

Fast Algorithms for SAT

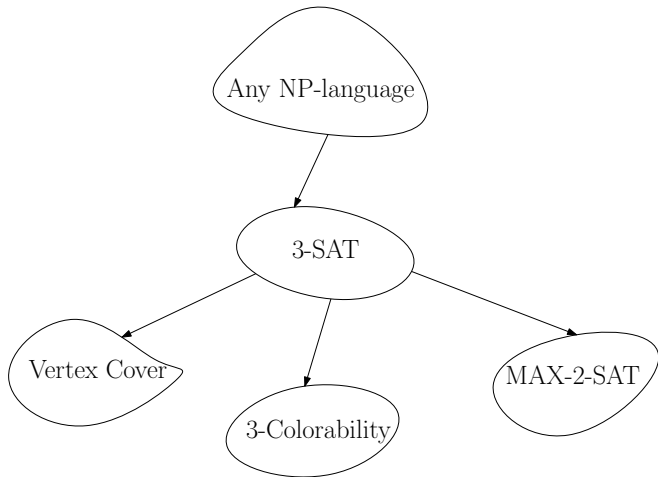
Dominik Scheder, ETH Zürich

FU Berlin, October 26, 2009

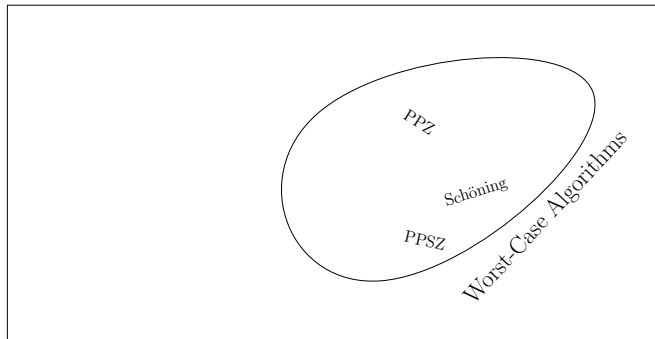
“Fast” Algorithms for SAT

Dominik Scheder, ETH Zürich

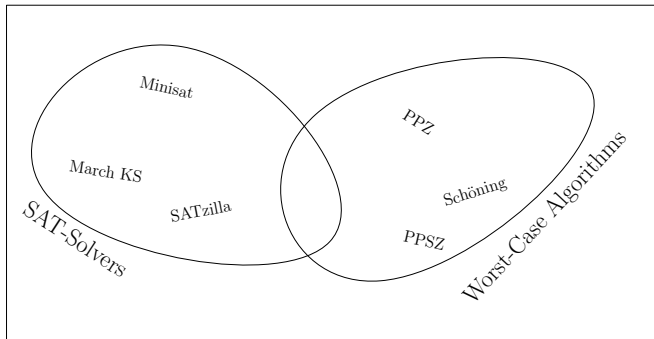
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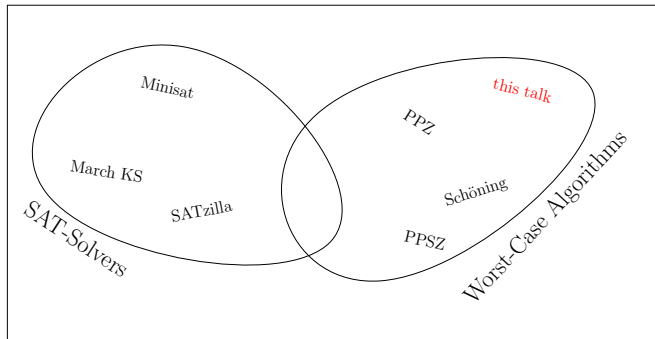
The World of SAT



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$$(x_2) \wedge (x_1 \vee \bar{x}_2 \vee \bar{x}_3) \wedge (x_1 \vee x_2 \vee \bar{x}_3) \wedge (x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3) \wedge (\bar{x}_3 \vee x_4)$$

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Assignment $(x_1, x_2, x_3, x_4) = (1, 1, 1, 1)$

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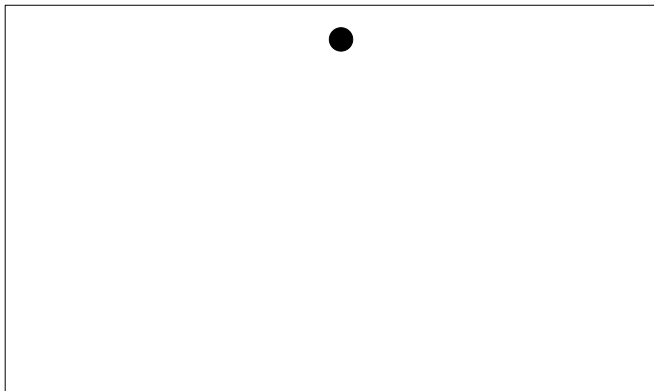
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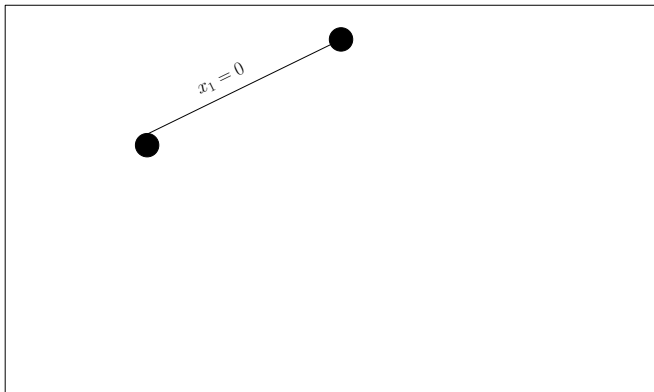
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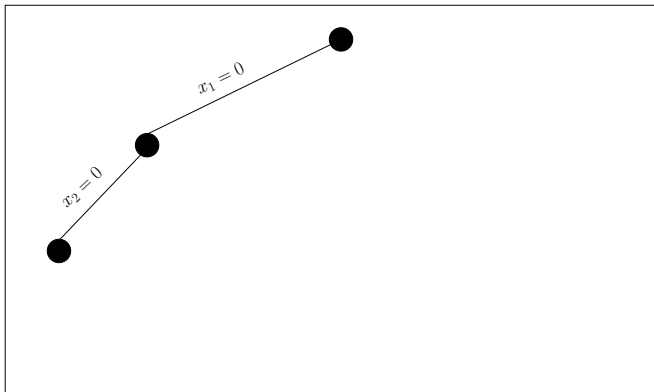
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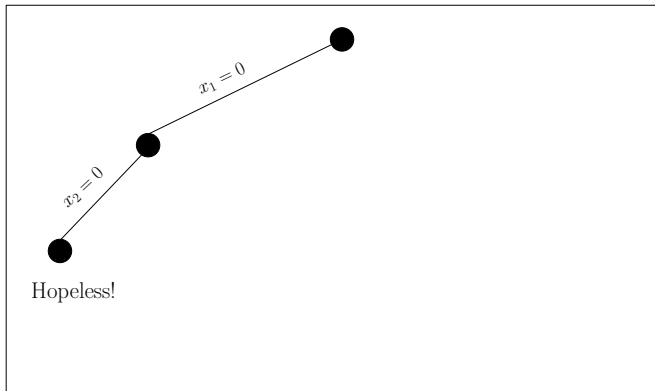
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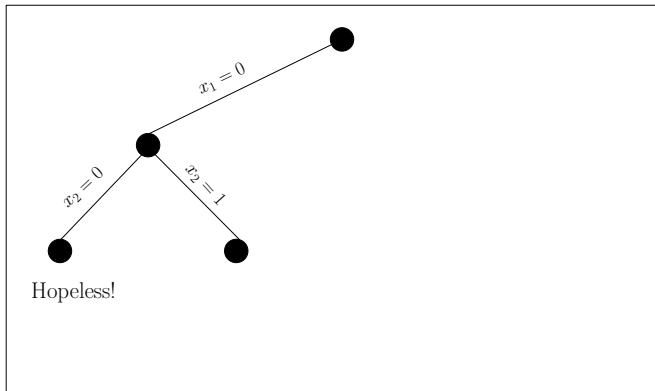
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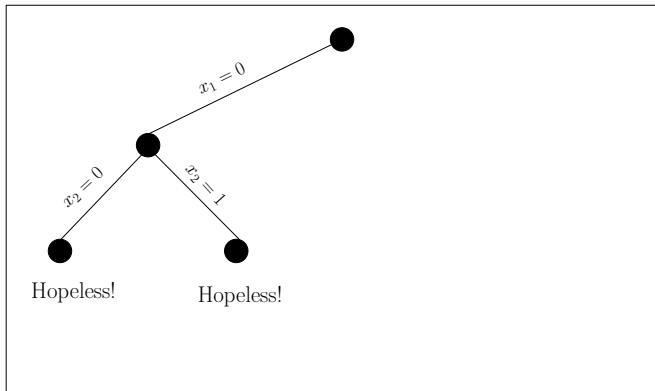
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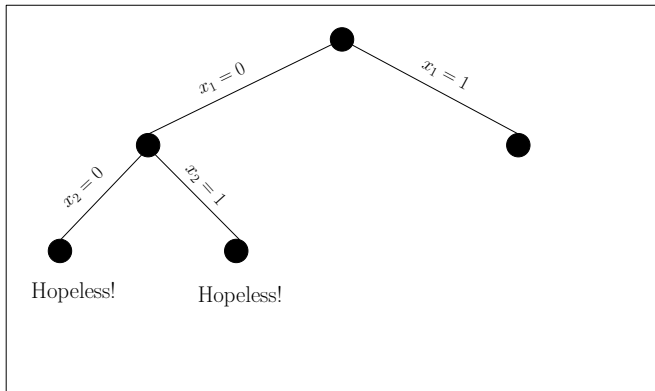
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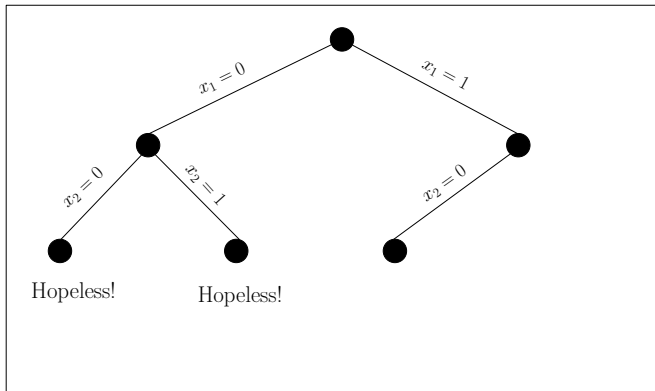
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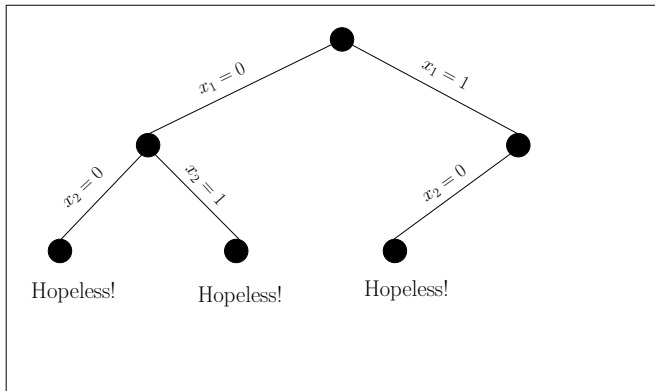
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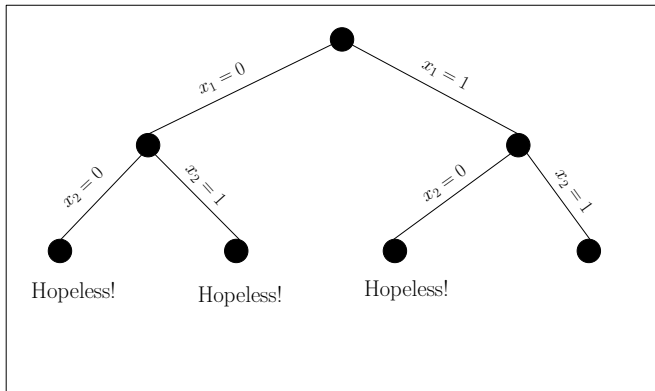
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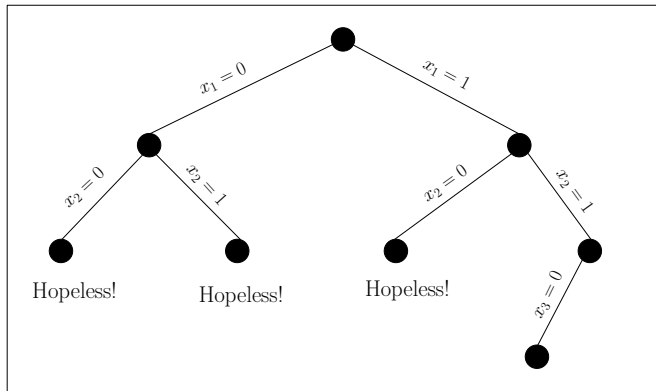
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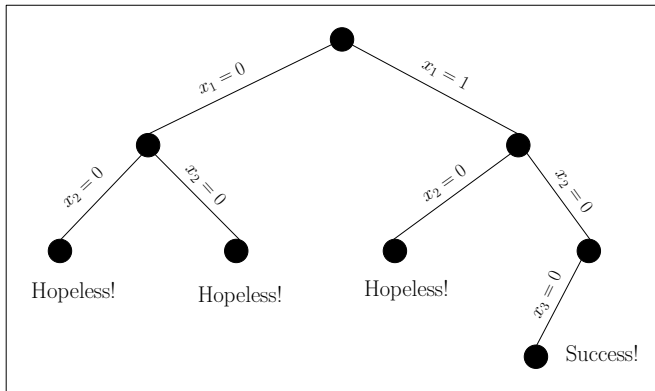
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At most 2^n steps

Algorithms for 3-SAT – Some History

2^n	What I just showed you		deterministic
1.618^n	Monien and Speckenmeyer	1985	deterministic
1.588^n	Paturi, Pudlák and Zane	1997	randomized
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The searchball algorithm

A Parametrized Problem

Let the following be given:

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We search the *Hamming ball* $B_r(\alpha)$

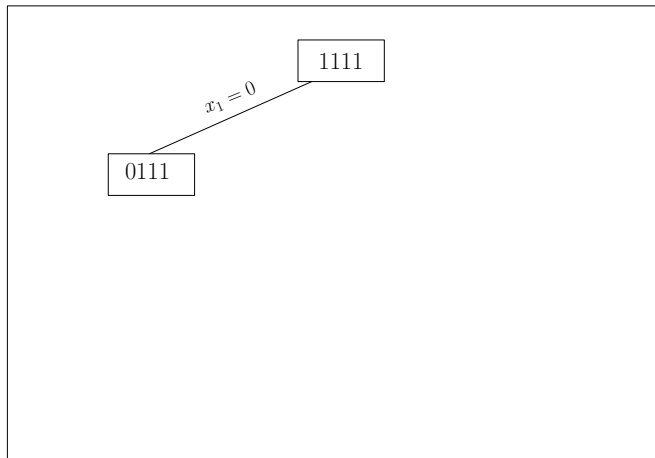
Example with $\alpha = (1, 1, 1, 1)$ and $r = 2$

$$(x_2) \wedge (x_1 \vee \bar{x}_2 \vee \bar{x}_3) \wedge (x_1 \vee x_2 \vee \bar{x}_3) \wedge (x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_4) \wedge (\bar{x}_3 \vee x_4)$$

1111

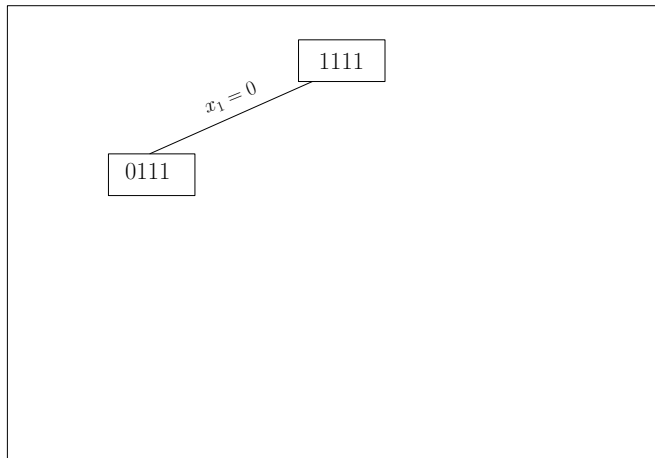
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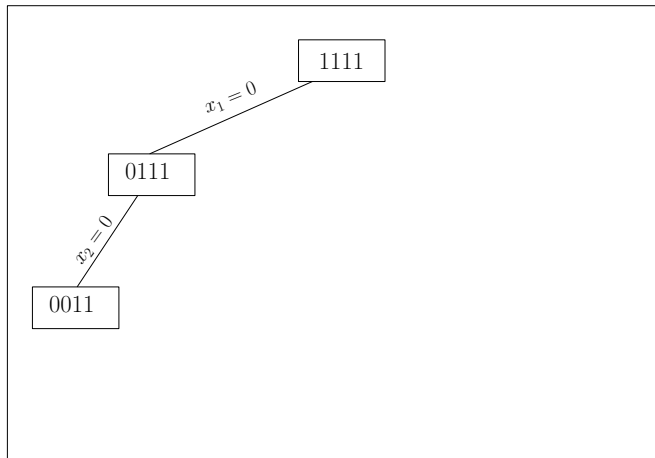
$$(x_2) \wedge (\bar{x}_2 \vee \bar{x}_3) \wedge (x_2 \vee \bar{x}_3) \wedge (\bar{x}_2) \wedge (\bar{x}_3 \vee x_4)$$



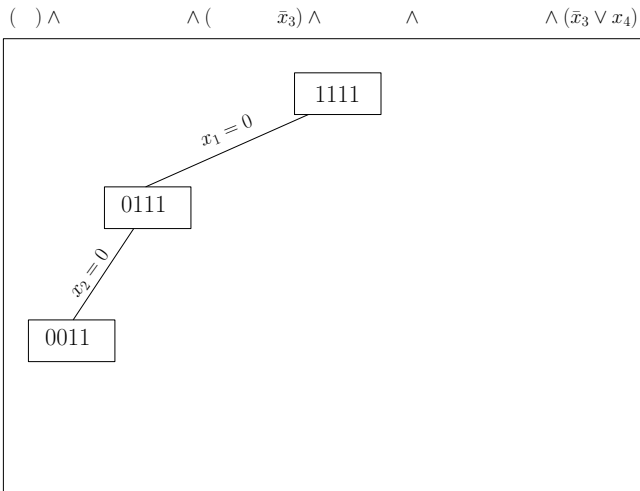
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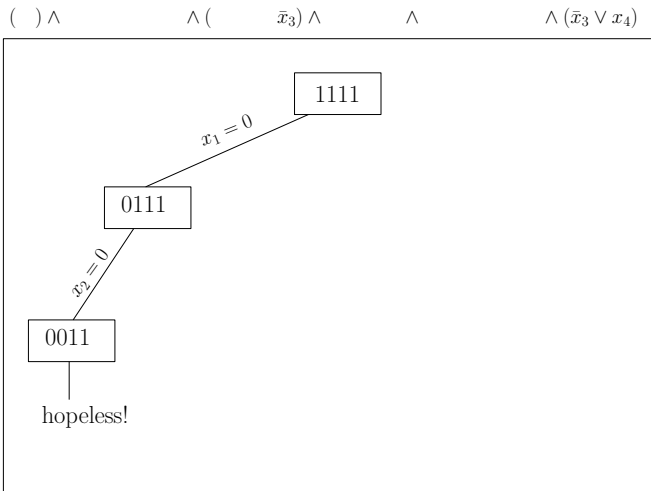
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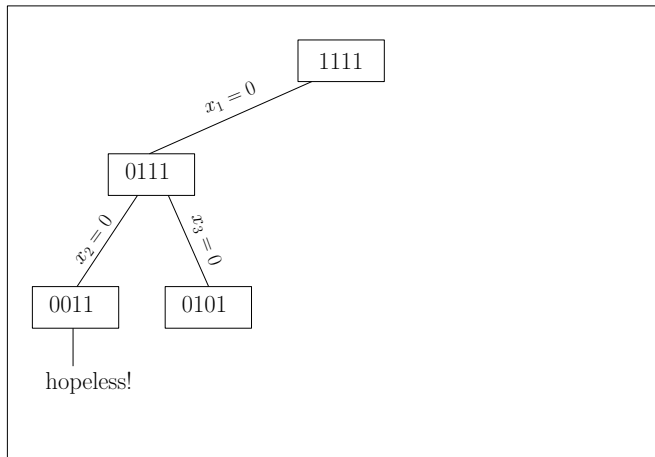
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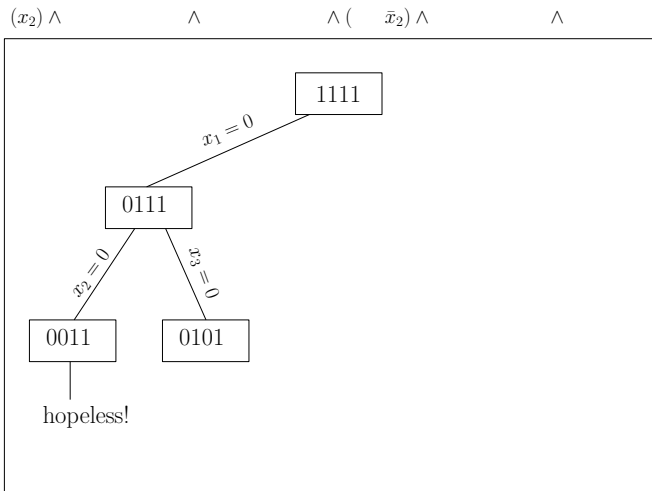
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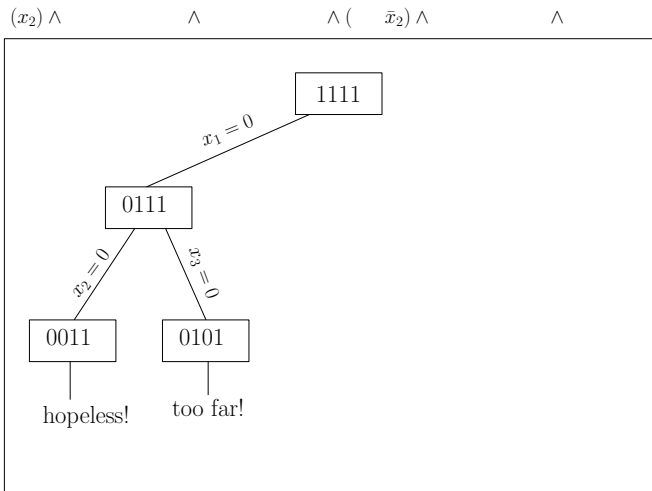
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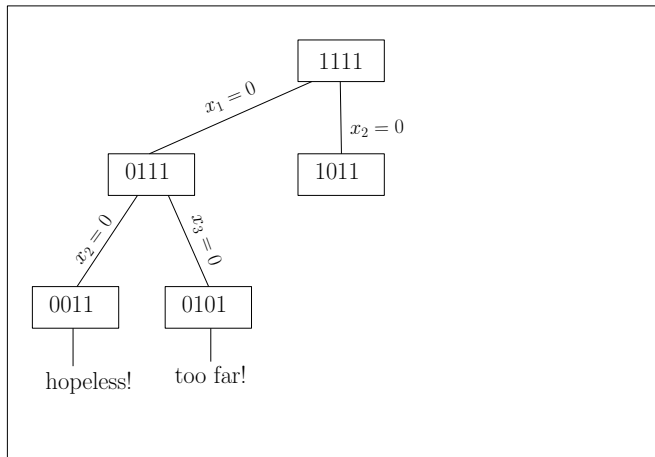


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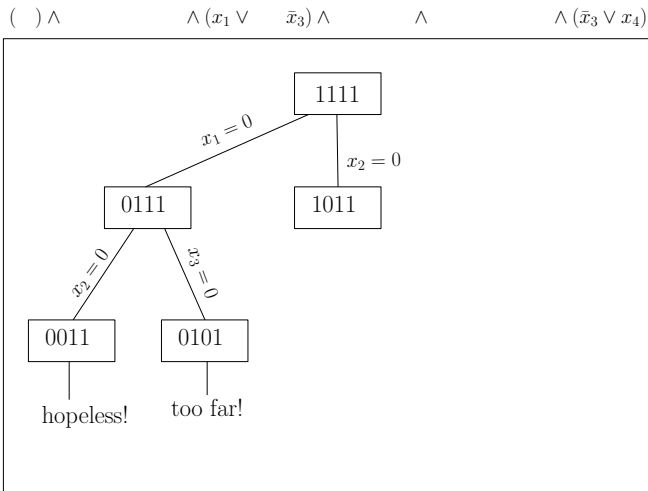


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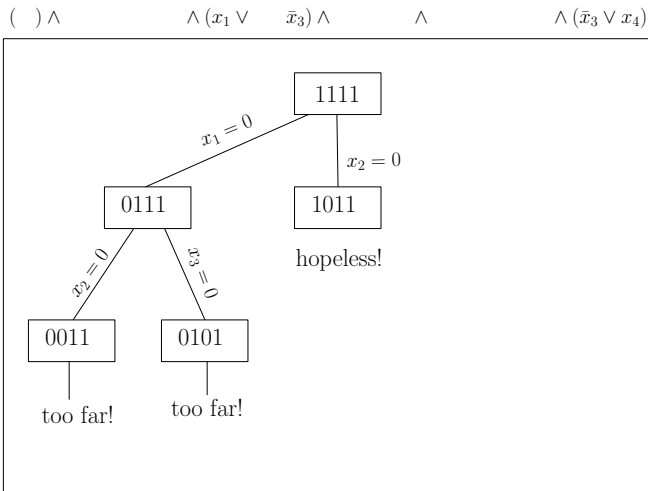
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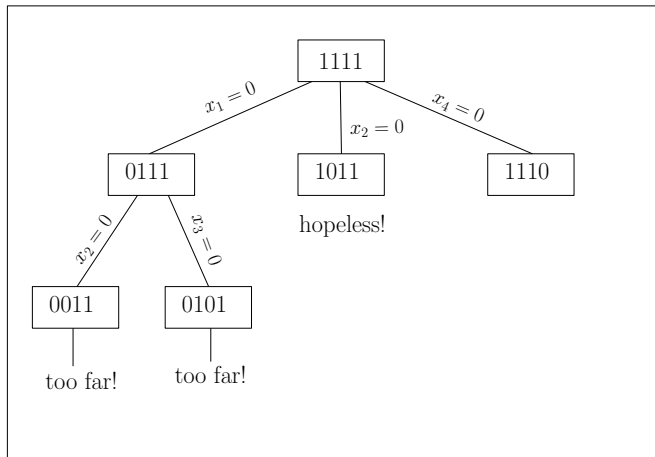


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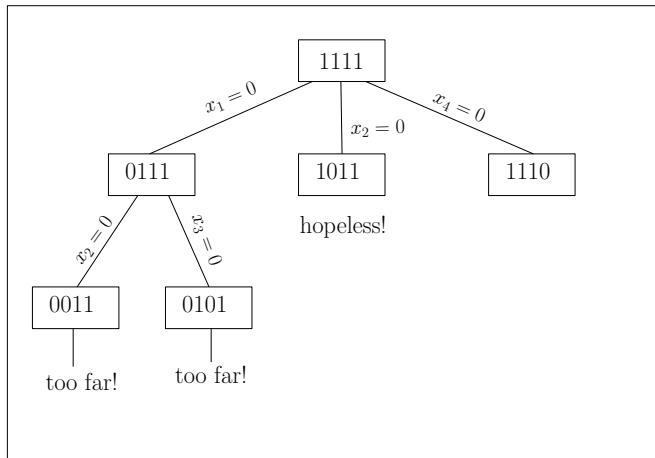
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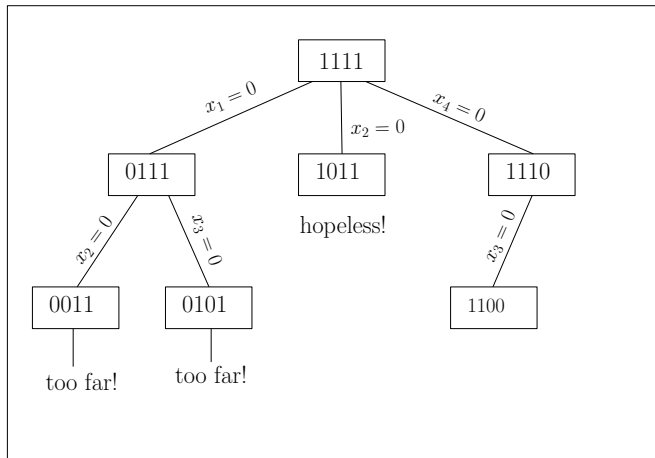
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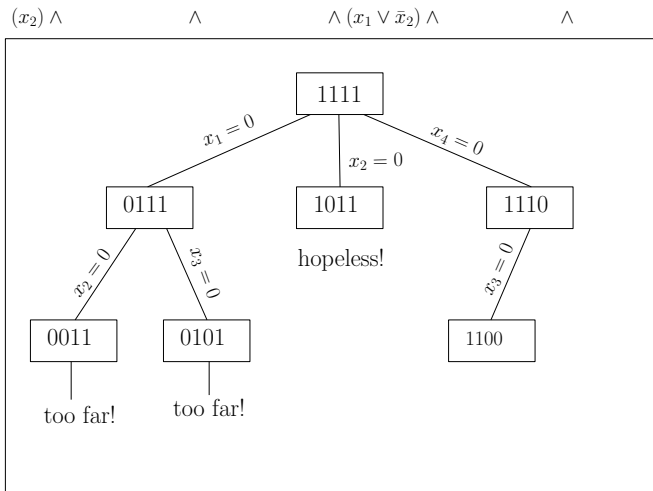


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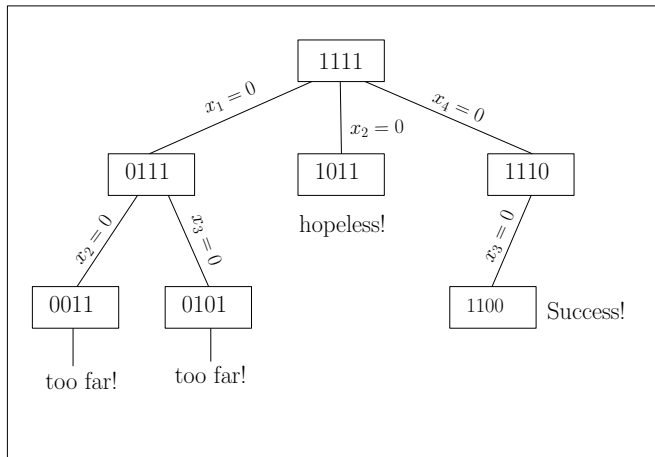


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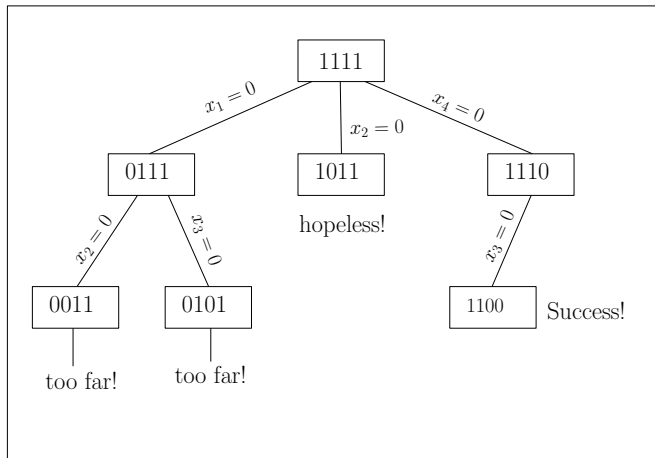
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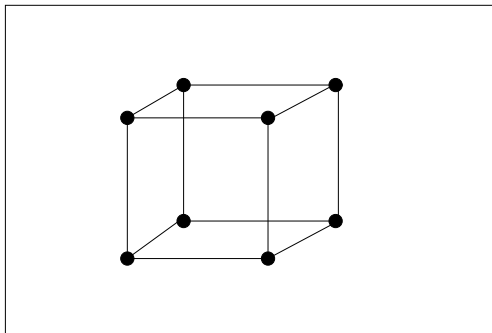
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At most 3^r steps for 3-SAT!

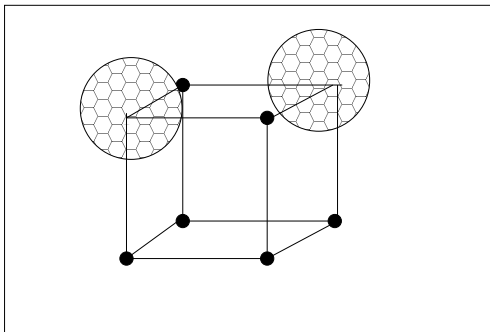
Idea of Local Search

The cube $\{0, 1\}^n$ of truth assignments



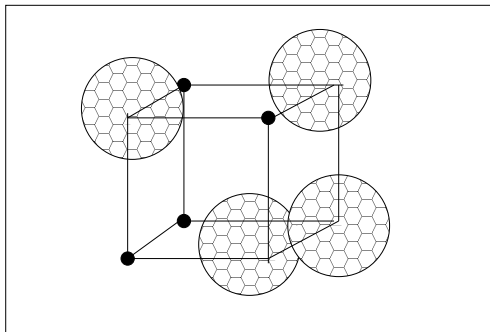
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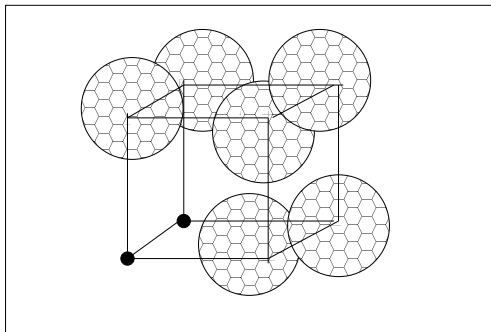
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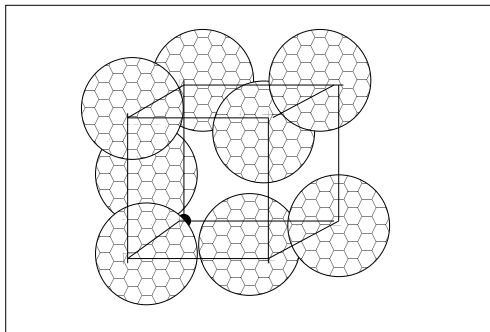
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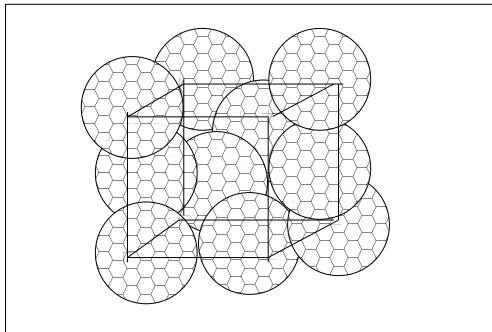
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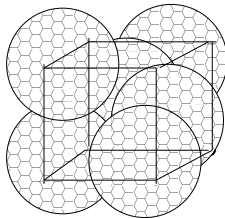
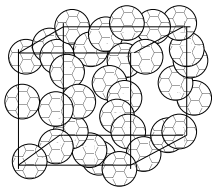


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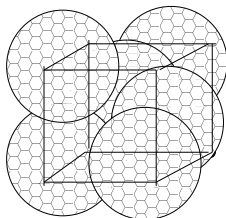
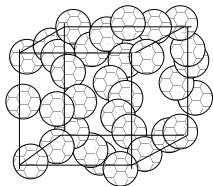
The cube $\{0, 1\}^n$ of truth assignments



Choosing the Optimal Radius



Choosing the Optimal Radius



a^r steps per ball \implies

$\left(2 - \frac{2}{a+1}\right)^n$ steps for the whole cube

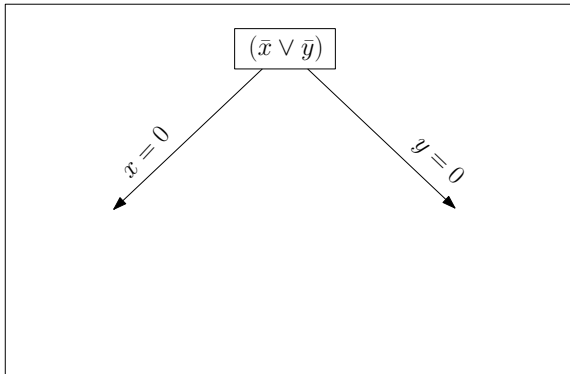
$3^r \implies 1.5^n$

$2.73^r \implies 1.465^n$

Improving searchball

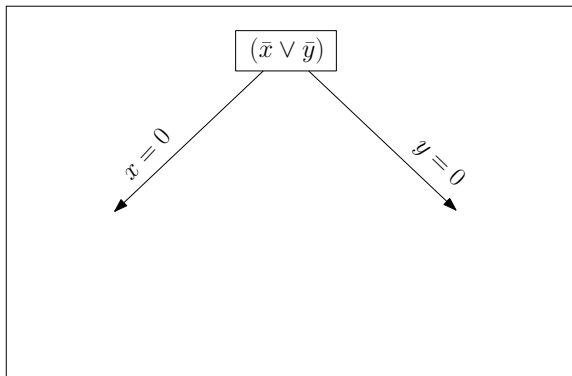
Four Simple Rules: Rule 1

$$F = (\bar{x} \vee \bar{y}) \wedge \dots$$



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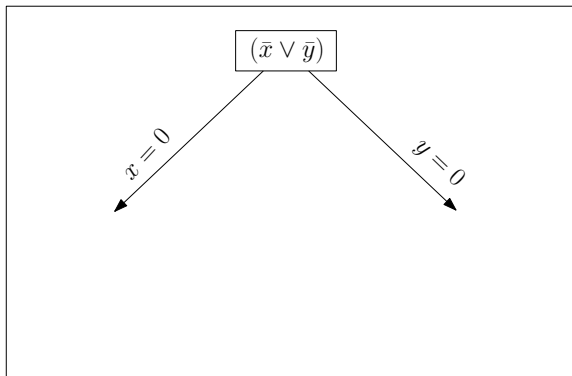
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$$L(r) \leq 2L(r-1)$$

Four Simple Rules: Rule 1

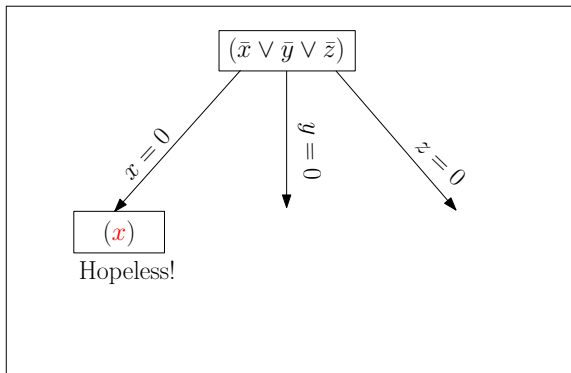
$$F = (\bar{x} \vee \bar{y}) \wedge \dots$$



$$L(r) \leq 2L(r-1) \implies L(r) \leq 2^r$$

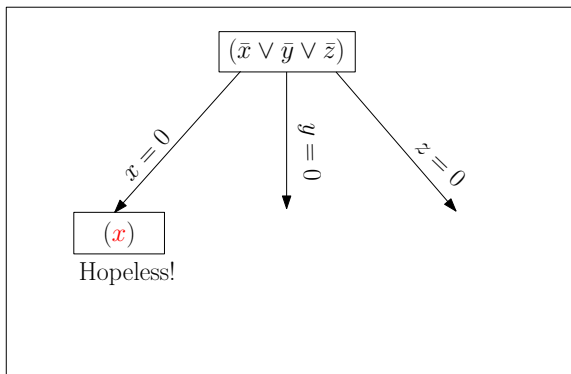
Four Simple Rules: Rule 2

$$F = (\bar{x} \vee \bar{y} \vee \bar{z}) \wedge (x) \wedge \dots$$



Four Simple Rules: Rule 2

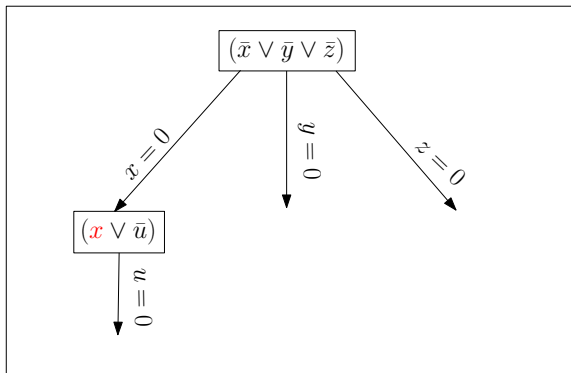
$$F = (\bar{x} \vee \bar{y} \vee \bar{z}) \wedge (x) \wedge \dots$$



$$L(r) \leq 2L(r-1) \implies L(r) \leq 2^r$$

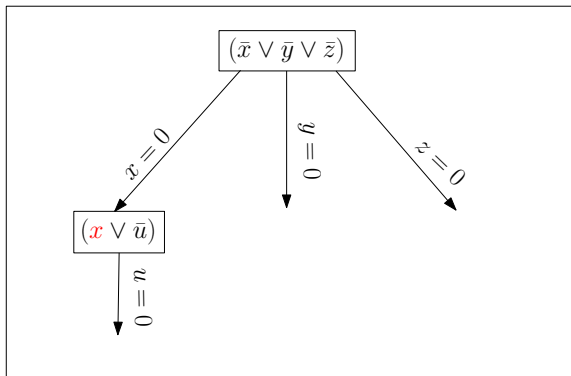
Four Simple Rules: Rule 3

$$F = (\bar{x} \vee \bar{y} \vee \bar{z}) \wedge (x \vee \bar{u}) \wedge \dots$$



Four Simple Rules: Rule 3

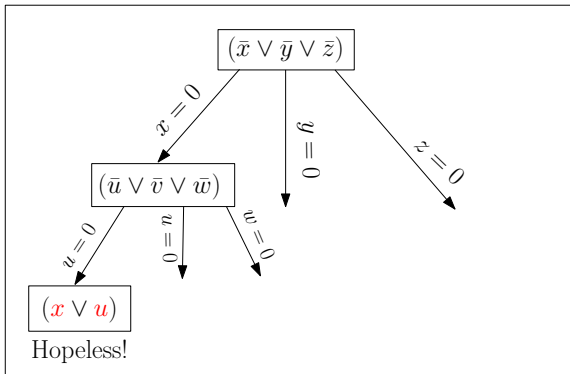
$$F = (\bar{x} \vee \bar{y} \vee \bar{z}) \wedge (x \vee \bar{u}) \wedge \dots$$



$$L(r) \leq 2L(r-1) + L(r-2) \implies L(r) \leq 2.414^r$$

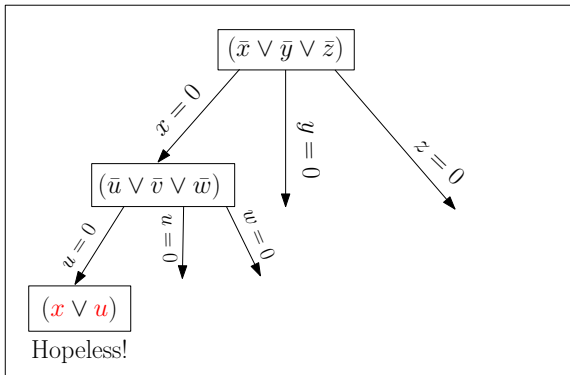
Four Simple Rules: Rule 4

$$F = (\bar{x} \vee \bar{y} \vee \bar{z}) \wedge (\bar{u} \vee \bar{v} \vee \bar{w}) \wedge (x \vee u) \wedge \dots$$



Four Simple Rules: Rule 4

$$F = (\bar{x} \vee \bar{y} \vee \bar{z}) \wedge (\bar{u} \vee \bar{v} \vee \bar{w}) \wedge (x \vee u) \wedge \dots$$



$$L(r) \leq 2L(r-1) + 2L(r-2) \implies L(r) \leq 2.73^r$$

Nice – But What Now?

Thrifty Search – $\alpha = (1, 1, 1, 1)$, $r = 2$

$$(\bar{x}_1 \vee \bar{y}_1 \vee \bar{z}_1) \wedge (\bar{x}_2 \vee \bar{y}_2 \vee \bar{z}_2) \wedge (\bar{x}_3 \vee \bar{y}_3 \vee \bar{z}_3) \wedge (x_1) \wedge (z_1 \vee x_2 \vee u) \wedge (y_1 \vee \bar{x}_3 \vee \bar{u})$$



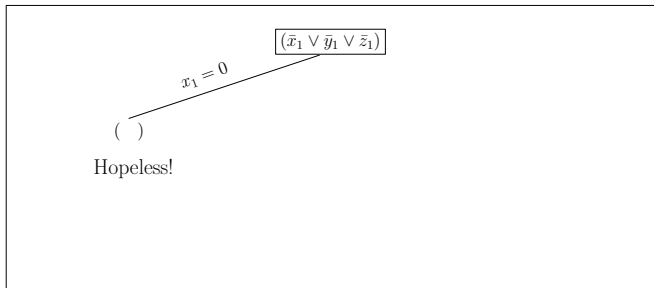
Thrifty Search – $\alpha = (1, 1, 1, 1)$, $r = 2$

$$(\bar{x}_1 \vee \bar{y}_1 \vee \bar{z}_1) \wedge (\bar{x}_2 \vee \bar{y}_2 \vee \bar{z}_2) \wedge (\bar{x}_3 \vee \bar{y}_3 \vee \bar{z}_3) \wedge (x_1) \wedge (z_1 \vee x_2 \vee u) \wedge (y_1 \vee \bar{x}_3 \vee \bar{u})$$

$$(\bar{x}_1 \vee \bar{y}_1 \vee \bar{z}_1)$$

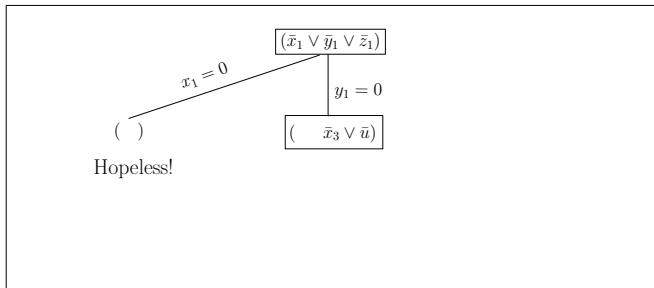
Thrifty Search – $\alpha = (1, 1, 1, 1)$, $r = 2$

$$\wedge (\bar{x}_2 \vee \bar{y}_2 \vee \bar{z}_2) \wedge (\bar{x}_3 \vee \bar{y}_3 \vee \bar{z}_3) \wedge () \wedge (z_1 \vee x_2 \vee u) \wedge (y_1 \vee \bar{x}_3 \vee \bar{u})$$



Thrifty Search – $\alpha = (1, 1, 1, 1)$, $r = 2$

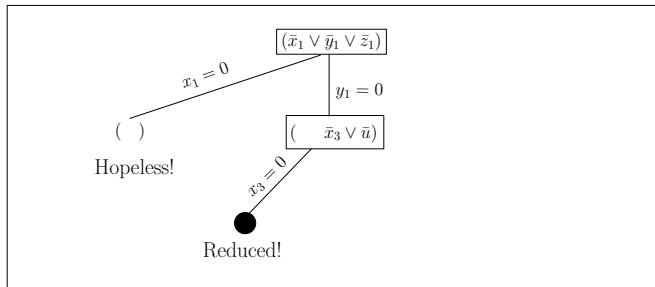
$$\wedge (\bar{x}_2 \vee \bar{y}_2 \vee \bar{z}_2) \wedge (\bar{x}_3 \vee \bar{y}_3 \vee \bar{z}_3) \wedge (x_1) \wedge (z_1 \vee x_2 \vee u) \wedge (\bar{x}_3 \vee \bar{u})$$



Thrifty Search – $\alpha = (1, 1, 1, 1)$, $r = 2$

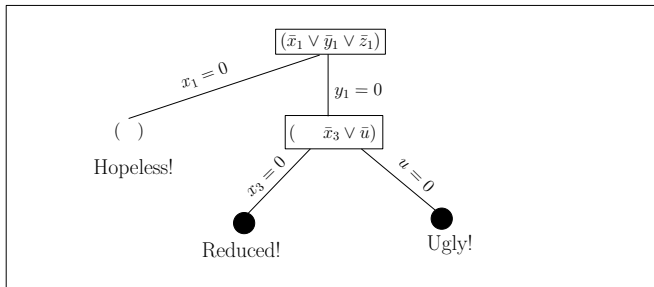
$$\wedge (\bar{x}_2 \vee \bar{y}_2 \vee \bar{z}_2) \wedge$$

$$\wedge (x_1) \wedge (z_1 \vee x_2 \vee u) \wedge$$



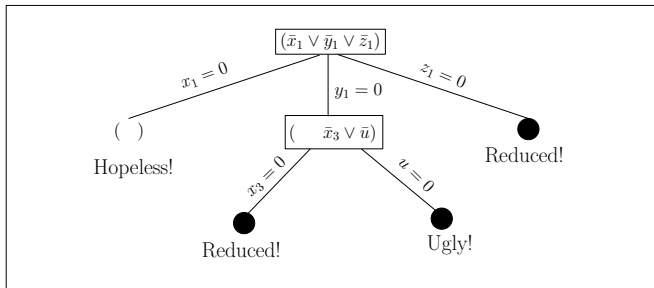
Thrifty Search – $\alpha = (1, 1, 1, 1)$, $r = 2$

$$\wedge (\bar{x}_2 \vee \bar{y}_2 \vee \bar{z}_2) \wedge (\bar{x}_3 \vee \bar{y}_3 \vee \bar{z}_3) \wedge (x_1) \wedge (z_1 \vee x_2) \wedge$$



Thrifty Search – $\alpha = (1, 1, 1, 1)$, $r = 2$

$$\wedge (\bar{x}_2 \vee \bar{y}_2 \vee \bar{z}_2) \wedge (\bar{x}_3 \vee \bar{y}_3 \vee \bar{z}_3) \wedge (x_1) \wedge (x_2 \vee u) \wedge (y_1 \vee \bar{x}_3 \vee \bar{u})$$

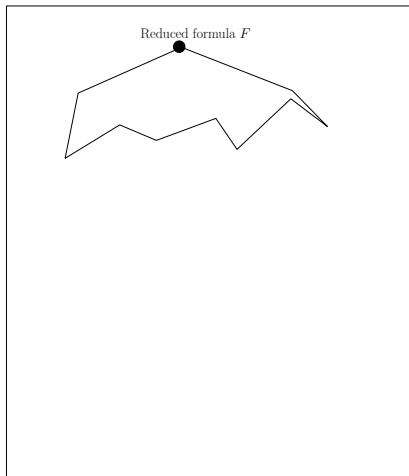


Iterating Thrifty Search

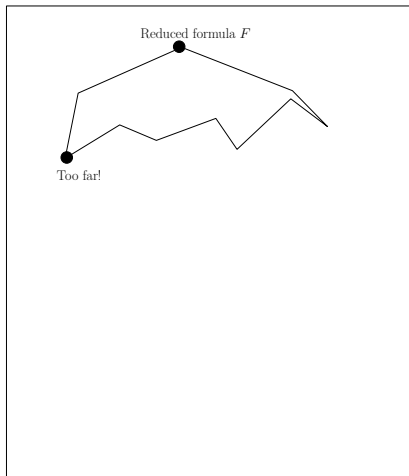
Reduced formula F



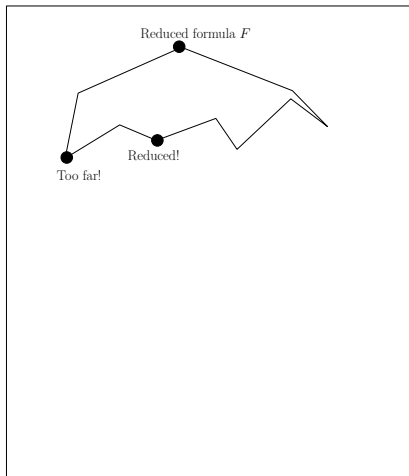
Iterating Thrifty Search



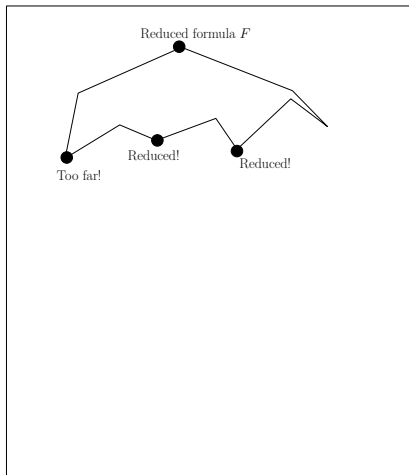
Iterating Thrifty Search



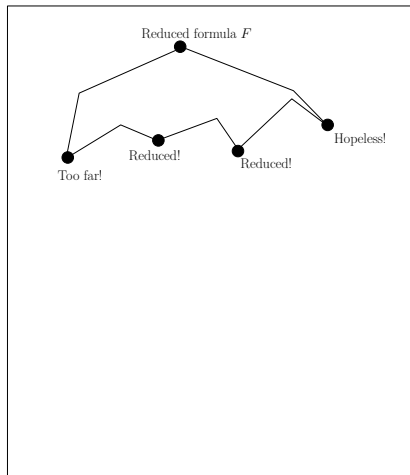
Iterating Thrifty Search



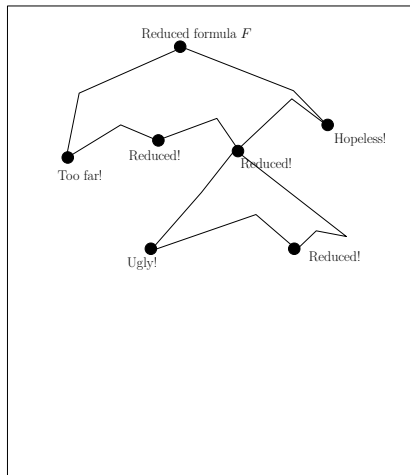
Iterating Thrifty Search



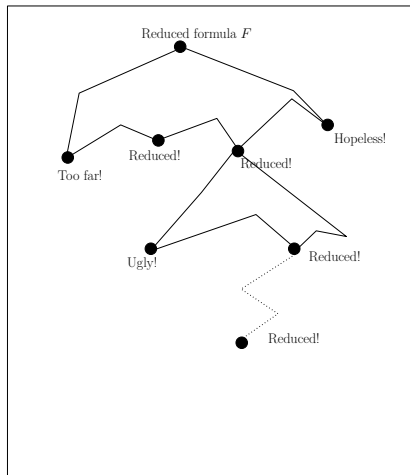
Iterating Thrifty Search



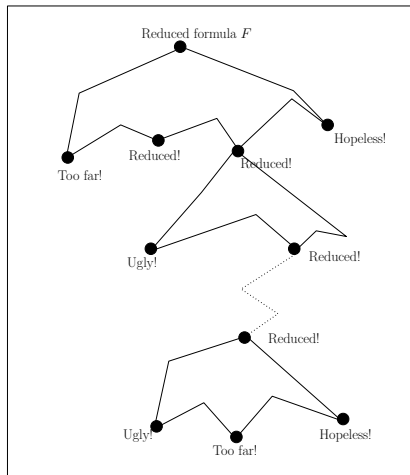
Iterating Thrifty Search



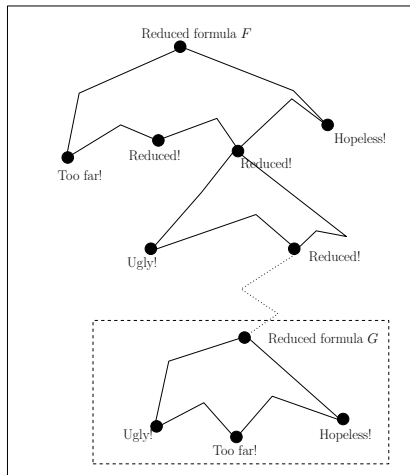
Iterating Thrifty Search



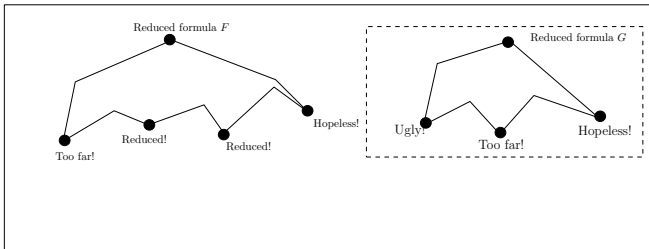
Iterating Thrifty Search



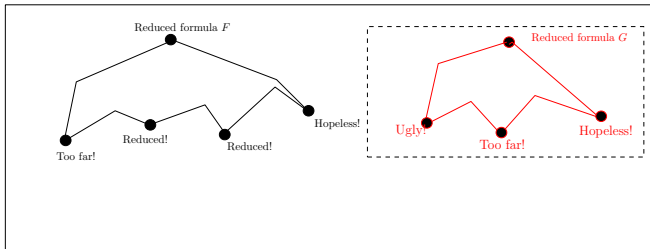
Iterating Thrifty Search



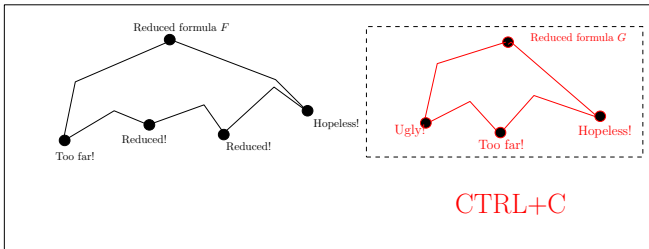
What Now?



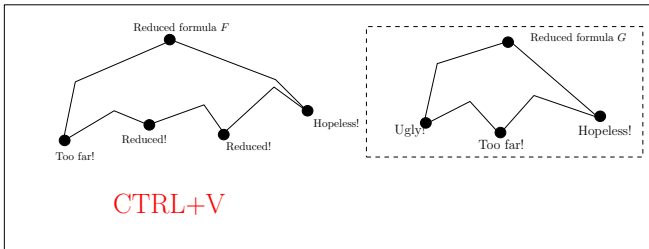
What Now?



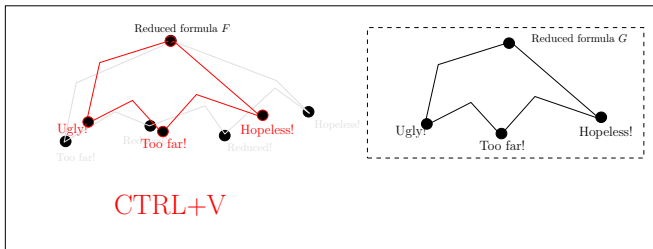
What Now?



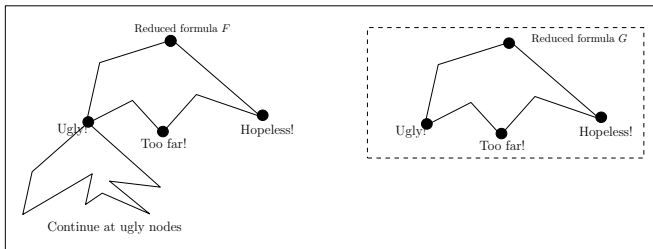
What Now?



What Now?



What Now?



Summary of the Algorithm

`searchball-improved`:

- Apply Rules 1–4
- Reduced Formula \Rightarrow Thrifty Search \Rightarrow Guide Formula
- CTRL+C, CTRL+V
- Use Guide Formula
- Recursively call `searchball-improved` on ugly nodes

Summary of the Algorithm

`searchball-improved`:

- Apply Rules 1–4
- Reduced Formula \Rightarrow Thrifty Search \Rightarrow Guide Formula
- CTRL+C, CTRL+V
- Use Guide Formula
- Recursively call `searchball-improved` on ugly nodes

Running Time for 3-SAT:

- dominated by Rule 4
- 2.73^r per ball
- 1.465^n for formula

Danke für Ihre Aufmerksamkeit!