Mathematical Foundations of Computer Science

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10 Network Flow

- Homework assignment published on Monday 2018-05-07
- Submit questions and first solution by Sunday, 2018-05-13, 12:00
- Submit final solution by Sunday, 2018-05-20.

Exercise 10.1. [From the video lecture] Recall the definition of the value of a flow: $\text{val}(f) = \sum_{v \in V} f(s, v)$. Let $S \subseteq V$ be a set of vertices that contains s but not t. Show that

$$\text{val}(f) = \sum_{u \in S, v \in V \setminus S} f(u, v) \ .$$

That is, the total amount of flow leaving s equals the total amount of flow going from S to $V \setminus S$. **Remark.** It sounds obvious. However, find a formal proof that works with the axiomatic definition of flows.

Exercise 10.2. Let $G = (V, E, c)$ be a flow network. Prove that flow is "transitive" in the following sense: If there is a flow from s to r of value k, and a flow from r to t of value k, then there is a flow from s to t of value k. **Hint.** The solution is extremely short. If you are trying something that needs more than 3 lines to write, you are on the wrong track.

10.1 An Algorithm for Maximum Flow

Recall the algorithm for Maximum Flow presented in the video. It is usually called the Ford-Fulkerson method.

Algorithm 1 Ford-Fulkerson Method

procedure FF$(G = (V, E), s, t, c)$
    Initialize f to be the all-0-flow.
    while there is a path p form s to t in the residual network G_f **do**
        $c_{\min} := \min\{c_f(e) \mid e \in p\}$
        let f_p be the flow in G_f that routes $c_{\min}$ flow along p
        $f := f + f_p$
    end while
    // now f is a maximum flow
    $S := \{v \in V \mid G_f$ contains a path from s to $v\}$
    // S is a minimum cut
    return (f, S)
end procedure

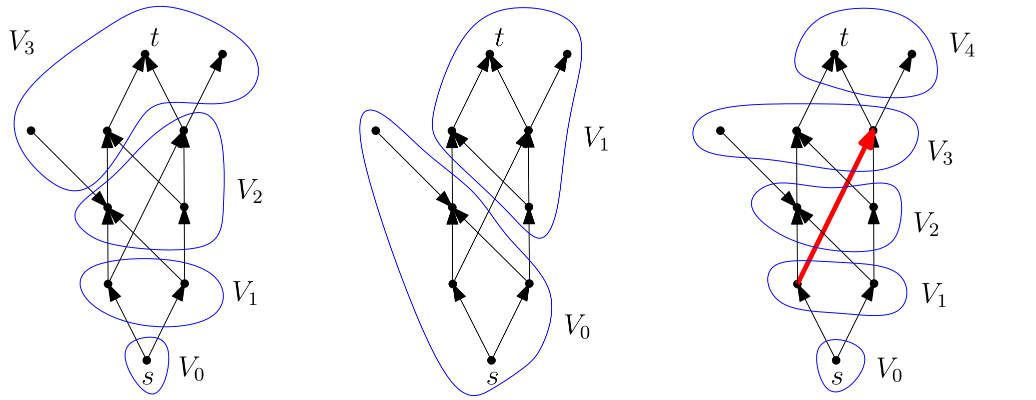
We proved in the lecture that f is a maximum flow and S is a minimum cut, by showing that upon termination of the while-loop, $\text{val}(f) = \text{cap}(S)$. The problem is that the while-loop might not terminate. In fact, there is an example with capacities in $\mathbb{R}$ for which the while loop does not terminate, and the value of f does not even converge to the value of a maximum flow. As indicated in the video, a little twist fixes this:

> **Edmonds-Karp Algorithm:** Execute the above Ford-Fulkerson Method, but in every iteration choose p to be a shortest s-t-path in G_f. Here, "shortest" means minimum number of edges.

In a series of exercises, you will now show that this algorithm always terminates after at most $n \cdot m$ iterations of the while loop (here $n = |V|$ and $m = |E|$).

Definition 10.3. *Let (G, s, t, c) be a flow network and $k \in \mathbb{N}_0$. A* k-layering *is a partition of $V = V_0 \cup \cdots \cup V_k$ such that (1) $s \in V_0$, (2) $t \in V_k$, (3) for every edge $(u, v) \in E$ the following holds: suppose $u \in V_i$ and $v \in V_j$. Then $j \leq i + 1$. In words, point (3) states that every edge moves at most one level forward.*

The figure below illustrates this concept: for one network we show two possible layerings and something that looks like a layering but is not:



A 3-layering. A 1-layering. Not a layering: the fat red edge "jumps ahead".

Exercise 10.4. Suppose the network (G, s, t, c) has a k-layering. Show that $\text{dist}(s, t) \geq k$. That is, every s-t-path in G has at most k edges.

Exercise 10.5. Conversely, suppose $\text{dist}(s, t) = k$. Show that (G, s, t, c) has a k-layering.

Let (G, s, t, c) be a flow network and $V_0, \ldots, V_k$ a k-layering. We call this layering *optimal* if $\text{dist}_G(s, t) = k$. Here, $\text{dist}_G(u, v)$ is the shortest-path distance from s to t (measured by number of edges). If there is no path from s to t, we set $\text{dist}_G(s, t) = \infty$. In this case, no layering is optimal. For example, the 3-layering in the above figure is optimal, but the 1-layering in the middle of the above figure is not. Let us explore how layerings and the Ford-Fulkerson Method interact.

Exercise 10.6. Let (G, s, t, c) be a flow network and $V_0, V_1, \ldots, V_k$ be an optimal layering (that is, $k = \text{dist}_G(s, t)$. Let p be a path from s to t of length k. Suppose we route some flow f along p (of some value $c_{\min} > 0$) and let (G_f, s, t, c_f) be the residual network. Show that $V_0, V_1, \ldots, V_k$ is a layering of (G_f, s, t, c_f), too. Obviously, condition (1) and (2) in the definition of k-layerings still hold, so you only have to check condition (3).

Exercise 10.7. Show that every network (G, s, t, c) has an optimal layering, provided there is a path from s to t.

Exercise 10.8. Imagine we are in some iteration of the while-loop of the Edmonds-Karp algorithm. Let $V_0, \ldots, V_k$ be an optimal layering of (G, s, t, c). Show that after at most m iterations of the while-loop, $V_0, \ldots, V_k$ ceases to be an optimal layering. **Remark.** Note that it is the *network* that changes from iteration to iteration of the while-loop, not the partition $V_0, \ldots, V_k$. We consider the partition $V_0, \ldots, V_k$ to be fixed in this exercise.

Exercise 10.9. Show that the Edmonds-Karp algorithm terminates after $n \cdot m$ iterations of the while-loop. **Hint.** Initially, compute an optimal k-layering (which?). Then keep this layering as long as its optimal. Once it ceases to be optimal, compute a new optimal layering. Note that the Edmonds-Karp algorithm does not actually need to compute any layering. It's us who compute it to show that $n \cdot m$ bound on the number of iterations.

Exercise 10.10. Show that every network has a maximum flow f. That is, a flow f such that $\mathrm{val}(f) \geq \mathrm{val}(f')$ for every flow f'. **Remark.** This sounds obvious but it is not. In fact, there might be an infinite sequence of flows $f_1, f_2, f_3, \ldots$ of increasing value that does not reach any maximum. Use the previous exercises!