How Many Conflicts Does It Need to be Unsatisfiable?

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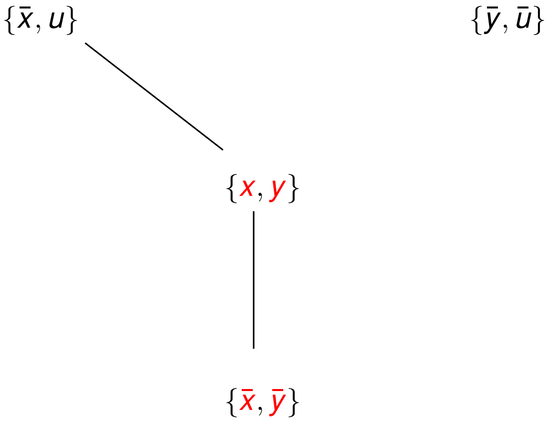

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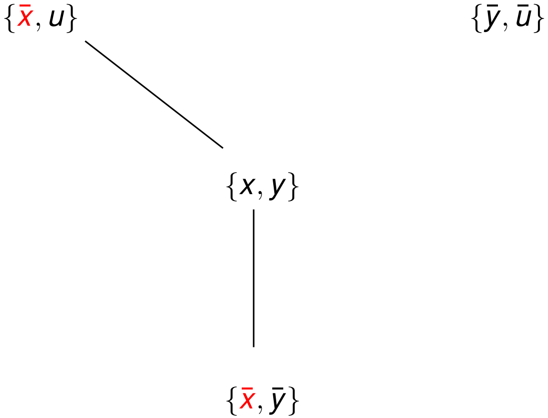

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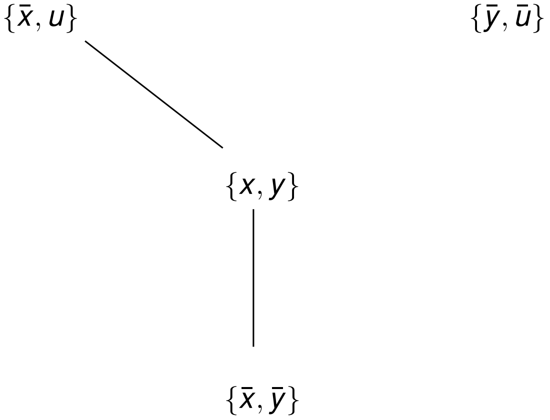

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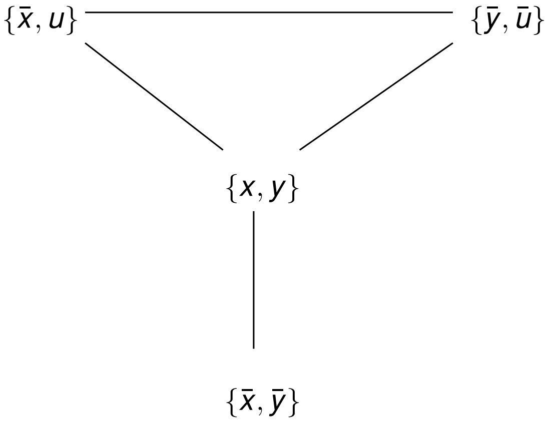

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4. *F has less than ??? conflicts.*

Number of Conflicts—A Lower Bound

Lemma

Let F be a k-CNF formula. If F has $< k2^{k-1}$ conflicts, then F is satisfiable.

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Instead of $k2^{k-1}$, we want something better... $2.5^k, 3^k, 4^k, 8^k$??
How big is possible?

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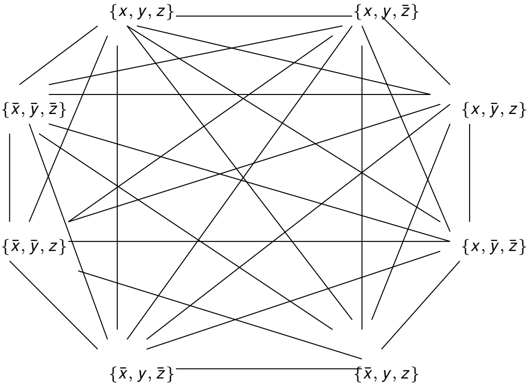

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- unsatisfiable k-CNF formula F
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Does there exist unsatisfiable k-CNF formulas with less conflicts?

Our main result

Theorem

Any k-CNF formula with less than $\mathcal{O}\left(2.69^k\right)$ conflicts is satisfiable.

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Proof. Apply the Lovász Local Lemma…

The Intuition Behind the Lovász Local Lemma

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- If each clause C has few neighbors, still okay.
- If C has many neighbors, but every neighbor is extremely likely to be satisfied, that's okay too.

The Lovász Local Lemma

Lemma

Let F be any CNF formula. Set every variable x of F to true with some probability $p(x)$, independently. If for every clause C, it holds that

$$\sum_{D \in N(C)} \Pr[D \text{ not satisfied}] \leq \frac{1}{4}$$

then F is satisfiable.

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For a literal u, let $\mathrm{occ}(u)$ denote the number of clauses in F containing u.

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If F is a k-CNF formula and $\text{occ}(u) \leq \frac{2^k}{4k}$ *for all literals u, then F is satisfiable.*

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- By the Lovász Local Lemma, F is satisfiable. $\square$

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Goal: F has "few" conflicts $\Longrightarrow$ F is satisfiable.

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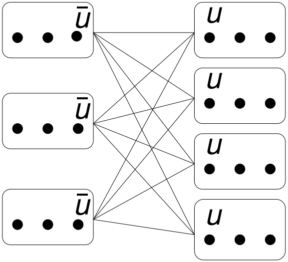

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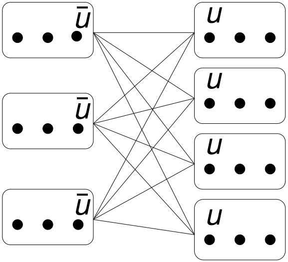

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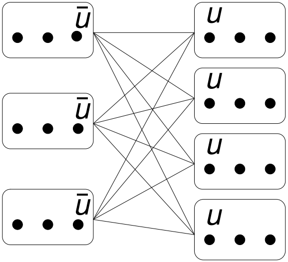

- F has at least $\text{occ}(u)\text{occ}(\bar{u})$ conflicts.
- If F has $\ll 4^k$ conflicts, $\text{occ}(\bar{u}) \ll 2^k$.

Exploiting the Unbalancedness of Variables

- Number of conflicts $\geq \mathrm{occ}(u)\mathrm{occ}(\bar{u})$.
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- Clause D is bad if $p(u) < \frac{1}{2}$ for *at least one* u.
  Note: $\Pr[D$ not satisfied$]$ may be "big".

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- $\sum_{D \in N(C)} \Pr[D \text{ not satisfied}]$ is small.

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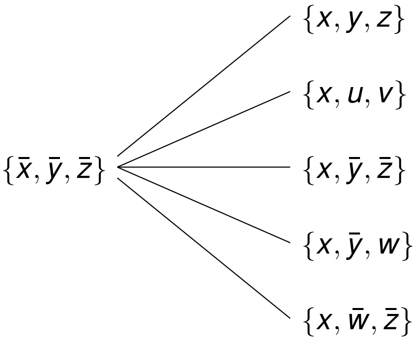

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- Define an appropriate probability distribution, exploiting the unbalancedness of certain variables.
- Remove certain literals $\Rightarrow$ new formula F'.
- This "sparsifies" the conflict structure of F.
- Use Lovász Local Lemma to show that F' is satisfiable.

Theorem (Lower Bound)

If a k-CNF F has at most $\mathcal{O}\left(2.69^k\right)$ conflicts, it is satisfiable.

Theorem (Upper Bound)

For every k, there is an unsatisfiable k-CNF formula with $\mathcal{O}\left(\frac{\log^2(k)4^k}{k}\right)$ conflicts.

That's It

Thank You For Your Attention!