as "the ith son of x is a big conjunction"). Let $\alpha \in$ strict-$\Gamma^*_{t+1,k} \cup$ strict-$\Delta^*_{t+1,k}$ with $t \geq 2$. We first define a τ-structure $\mathcal{A}_0(\alpha)$, the *pre-parse structure of* α. We obtain it from the parse tree of α, where $E^{\mathcal{A}_0(\alpha)}$ is the edge relation with edges directed from the root to the leaves, by fixing the interpretation in $\mathcal{A}_0(\alpha)$ of the remaining symbols of τ by the following clauses: Let u be a node of $\mathcal{A}_0(\alpha)$ and β the subformula of α corresponding to the node u.

- If u is the root of the tree, then $\mathrm{Root}^{\mathcal{A}_0(\alpha)} u$.
- If $\beta = \vee_{i \in [k]} \beta_i$ or $\beta = \wedge_{i \in [k]} \beta_i$, where $\beta_1, \dots, \beta_k \in$ strict-$\Gamma^*_{s,k} \cup$ strict-$\Delta^*_{s,k}$ for some $s \geq 2$, then $S_j^{\mathcal{A}_0(\alpha)} u_j$ for $j \in [k]$ where u_j is the son of u corresponding to β_j.
- If $\beta = \vee_{i \in [k]} \beta_i$ or $\beta = \wedge_{i \in [k]} \beta_i$, where $\beta_1, \dots, \beta_k \in$ strict-$\Gamma^*_{s,k} \cup$ strict-$\Delta^*_{s,k}$ for some $s \geq 2$, and if $j \in [k]$ and $\beta_j \in$ strict-$\Gamma^*_{s,k}$, then $C_j^{\mathcal{A}_0(\alpha)} u$. (Note that we encode the information on whether a subformula in strict-$\Gamma^*_{s,k} \cup$ strict-$\Delta^*_{s,k}$ is in strict-$\Gamma^*_{s,k}$ or in strict-$\Delta^*_{s,k}$ by putting or not putting the parent into the corresponding relation C_j. The reason that we choose such a counter-intuitive encoding is that in the formulas we are going to introduce we need the information about the son at the parent in order to pick the right quantifier to access the son.)

Finally we obtain $\mathcal{A}(\alpha)$, the *parse structure of* α, from $\mathcal{A}_0(\alpha)$ by the following manipulations:

- first we contract the edges between a negative literal and its variable and identify the node obtained thereby with the node of the variable;
- we identify the nodes corresponding to the same propositional variable and identify the node obtained thereby with the variable itself.

We define formulas $\psi^s_{\bigwedge}(y, x_1, \dots, x_k)$ and $\psi^s_{\bigvee}(y, x_1, \dots, x_k)$ for $2 \leq s \leq t+1$ and $\psi^s_{\wedge}(y, x_1, \dots, x_k)$ and $\psi^s_{\vee}(y, x_1, \dots, x_k)$ for $3 \leq s \leq t+1$ such that for every node $u \in T$ corresponding to a subformula β of α and for all variables $X_1, \dots, X_k$ of α we have (recall that we identified a node corresponding to a variable with the variable itself):

(i) If $\beta \in$ strict-$\Gamma^*_{s,k}$, then $\big(\{X_1, \dots, X_k\}$ satisfies $\beta \iff \mathcal{A}(\alpha) \models \psi^s_{\bigwedge}(u, X_1, \dots, X_k)\big)$.

(ii) If $\beta \in$ strict-$\Delta^*_{s,k}$, then $\big(\{X_1, \dots, X_k\}$ satisfies $\beta \iff \mathcal{A}(\alpha) \models \psi^s_{\bigvee}(u, X_1, \dots, X_k)\big)$.

(iii) If $\beta = \wedge_{i=1}^k \beta_i$, where $\beta_1, \dots, \beta_k \in$ strict-$\Gamma^*_{s,k} \cup$ strict-$\Delta^*_{s,k}$, then

$$\{X_1, \dots, X_k\} \text{ satisfies } \beta \iff \mathcal{A}(\alpha) \models \psi^{s+1}_{\wedge}(u, X_1, \dots, X_k).$$

(iv) If $\beta = \vee_{i=1}^k \beta_i$, where $\beta_1, \dots, \beta_k \in$ strict-$\Gamma^*_{s,k} \cup$ strict-$\Delta^*_{s,k}$, then

$$\{X_1, \dots, X_k\} \text{ satisfies } \beta \iff \mathcal{A}(\alpha) \models \psi^{s+1}_{\vee}(u, X_1, \dots, X_k).$$

We let

$$\psi^2_{\bigwedge}(y, x_1, \dots, x_k) := \forall z(Eyz \to (Ezx_1 \vee \dots \vee Ezx_k)),$$
$$\psi^2_{\bigvee}(y, x_1, \dots, x_k) := \exists z(Eyz \wedge (\neg Ezx_1 \wedge \dots \wedge \neg Ezx_k)).$$

For $s \geq 2$, we let

$$\begin{aligned}
\psi^{s+1}_{\bigwedge}(y, x_1, \dots, x_k) &:= \forall z\big(Eyz \to \psi^{s+1}_{\vee}(z, x_1, \dots, x_k)\big),\\
\psi^{s+1}_{\bigvee}(y, x_1, \dots, x_k) &:= \exists z\big(Eyz \wedge \psi^{s+1}_{\wedge}(z, x_1, \dots, x_k)\big),\\
\psi^{s+1}_{\wedge}(y, x_1, \dots, x_k) &:= \bigwedge_{i \in [k]} \Big(\quad \big(C_i y \to \forall z\big((S_i z \wedge Eyz) \to \psi^s_{\bigwedge}(z, x_1, \dots, x_k)\big)\big)\\
&\qquad \wedge \big(\neg C_i y \to \exists z\big(S_i z \wedge Eyz \wedge \psi^s_{\bigvee}(z, x_1, \dots, x_k)\big)\big)\Big),\\
\psi^{s+1}_{\vee}(y, x_1, \dots, x_k) &:= \bigvee_{i \in [k]} \Big(\quad \big(C_i y \wedge \forall z\big((S_i z \wedge Eyz) \to \psi^s_{\bigwedge}(z, x_1, \dots, x_k)\big)\big)\\
&\qquad \vee \big(\neg C_i y \wedge \exists z\big(S_i z \wedge Eyz \wedge \psi^s_{\bigvee}(z, x_1, \dots, x_k)\big)\big)\Big).
\end{aligned}$$

It is easy to see that these formulas satisfy (i) – (iv).

In the following sections we are going to use variants of the parse structure and of these formulas.

5. Fagin-definability

We use the fact that, for fixed t, the structures $\mathcal{A}(\alpha)$ for $\alpha \in$ strict-$\Gamma^*_{t,k}$ all have the same "skeleton" to obtain a characterization of $\mathrm{W}^*[t]$ in terms of Fagin-definable problems.

Let Z be a fixed set variable (that is, unary relation variable). We consider first-order formulas that may contain atomic subformulas of the form Zx. We abbreviate $\exists x(Zx \wedge \psi)$ and $\forall x(Zx \to \psi)$ by $(\exists x \in Z)\psi$ and $(\forall x \in Z)\psi$, respectively. Recall that a first-order formula $\varphi = \varphi(Z)$ defines the problem:

p-WD$_\varphi$		
	Input:	A structure $\mathcal{A}$ and $k \in \mathbb{N}$.
	Parameter:	k.
	Problem:	Decide whether there is a subset $S \subseteq A$ of cardinality $\lvert S\rvert = k$ such that $\mathcal{A} \models \varphi(S)$.

We say that $\varphi(Z)$ *Fagin-defines* p-WD$_\varphi$. For a class Φ of first-order formulas, we let p-WD-Φ be the class of all parameterized problems p-WD$_\varphi$, where $\varphi(Z) \in \Phi$.

If Φ is a class of formulas of first-order logic, let

– BB(Φ) be the smallest set of formulas containing the formulas in Φ and closed under boolean combinations and bounded quantification (that is, $(\exists x \in Z)$ and $(\forall x \in Z)$).

The following results are from [6] and [9], respectively:

Theorem 15. *Let* $t \geq 1$. *Then:*

(1) $\mathrm{W}[t] = [p\text{-WD-}\Pi_t]^{\mathrm{fpt}}$.

(2) $\mathrm{W}[t] = [p\text{-WD-}\Pi^b_{t-1}]^{\mathrm{fpt}}$. *Here* Π^b_{t-1} *denotes the set of formulas of the form* $\forall \bar{y}_1 \exists \bar{y}_2 \forall \bar{y}_3 \dots Q\bar{y}_{t-1}\psi$ *with* $\psi \in \mathrm{BB}(\textsc{Atom})$.

Definition 16. We define the sets B_n of first-order formulas by induction on n:

– $B_0 := \mathrm{BB}(\textsc{Atom})$;

– $B_{n+1} := \forall^*(\mathrm{BB}(B_n))$.

Note that $\Pi^b_0 = B_0$ and $\Pi^b_1 = B_1$. In view of the equalities $\mathrm{W}[1] = \mathrm{W}^*[1]$ and $\mathrm{W}[2] = \mathrm{W}^*[2]$ we see that the characterization of the W*-hierarchy in terms of Fagin-definable problems given by the following theorem extends the characterization of $\mathrm{W}^*[1]$ and $\mathrm{W}^*[2]$ contained in Theorem 15 in a natural way.

Theorem 17. *For* $t \geq 1$,

$$\mathrm{W}^*[t] = [p\text{-WD-}B_{t-1}]^{\mathrm{fpt}}.$$

Proof: Fix $t \geq 1$. We first prove p-WD-$B_{t-1} \subseteq \mathrm{W}^*[t]$. Let $\varphi(Z) \in B_{t-1}$. For every $k \geq 1$, choose new variables $x_1, \dots, x_k$ and let φ_k be the sentence

$$\exists x_1 \dots \exists x_k (\bigwedge_{1 \leq i < j \leq k} x_i \neq x_j \wedge \varphi'_k),$$

where φ'_k is obtained from $\varphi(Z)$ by inductively replacing

– atoms Zy by $\bigvee_{i \in [k]} y = x_i$

– every quantifier $(\forall y \in Z)\chi(y, \dots)$ by $\bigwedge_{i \in [k]} \chi(x_i, \dots)$

– every quantifier $(\exists y \in Z)\chi(y, \dots)$ by $\bigvee_{i \in [k]} \chi(x_i, \dots)$.

Then for every structure $\mathcal{A}$, there is a subset $S \subseteq A$ with $\lvert S\rvert = k$ such that $\mathcal{A} \models \varphi(S)$ if and only if $\mathcal{A} \models \varphi_k$. Moreover a simple induction on t shows that $\varphi'_k \in Q_{t-1,u}$ for some u and hence $\varphi_k \in \Sigma^*_{t,u}$. Thus the mapping $(\mathcal{A}, k) \mapsto (\mathcal{A}, \varphi_k)$ is an fpt-reduction from p-WD$_\varphi$ to p-MC($\Sigma^*_{t,u}$).

We turn to the converse inclusion $\mathrm{W}^*[t] \subseteq [p\text{-WD-}B_{t-1}]^{\mathrm{fpt}}$. Since $\mathrm{W}[1] = \mathrm{W}^*[1]$ and $\mathrm{W}[2] = \mathrm{W}^*[2]$ and $\Pi^b_0 = B_0$ and $\Pi^b_1 = B_1$, we know that this inclusion holds for $t = 1, 2$ by Theorem 15(2). So assume $t \geq 3$. We will make use of the formulas $\psi^s_{\bigwedge}(y, x_1, \dots, x_k), \psi^s_{\bigvee}(y, x_1, \dots, x_k), \dots$ introduced in Section 4.1. We shall replace the variables $x_1, \dots, x_k$ by the set variable Z (representing the set $\{x_1, \dots, x_k\}$). But we also have to get rid of the conjunctions and disjunctions in these formulas ranging over $[k]$, as the formula we are looking for must be independent of k. Since in its relevant interpretations Z will have k elements, we replace the conjunctions and disjunctions by appropriate bounded quantifications.

We show p-MC$(\Sigma^*_{t,1}) \in [p\text{-WD-}B_{t-1}]^{\text{fpt}}$. We make use of Proposition 12 and start with an instance (α, k) of p-WSAT(strict-$\Gamma^*_{t,k}$).

Denote by Var(α) the set of propositional variables of α. Let $\mathcal{A} = \mathcal{A}(\alpha)$ be the parse structure of α introduced in Section 4.1. We obtain the set $A(\alpha, k)$ from A by replacing every variable in Var(α) by k copies of it; more precisely, we let

$$A(\alpha, k) := (A \setminus \text{Var}(\alpha)) \cup (\text{Var}(\alpha) \times [k]).$$

We define the $\tau' := \{E, \mathit{Root}, S, C, \mathit{Fir}, \mathit{Sec}\}$-structure $\mathcal{A}(\alpha, k)$ with universe $A(\alpha, k)$ as follows:

$$\begin{aligned} E^{\mathcal{A}(\alpha,k)} &:= \{(u, v) \mid u, v \in A \setminus \text{Var}(\alpha), \text{ and } E^{\mathcal{A}}uv\} \cup \\ & \{(u, (X, i)) \mid u \in A \setminus \text{Var}(\alpha),\ (X, i) \in \text{Var}(\alpha) \times [k], \text{ and } E^{\mathcal{A}}uX\}, \\ \mathit{Root}^{\mathcal{A}(\alpha,k)} &:= \{u \mid \mathit{Root}^{\mathcal{A}}\, u\}. \end{aligned}$$

The binary relations $S^{\mathcal{A}(\alpha,k)}$ and $C^{\mathcal{A}(\alpha,k)}$ will encode the information contained in the $S_i^{\mathcal{A}}$s and the $C_i^{\mathcal{A}}$s, respectively:

$$\begin{aligned} S^{\mathcal{A}(\alpha,k)} &:= \{((X, i), u) \mid (X, i) \in \text{Var}(\alpha) \times [k],\ u \in A \setminus \text{Var}(\alpha), \text{ and } S_i^{\mathcal{A}}u\}, \\ C^{\mathcal{A}(\alpha,k)} &:= \{((X, i), u) \mid (X, i) \in \text{Var}(\alpha) \times [k],\ u \in A \setminus \text{Var}(\alpha), \text{ and } C_i^{\mathcal{A}}u\}. \end{aligned}$$

The remaining unary relations $\mathit{Fir}^{\mathcal{A}(\alpha,k)}$ and $\mathit{Sec}^{\mathcal{A}(\alpha,k)}$ will allow us to access the components of elements of Var$(\alpha) \times [k]$:

$$\begin{aligned} \mathit{Fir}^{\mathcal{A}(\alpha,k)} &:= \{((X, i), (Y, j)) \mid (X, i), (Y, j) \in \text{Var}(\alpha) \times [k] \text{ and } X \neq Y\}, \\ \mathit{Sec}^{\mathcal{A}(\alpha,k)} &:= \{((X, i), (Y, j)) \mid (X, i), (Y, j) \in \text{Var}(\alpha) \times [k] \text{ and } i \neq j\} \end{aligned}$$

We adapt the formulas $\psi^s_{\bigwedge}(y, x_1, \ldots, x_k), \psi^s_{\bigvee}(y, x_1, \ldots, x_k), \ldots$ to the new framework thereby obtaining formulas $\rho^s_{\bigwedge}(y, Z), \rho^s_{\bigvee}(y, Z), \ldots$ such that for every node $u \in A(\alpha, k)$ corresponding to a subformula β and for all $X_1, \ldots, X_k \in \text{Var}(\alpha)$ we have for $s \geq 2$:

(i) If $\beta \in$ strict-$\Gamma^*_{s,k}$, then $\big(\{X_1, \ldots, X_k\}$ satisfies $\beta \iff \mathcal{A}(\alpha, k) \models \rho^s_{\bigwedge}(u, \{(X_1, 1), \ldots, (X_k, k)\})\big)$.

(ii) If $\beta \in$ strict-$\Delta^*_{s,k}$, then $\big(\{X_1, \ldots, X_k\}$ satisfies $\beta \iff \mathcal{A}(\alpha, k) \models \rho^s_{\bigvee}(u, \{(X_1, 1), \ldots, (X_k, k)\})\big)$.

(iii) If $\beta = \wedge_{i \in [k]} \beta_i$, where $\beta_1, \ldots, \beta_k \in$ strict-$\Gamma^*_{s,k} \cup$ strict-$\Delta^*_{s,k}$, then

$$\{X_1, \ldots, X_k\} \text{ satisfies } \beta \iff \mathcal{A}(\alpha, k) \models \rho^{s+1}_{\wedge}(u, \{(X_1, 1), \ldots, (X_k, k)\}).$$

(iv) If $\beta = \vee_{i \in [k]} \beta_i$, where $\beta_1, \ldots, \beta_k \in$ strict-$\Gamma^*_{s,k} \cup$ strict-$\Delta^*_{s,k}$, then

$$\{X_1, \ldots, X_k\} \text{ satisfies } \beta \iff \mathcal{A}(\alpha, k) \models \rho^{s+1}_{\vee}(u, \{(X_1, 1), \ldots, (X_k, k)\}).$$

We let

$$\begin{aligned} \rho^2_{\bigwedge}(y, Z) &:= \forall z\big(Eyz \to (\exists x \in Z) Ezx\big), \\ \rho^2_{\bigvee}(y, Z) &:= \exists z\big(Eyz \wedge (\forall x \in Z) \neg Ezx\big). \end{aligned}$$

For $s \geq 2$, we let

$$\begin{aligned} \rho^{s+1}_{\bigwedge}(y, Z) &:= \forall z\big(Eyz \to \rho^{s+1}_{\vee}(z, Z)\big), \\ \rho^{s+1}_{\bigvee}(y, Z) &:= \exists z\big(Eyz \wedge \rho^{s+1}_{\wedge}(z, Z)\big), \\ \rho^{s+1}_{\wedge}(y, Z) &:= (\forall u \in Z)\Big(\quad \big(Cuy \to \forall z\big((Suz \wedge Eyz) \to \rho^s_{\bigwedge}(z, Z)\big)\big) \\ &\qquad\qquad\qquad \wedge \big(\neg Cuy \to \exists z\big(Suz \wedge Eyz \wedge \rho^s_{\bigvee}(z, Z)\big)\big)\Big), \\ \rho^{s+1}_{\vee}(y, Z) &:= (\exists u \in Z)\Big(\quad \big(Cuy \wedge \forall z\big((Suz \wedge Eyz) \to \rho^s_{\bigwedge}(z, Z)\big)\big) \\ &\qquad\qquad\qquad \vee \big(\neg Cuy \wedge \exists z\big(Suz \wedge Eyz \wedge \rho^s_{\bigvee}(z, Z)\big)\big)\Big). \end{aligned}$$

Merging in $\rho_\wedge^{s+1}$ and $\rho_\vee^{s+1}$ the quantifiers $\forall z$ and $\exists z$ with the leading $\forall$-quantifier of $\rho_{\bigwedge}^s$ and the leading $\exists$-quantifier of $\rho_{\bigvee}^s$, respectively, we see by a simple induction that for $s \geq 2$ the formulas $\rho_{\bigwedge}^s$, $\rho_{\bigvee}^s$ and $\rho_\wedge^{s+1}$, $\rho_\vee^{s+1}$ are (equivalent to) formulas in B_{s-1} and $\mathrm{BB}(B_{s-1})$, respectively.

Now

$$(\alpha, k) \in p\text{-WS}\textsc{at}(\text{strict-}\Gamma_{t,k}^*) \iff (\mathcal{A}(\alpha, k), k) \in p\text{-WD}_{\varphi_t},$$

where $\varphi_t(Z)$ is a formula in B_{t-1} equivalent to the following formula:

$$(\forall x \in Z)\big(\forall y \in Z)(y \neq x \to (\mathit{Fir}\, xy \wedge \mathit{Sec}\, xy)\big) \wedge \forall y(\mathit{Root}\, y \to \rho_{\bigwedge}^t(y, Z)). \qquad \square$$

6. Model-checking on graphs

The following result generalizes the second part of Theorem 4(2) to the W*-hierarchy. Based on [10], a weaker result is shown in [12]: the parameterized model-checking for $\Sigma_{t,1}^*$ on the class of colored graphs is $\mathrm{W}^*[t]$-complete.

Theorem 18. *p-MC(GRAPH, $\exists^*(Q_{t,1})$) and hence, p-MC(GRAPH, $\Sigma_{t,1}^*$), are $\mathrm{W}^*[t]$-complete under fpt-reductions.*

Proof: Again using Proposition 12 it suffices to show that there is an fpt-reduction associating with (α, k) where $\alpha \in$ strict-$\Gamma_{t+1,k}^*$ an equivalent instance $(\mathcal{G}, \varphi)$ of p-MC(GRAPH, $\Sigma_{t,1}^*$).

So let $\alpha \in$ strict-$\Gamma_{t+1,k}^*$. We consider the parse structure $\mathcal{A}(\alpha)$ (see Section 4.1) and show how we can pass to a graph $\mathcal{G}$: We start by letting $\mathcal{G}$ be the undirected $\{E\}$-graph underlying $\mathcal{A}(\alpha)$ but we are going to contract edges, and to add vertices and edges.

For every vertex u of $\mathcal{G}$ corresponding to a formula $\beta = \wedge_{i\in[k]}\beta_i$ or $\beta = \vee_{i\in[k]}\beta_i$, we contract, for all $i \in [k]$, the edge between u and u_i, where u_i is the vertex corresponding to β_i; we identify the vertex obtained thereby with u (so we say that it is the vertex corresponding to β and the vertices u_i are not present in $\mathcal{G}$).

We add further vertices $s_1, \ldots, s_k, c_1, \ldots, c_k$ to $\mathcal{G}$ in order to have available the information given in $\mathcal{A}(\alpha)$ by the relations $S_i^{\mathcal{A}(\alpha)}$ and $C_i^{\mathcal{A}(\alpha)}$. Moreover:

- If $C_j^{\mathcal{A}(\alpha)}u$, then ($u$ corresponds to a formula $\beta = \wedge_{i\in[k]}\beta_i$ or $\beta = \vee_{i\in[k]}\beta_i$, and thus is still present in $\mathcal{G}$) we add an (undirected) edge from c_j to u in $\mathcal{G}$.

- If $S_j^{\mathcal{A}(\alpha)}u$, then in the parse structure $\mathcal{A}(\alpha)$ the father u_0 of u corresponds to a formula $\beta = \wedge_{i\in[k]}\beta_i$ or $\beta = \vee_{i\in[k]}\beta_i$ and u corresponds to β_j. In particular, u is not present in $\mathcal{G}$, but all sons w of u in $\mathcal{A}(\alpha)$ are present in $\mathcal{G}$; we add an (undirected) edge from each such w to s_j in $\mathcal{G}$ (so this edge keeps the information that w is a grandson of u_0 via the jth son).

Furthermore we have to add vertices and edges to be able to identify the root (without the relation symbol *Root*) and the distinguished vertices $s_1, \ldots, s_k, c_1, \ldots, c_k$. Note that so far $\mathcal{G}$ does not contain cliques of size 4.

To identify the root we "link it with a clique of size 4" that is we add a clique of size 4 and an edge between the root and exactly one vertex of the clique. Let

$$\mathit{root}(x, y_1, y_2, y_3, y_4) := Exy_1 \wedge \text{“}y_1, y_2, y_3, y_4 \text{ is a clique.”}$$

Similarly we link c_i with $(1+i)$-many cliques of size 4 and s_i with $(1+k+i)$-many cliques of size 4 and set

$$\mathit{ci}(x, \bar{y}_1, \ldots, \bar{y}_{1+i}) := Exy_{11} \wedge \ldots \wedge Exy_{1+i1} \wedge \text{“}\bar{y}_1, \ldots, \bar{y}_{1+i} \text{ are cliques”}$$

(where $\bar{y}_j = y_{j1}y_{j2}y_{j3}y_{j4}$) and we define $\mathit{si}(x, \bar{y}_1, \ldots, \bar{y}_{1+k+i})$ analogously. Finally, we add a vertex v, link it with $(1+2k+1)$-many cliques, and add edges from v to all vertices representing variables. Define $\mathit{variable}(x, \bar{y}_1, \ldots, \bar{y}_{1+2k+1})$ in the obvious way. Let $\mathcal{G}$ be the graph obtained in this way.

The sentence φ we aim at will be obtained from the following formula by adding a conjunct expressing that all the variables displayed in the formula are pairwise distinct and by then existentially quantifying all these variables (for easier reading we use the letters $r, c_1, \ldots, c_, s_1, \ldots, s_k$ also as variables)

$$\begin{aligned} & \mathit{root}(r, \bar{y}) \wedge \bigwedge_{i \in [k]} \Big(\mathit{ci}(c_i, \bar{y}_{i1}, \ldots, \bar{y}_{i1+i}) \wedge \mathit{si}(s_i, \bar{w}_{i1}, \ldots, \bar{w}_{i1+k+i}) \Big) \wedge \\ & \quad \mathit{variable}(v, \bar{z}_1, \ldots, \bar{z}_{1+2k+1}) \wedge \bigwedge_{i \in [k]} Evx_i \wedge \rho \end{aligned}$$

with

$$\rho := \forall z (Erz \rightarrow \chi^t_\vee(r, z, \bar{x}))$$

and where the formulas $\chi^s_\wedge, \chi^s_\vee$ are obtained from the formulas $\psi^s_\wedge, \psi^s_\vee$ of Section 4.1 by taking into account on the syntactic side the changes that led us from $\mathcal{A}(\alpha)$ to $\mathcal{G}$. In particular, since the edges are undirected we have to be sure that the quantifications only range over the sons of a vertex y; therefore we have an additional variable w representing the corresponding parent. We set (with $\bar{x} = x_1 \ldots x_k$)

$$\begin{aligned} \chi^3_\wedge(w, y, \bar{x}) := \bigwedge_{i \in [k]} \Big(& \quad \big(Ec_i y \rightarrow \forall z \big((Eyz \wedge z \neq w \wedge Es_i z) \rightarrow (Ezx_1 \vee \ldots \vee Ezx_k)\big) \\ & \wedge \big(\neg Ec_i y \rightarrow \exists z \big(Eyz \wedge z \neq w \wedge Es_i z \wedge (\neg Ezx_1 \wedge \ldots \wedge \neg Ezx_k)\big)\big)\Big), \\ \chi^3_\vee(w, y, \bar{x}) := \bigvee_{i \in [k]} \Big(& \quad \big(Ec_i y \rightarrow \forall z \big((Eyz \wedge z \neq w \wedge Es_i z) \rightarrow (Ezx_1 \vee \ldots \vee Ezx_k)\big) \\ & \wedge \big(\neg Ec_i y \rightarrow \exists z \big(Eyz \wedge z \neq w \wedge Es_i z \wedge (\neg Ezx_1 \wedge \ldots \wedge \neg Ezx_k)\big)\big)\Big), \end{aligned}$$

For $3 < s \leq t$, we let

$$\begin{aligned} \chi^{s+1}_\wedge(w, y, \bar{x}) := \bigwedge_{i \in [k]} \Big(& \quad \big(Ec_i y \rightarrow \forall z \big((Eyz \wedge z \neq w \wedge Es_i z) \rightarrow \chi^s_\vee(y, z, \bar{x})\big)\big) \\ & \wedge \big(\neg Ec_i y \rightarrow \exists z \big(Eyz \wedge z \neq w \wedge Es_i z \wedge \chi^s_\wedge(y, z, \bar{x})\big)\big)\Big), \\ \chi^{s+1}_\vee(w, y, \bar{x}) := \bigvee_{i \in [k]} \Big(& \quad \big(Ec_i y \wedge \forall z \big((Eyz \wedge z \neq w \wedge Es_i z) \rightarrow \chi^s_\vee(y, z, \bar{x})\big)\big) \\ & \vee \big(\neg Ec_i y \wedge \exists z \big(Eyz \wedge z \neq w \wedge Es_i z \wedge \chi^s_\wedge(z, x_1, \ldots, x_k)\big)\big)\Big). \end{aligned}$$

By induction on s one easily verifies that $\chi^s_\wedge, \chi^s_\vee \in \mathrm{BC}(Q_{s-2,1})$. Therefore, φ is (equivalent to a) $\exists^*(Q_{t-1,1})$-formula. Moreover

$$\alpha \text{ is } k\text{-satisfiable} \iff \mathcal{G} \models \varphi$$

which gives the desired reduction. □

7. A W*[3]-complete problem for hypergraphs

In [1] and [11] it is shown that the problems p-MAXIMAL-IRREDUNDANT-SET and p-MAXIMAL-SHATTERED-SET, respectively, are in W*[3]. It is not known whether one of them lies in W[3]. Since both problems are also contained in A[2], we do not believe that they are W*[3]-complete. In this section we present a W*[3]-complete problem.

Recall that a hypergraph $\mathcal{H} = (V, E)$ consists of a (finite) set V, the set of *vertices* and a set E of *hyperedges* (or *edges*). Each hyperedge is a subset of V. Hence, graphs are hypergraphs with all hyperedges of size 2. If all edges of $\mathcal{H}$ have size 3, then we say that $\mathcal{H}$ is a 3-*hypergraph*. If $\mathcal{H} = (V, E)$ is 3-hypergraph, every $a \in V$ induces a graph $\mathcal{H}^a = (V^a, E^a)$ given by

$$V^a := \big\{ v \in V \mid v \neq a \text{ and there is } e \in E \text{ with } a, v \in e \big\} \quad \text{and} \quad E^a := \big\{ \{u, v\} \mid \{a, u, v\} \in E \big\}.$$

The proof of the main result of this section will show implicitly the following:

Theorem 19. *The problem*

p-HYPERGRAPH-(NON)-DOMINATING-SET

Input: A 3-hypergraph $\mathcal{H} = (V, E)$, a set $M \subseteq V$, and $k \geq 1$.

Parameter: k.

Problem: Decide whether there exists a set $D \subseteq V$ of cardinality k such that

- if $a \in M$, then D is a dominating set[2] of $\mathcal{H}^a$,
- if $a \notin M$, then D is not a dominating set of $\mathcal{H}^a$.

is W[3]*-complete under fpt-reductions.*

We turn to a generalization of p-HYPERGRAPH-(NON)-DOMINATING-SET that is W*[3]-complete. A *colored hypergraph* $\mathcal{H}$ is a tuple $(V, E_1, \dots, E_k)$ for some $k \geq 1$, where each $\mathcal{H}_i := (V, E_i)$ is a hypergraph. Intuitively, we view E_i as the hyperedges of color i of the hypergraph $(V, \bigcup_{i \in [k]} E_i)$.

Let $\mathcal{H} = (V, E_1, \dots, E_k)$ be a colored hypergraph. If all $\mathcal{H}_i$ are 3-hypergraphs (or equivalently, if $(V, \bigcup_{i \in [k]} E_i)$ is a 3-hypergraph), then we say that $\mathcal{H}$ is a colored 3-hypergraph. For $i \in [k]$ and $a \in V$ we denote $(\mathcal{H}_i)^a$, $(V_i)^a$, and $(E_i)^a$ by $\mathcal{H}_i^a$, V_i^a, and E_i^a, respectively.

Theorem 20. *The problem*

p-COLORED-HYPERGRAPH-(NON)-DOMINATING-SET

Input: A colored 3-hypergraph $\mathcal{H} = (V, E_1, \dots, E_k)$ with $k \geq 1$, and a mapping $\ell : V \to \mathbb{N}$ with $\ell(a) \leq k$ for all $a \in V$.

Parameter: k.

Problem: Decide whether there exists a set $D \subseteq V$ of cardinality k such that for all $a \in V$ there is a color $i \in [k]$ such that

- if $i \leq \ell(a)$, then D is a dominating set of $\mathcal{H}_i^a$,
- if $i > \ell(a)$, then D is not a dominating set of $\mathcal{H}_i^a$.

is W*[3]*-complete under fpt-reductions.*

Proof: First we show that p-COLORED-HYPERGRAPH-(NON)-DOMINATING-SET is in W*[3] by reducing it to p-MC($\Sigma^*_{3,1}$). Let $((\mathcal{H}, \ell), k)$ be an instance with $\mathcal{H} = (V, E_1, \dots, E_k)$. We define a structure $\mathcal{A}$ over the vocabulary $\tau = \{O_1, \dots, O_k, R_1, \dots, R_k, D_1, \dots, D_k\}$, where each O_i is binary, each R_i is ternary and each I_i is unary. We let $A := V$ be the universe of $\mathcal{A}$. For $i \in [k]$ we set

$$O_i^{\mathcal{A}} := \big\{(a, v) \mid v \neq a \text{ and there is } e \in E_i \text{ with } v, a \in e\big\},$$

$$R_i^{\mathcal{A}} := \big\{(a, u, v) \mid \{a, u, v\} \in E_i\big\},$$

$$D_i^{\mathcal{A}} := \big\{a \mid a \in V \text{ and } i \leq \ell(a)\big\}.$$

Now it is easy to check that

$$((\mathcal{H}, \ell), k) \in p\text{-COLORED-HYPERGRAPH-(NON)-DOMINATING-SET} \quad \iff \quad \mathcal{A} \models \varphi,$$

where

$$\varphi := \exists x_1 \dots \exists x_k \Big(\bigwedge_{1 \leq i < j \leq k} x_i \neq x_j \ \wedge\ \forall y \Big(\bigvee_{i \in [k]} \big(D_i y \wedge \forall z \big(O_i yz \to \bigvee_{i \in [k]} (z = x_i \vee R_i yzx_i)\big)\big) \\ \vee \big(\neg D_i y \wedge \exists z \big(O_i yz \wedge \bigwedge_{i \in [k]} (z \neq x_i \wedge \neg R_i yzx_i)\big)\big)\Big)\Big).$$

[2]Note that D may contain vertices not in V^a, thus more precisely we should say that $D \cap V^a$ is a dominating set of $\mathcal{H}^a$.

Since $\varphi \in \Sigma^*_{3,1}$, this gives the desired reduction.

Now we show p-MC$(\Sigma^*_{3,1}) \leq^{\text{fpt}}$ p-COLORED-HYPERGRAPH-(NON)-DOMINATING-SET. We make use of Corollary 14 and start with an instance (α, k) of p-WSAT(strict-$\Gamma^*_{3,k}$) with the property that every assignment satisfying α has weight k.

Denote by Var(α) the set of propositional variables of α. Without loss of generality we assume that

$$\alpha = \bigwedge_{a \in I} \vee_{i \in [k]} \beta_{a,i},$$

where for some index set J with $|J| \geq 3$ and some mapping $a \mapsto \ell_a$ with $a \in I$ and $0 \leq \ell_a \leq k$

– if $i \leq \ell_a$, then
$$\beta_{a,i} = \bigwedge_{u \in J} \bigvee_{v \in J} X_{a,i,u,v},$$

– if $\ell_a < i \leq k$, then
$$\beta_{a,i} = \bigvee_{u \in J} \bigwedge_{v \in J} \neg X_{a,i,u,v},$$

with propositional variables $X_{a,i,u,v}$. Moreover, we assume that $|\text{Var}(\alpha)| \geq k + 2$.

In a first step we construct a colored 3-hypergraph $\mathcal{H} = (V, E_1, \ldots, E_k)$ as follows

$$\begin{aligned}
V &:= I \dot\cup J \dot\cup \text{Var}(\alpha), \\
E_i &:= \big\{\{a, u, X_{a,i,u,v}\} \mid a \in I \text{ and } u, v \in J\big\} \\
&\quad \cup \big\{\{a, X, X'\} \mid a \in I \text{ and } X, X' \in \text{Var}(\alpha),\ X \neq X'\big\}, \qquad \text{for } i \in [k-1], \\
E_k &:= \big\{\{a, u, X_{a,k,u,v}\} \mid a \in I \text{ and } u, v \in J\big\} \\
&\quad \cup \big\{\{a, X, X'\} \mid a \in I \text{ and } X, X' \in \text{Var}(\alpha),\ X \neq X'\big\} \\
&\quad \cup \big\{\{u, u', u''\} \mid\ u, u', u'' \in J,\ u, u', u'' \text{ pairwise distinct } \big\}.
\end{aligned}$$

Moreover we define a mapping $\ell : V \to \mathbb{N}$ by

$$\ell(a) := \begin{cases} \ell_a, & \text{if } a \in I, \\ 0, & \text{otherwise.} \end{cases}$$

Then for all $S \subseteq \text{Var}(\alpha)$

$$S \text{ satisfies } \alpha \iff \begin{array}{l} \text{for all } a \in I \text{ there exists } i \in [k] \text{ such that} \\ \text{– if } i \leq \ell(a), \text{ then } S \text{ is a dominating set of } \mathcal{H}^a_i, \\ \text{– if } i > \ell(a), \text{ then } S \text{ is not a dominating set of } \mathcal{H}^a_i. \end{array} \tag{3}$$

Let $S \subseteq \text{Var}(\alpha)$ be of size $\leq k$. Since $|\text{Var}(\alpha)| \geq k + 2$, for any $X \in \text{Var}(\alpha)$ there exists some

$$X' \in \text{Var}(\alpha) \setminus \big(S \cup \{X\}\big).$$

Then X' witnesses that S is *not* a dominating set of $\mathcal{H}^X_k$. Similarly, since $|J| \geq 3$, for every $u, u' \in J$ with $u \neq u'$, the vertex u' is a vertex of the graph $\mathcal{H}^u_k$ that has no neighbor in Var(α); therefore no $S \subseteq \text{Var}(\alpha)$ is a dominating set of $\mathcal{H}^u_k$.

Altogether, we see that for $S \subseteq \text{Var}(\alpha)$ of size $\leq k$, the equivalence (3) remains true, if we replace "for all $a \in I$" by "for all $a \in V$."

There still remains one problem: It is not guaranteed that $S \subseteq \text{Var}(\alpha)$ for every "solution" $S \subseteq V$, that is, for every $S \subseteq V$ witnessing that $((\mathcal{H}, \ell), k) \in$ p-COLORED-HYPERGRAPH-(NON)-DOMINATING-SET. For that purpose we have to modify $\mathcal{H}$. We consider the colored 3-hypergraph $\bar{\mathcal{H}}$, which consists of $(k+1)$-many copies of $\mathcal{H}$, yet thereby not duplicating vertices in Var(α). And we extend ℓ to a mapping $\bar{\ell}$ accordingly. Then for every solution S of $((\bar{\mathcal{H}}, \bar{\ell}), k)$, there exists a copy such that the restriction S' of S to that copy contains no vertices other than variables. Thus, by (3), S' is a satisfying assignment of α of weight $\leq k$. Hence, by our assumption on α we have $|S'| = k$ and therefore, $S' = S$.

More precisely, we let

$$\bar{\mathcal{H}} := \big(\bar{V}, \bar{E}_1, \ldots, \bar{E}_k\big)$$

be the colored 3-hypergraph with

$$\begin{aligned}
\bar{V} &:= \Big(\big(V \setminus \mathrm{Var}(\alpha)\big) \times [k+1]\Big) \cup \mathrm{Var}(\alpha),\\
\bar{E}_i &:= \big\{\{(a,m),(u,m),X\} \mid \{a,u,X\} \in E_i,\ a,u \in I \cup J,\ X \in \mathrm{Var}(\alpha),\ m \in [k+1]\big\}\\
&\quad \cup \big\{\{(a,m),X,X'\} \mid \{a,X,X'\} \in E_i,\ a \in I,\ X,X' \in \mathrm{Var}(\alpha),\ m \in [k+1]\big\},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } i \in [k-1]\\
\bar{E}_k &:= \big\{\{(a,m),(u,m),X\} \mid \{a,u,X\} \in E_k,\ a,u \in I \cup J,\ X \in \mathrm{Var}(\alpha),\ m \in [k+1]\big\}\\
&\quad \cup \big\{\{(a,m),X,X'\} \mid \{a,X,X'\} \in E_k,\ a \in I,\ X,X' \in \mathrm{Var}(\alpha),\ m \in [k+1]\big\}\\
&\quad \cup \big\{\{(u,m),(u',m),(u'',m)\} \mid \{u,u',u''\} \in E_k,\ u,u',u'' \in J,\ m \in [k+1]\big\}.
\end{aligned}$$

We set $\bar{\ell}((a,m)) := \ell(a)$ for $a \in V \setminus \mathrm{Var}(\alpha)$ and $m \in [k+1]$, and $\bar{\ell}(X) := \ell(X)$ for $X \in \mathrm{Var}(\alpha)$. Now with the above-mentioned argument it is routine to check that

$$\alpha \text{ is } k\text{-satisfiable} \iff ((\bar{\mathcal{H}}, \bar{\ell}), k) \in p\text{-COLORED-HYPERGRAPH-(NON)-DOMINATING-SET}. \quad \Box$$

8. Conclusions

In the unwieldy multitude of parameterized complexity classes, results that bring classes or hierarchies of classes closer together are particularly welcome. Our theorem relating the W*-hierarchy and the W-hierarchy means some progress here.

It is still open whether the two hierarchies coincide level wise, that is, whether $\mathrm{W}[t] \subseteq \mathrm{W}^*[t]$ for all (or for any) $t \geq 3$.

References

[1] J.F. Buss and T. Islam. Algorithms in the W-Hierarchy. Submitted for publication, 2005. Available at `http://www.cs.uwaterloo.ca/~jfbuss/`.

[2] R.G. Downey and M.R. Fellows. Fixed-parameter tractability and completeness I: Basic results. *SIAM Journal on Computing*, 24:873–921, 1995.

[3] R.G. Downey and M.R. Fellows. Fixed-parameter tractability and completeness II: On completeness for $W[1]$. *Theoretical Computer Science*, 141:109–131, 1995.

[4] R.G. Downey and M.R. Fellows. Threshold dominating sets and an improved characterization of $W[2]$. *Theoretical Computer Science*, 209:123–140, 1998.

[5] R.G. Downey and M.R. Fellows. *Parameterized Complexity*. Springer-Verlag, 1999.

[6] R.G. Downey, M.R. Fellows, and K. Regan. Descriptive complexity and the W-hierarchy. In P. Beame and S. Buss, editors, *Proof Complexity and Feasible Arithmetic*, volume 39 of *AMS-DIMACS Volume Series*, pages 119–134. AMS, 1998.

[7] R.G. Downey and M.R. Fellows. Fixed-parameter tractability and completeness I: Basic Results. *SIAM Journal on Computing*, 24:873–921, 1995.

[8] R.G. Downey, M.R. Fellows, and U. Taylor. The parameterized complexity of relational database queries and an improved characterization of $W[1]$. In D.S. Bridges, C. Calude, P. Gibbons, S. Reeves, and I.H. Witten, editors, *Combinatorics, Complexity, and Logic – Proceedings of DMTCS '96*, pages 194–213. Springer-Verlag, 1996.

[9] J. Flum and M. Grohe. Fixed-parameter tractability, definability, and model checking. *SIAM Journal on Computing*, 31(1):113–145, 2001.

[10] J. Flum and M. Grohe. Model-checking problems as a basis for parameterized intractability. *Logical Methods in Computer Science*, 1(1), 2005.

[11] J. Flum and M. Grohe. *Parameterized Complexity Theory*. Springer-Verlag, 2006.

[12] J. Landes. Die W*-Hierarchie. Diplomarbeit, Universität Freiburg, 2005.