Mathematical Foundations
of
Computer Science

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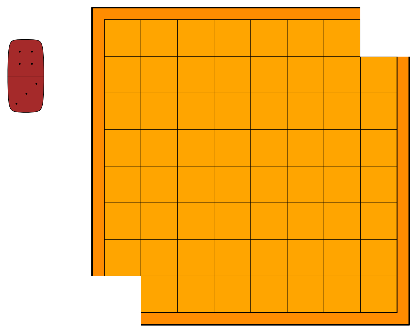
- Homework assignment published on Monday, 2018-02-26
- Submit questions or first draft solutions by Sunday, 2018-03-04, 18:00 by email to the TA and to me (dominik.scheder@gmail.com)
- We will discuss some problems on Monday, 2018-03-05.
- You will receive feedback by Wednesday, 2018-03-08.
- Revise your solution and hand in your final submission by Sunday, 2018-03-11.

1 Broken Chessboard and Jumping With Coins

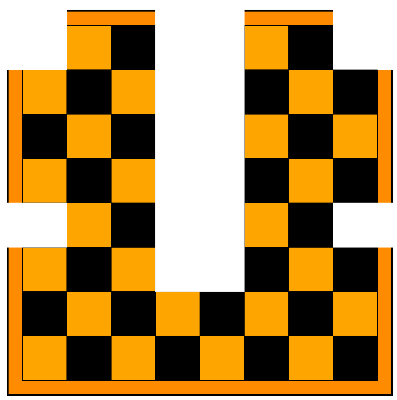
1.1 Tiling a Damaged Checkerboard

In the video lecture you have seen a proof that one cannot tile the "damaged" 8×8 checkerboard with domino stones:



Exercise 1.1. Re-write the proof in your own way, using simple English sentences.

Exercise 1.2. Look at the seriously damaged 8×8 checkerboard. For convenience I already colored it black and white (or rather black and beige):



Try to tile it with domino stones and you will fail. However, since there are 24 black and 24 beige squares, the simple argument from the lecture will fail.

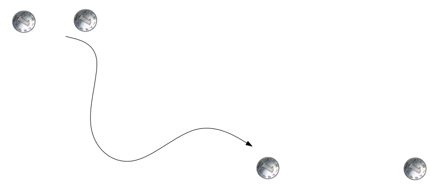
Prove that the above board cannot be tiled. Try to find a short and simple argument!

1.2 Jumping with Coins

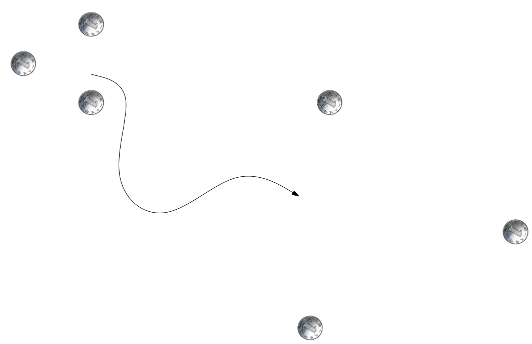
This is simply a reminder of the exercises I posed in the video lecture. For details, refer to the video.

Remark. The following exercises are of different levels of difficulty, but they all have a very simple proof (altough the proof might not be easy to find).

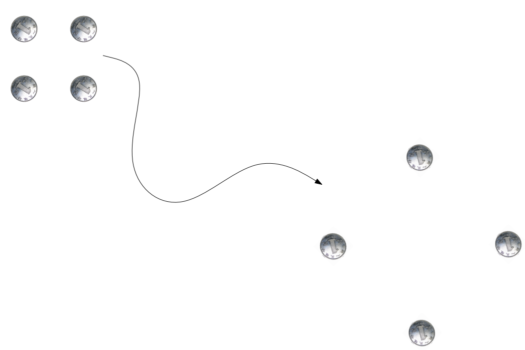
Exercise 1.3. You jump around with two coins. Show that you cannot increase the distance between the two coins.



Exercise 1.4. You jump around with three coins. Show that you cannot start with an equilateral triangle and end up with a bigger equilateral triangle. Give a simple proof!



You jump around with four coins which in the beginning form a square of side length 1.



Exercise 1.5. Show that you cannot form a square of side length 2.

Exercise 1.6. Show that you cannot achieve a position in which two coins are at the same position.

Exercise 1.7. Show that you cannot form a larger square.

2 Exclusion-Inclusion

2.1 Sets

Exercise 2.1. Let A, B, C be finite sets.

1. Prove that $|A \cup B| = |A| + |B| - |A \cap B|$.
2. What about $|A \cup B \cup C|$? Find a formula in terms of pairwise and three-wise intersections.
3. What about $|A \cup B \cup C \cup D|$? Find a formula in terms of pairwise, three-wise, and four-wise intersections.

Exercise 2.2. [The Exclusion-Inclusion Formula] Maybe you have noticed a pattern. Find a general formula, i.e., for $|A_1 \cup \cdots \cup A_n|$ in terms of the size of intersections $A_I := \bigcap_{i \in I} A_i$.

Exercise 2.3. Justify the formula you found in the previous exercise. **Hint.** There is a proof using induction on n. **Hint.** There is a proof that does not need induction on n.

3 Feasible Intersection Patterns

Exercise 3.1. Find sets A_1, A_2, A_3, A_4 such that all pairwise intersections have size 3 and all three-wise intersections have size 1. Formally,

1. $|A_i \cap A_j| = 3$ for all $\{i, j\} \in \binom{[4]}{2}$,
2. $|A_i \cap A_j \cap A_k| = 1$ for all $\{i, j, k\} \in \binom{[4]}{3}$.

Exercise 3.2. Show that if we insist that $|A_i| = 5$ for all i, then the task from the above exercise cannot be solved.

In the spirit of the previous questions, let us call a sequence $(a_1, a_2, \ldots, a_n) \in \mathbb{N}_0$ *feasible* if there are sets $A_1, \ldots, A_n$ such that all k-wise intersections have size a_k. That is, $|A_i| = a_1$ for all i, $|A_i \cap A_j| = a_2$ for all $i \neq j$ and so on. The previous exercise would thus state that $(5, 3, 1, 0)$ is not feasible, but $(6, 3, 1, 0)$ is, as one solution of Exercise 3.1 shows.

***Exercise 3.3.** *Suppose I give you a sequence* $(a_1, \ldots, a_n)$*. Find a way to determine whether such a sequence is feasible or not.*

Since this exercise might be too difficult, so I will talk about it in our Thursday class and break it into more manageable parts.