Algorithms (II)

Difficult Problems

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Efficient Problems, Difficult Problems
Efficient Algorithms

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  • Finding shortest paths in graphs,
  • Minimum spanning trees in graphs,
  • Matchings in bipartite graphs,
  • Maximum increasing subsequences,
  • Maximum flows in networks,
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• All these algorithms are efficient, because in each case their time requirement grows as a polynomial function (such as $n$, $n^2$, or $n^3$) of the size of the input.
Exponential Search Space

• In all these problems we are searching for a solution (path, tree, matching,) from among an exponential population of possibilities.

Other "search problems" in which again we are seeking a solution with particular properties among an exponential chaos of alternatives.

The fastest algorithms we know for them are all exponential.
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Minimum Spanning Trees
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- The goal is to pick enough of these edges that the nodes are connected, the total maintenance cost is minimum.

One immediate observation is that the optimal set of edges cannot contain a cycle.
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![Graph diagram with labeled nodes and edges]
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A Greedy Approach

- Kruskal’s minimum spanning tree algorithm starts with the empty graph and then selects edges from $E$ according to the following rule.

Example:
Starting with an empty graph and then attempt to add edges in increasing order of weight:
- $B \rightarrow C$
- $C \rightarrow D$
- $B \rightarrow D$
- $C \rightarrow F$
- $D \rightarrow F$
- $E \rightarrow F$
- $A \rightarrow D$
- $A \rightarrow B$
- $C \rightarrow E$
- $A \rightarrow C$
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- **Repeatedly add the next lightest edge that doesn’t produce a cycle.**
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$B - C; C - D; B - D; C - F; D - F; E - F; A - D; A - B; C - E; A - C$
A General Kruskal’s Algorithm

\[ X = \{ \}; \]
repeat until \(|X| = |V| - 1; \]
\quad pick a set \(S \subset V\) for which \(X\) has no edges between \(S\) and \(V - S\);
\quad let \(e \in E\) be the minimum-weight edge between \(S\) and \(V - S\);
\[ X = X \cup \{e\}; \]
Prim’s Algorithm

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• On each iteration, the subtree defined by $X$ grows by one edge, namely, the lightest edge between a vertex in $S$ and a vertex outside $S$. We can equivalently think of $S$ as growing to include the vertex $v \notin S$ of smallest cost:

$$\text{cost}(v) = \min_{u \in S} w(u, v)$$
A Little Change of the MST

What if the tree is not allowed to branch?
Satisfiability Problem
Satisfiability

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The instances of *Satisfiability* or *SAT*:

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- Given a Boolean formula in conjunctive normal form, either find a satisfying truth assignment or else report that none exists.
2-SAT

Given a set of clauses, where each clause is the disjunction of two literals. You are looking for a way to assign a value true or false to each of the variables so that all clauses are satisfied. That is, there is at least one true literal in each clause.

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- \(G_I\) has \(2m\) edges: for each clause \((\alpha \lor \beta)\) of \(I\), \(G_I\) has an edge from the negation of \(\alpha\) to \(\beta\), and one from the negation of \(\beta\) to \(\alpha\).
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• Show that if $G_I$ has a strongly connected component containing both $x$ and $\bar{x}$ for some variable $x$, then $I$ has no satisfying assignment.

• If none of $G_I$'s strongly connected components contain both a literal and its negation, then the instance $I$ must be satisfiable.

• Conclude that there is a linear-time algorithm for solving 2-SAT.
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Search Problems (Decision Problems)
Satisfiability

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  - an object that meets a particular specification,
  - in this case an assignment that satisfies each clause.
- **If no such solution exists, we must say so.**
Search Problems

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• There is a polynomial-time algorithm that takes as input $I$ and $S$ and decides whether or not $S$ is a solution of $I$.
  • For SAT, this is easy as it just involves checking whether the assignment specified by $S$ indeed satisfies every clause in $I$. 
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We say $S$ is a solution to $I$ if and only if $C(I, S) = true$. 
Satisfiability Revisit

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  • 2-SAT, however, can be solved in linear time.
  • If all clauses contain at most one positive literal, then the Boolean formula is called a Horn formula, and a satisfying truth assignment, if one exists, can be found by the greedy algorithm.
Traveling Salesman Problem
The Traveling Salesman Problem

- In the traveling salesman problem (TSP), we are given $n$ vertices and all $n(n-1)/2$ distances between them, and a budget $b$.
- To find a cycle that passes through every vertex exactly once, of total cost $b$ or less - or to report that no such cycle.
- A permutation $\tau(1), \ldots, \tau(n)$ of the vertices such that when they are toured in this order, the total distance covered is at most $b$:
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But why are we expressing the TSP in this way, when in reality it is an optimization problem, in which the shortest possible tour is sought?
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• Any algorithm that solves the optimization also readily solves the search problem:

  • First suppose that we somehow knew the cost of the optimum tour; then we could find this tour by calling the algorithm for the search problem, using the optimum cost as the budget.
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- Given a potential solution to the TSP, it is easy to check the properties “is a tour” (just check that each vertex is visited exactly once) and “has total length $\leq b$.”
- But how could one check the property ”is optimal”?
TSP Revisit

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• The TSP can be thought of as a tough cousin of the MST problem, in which the tree is not allowed to branch and is therefore a path.
• This extra restriction on the structure of the tree results in a much harder problem.
Euler and Rudrata
Euler Path

Euler path:
Given a graph, find a path that contains each edge exactly once.
Euler Path

- The answer is yes if and only if:
  1. the graph is connected and
  2. every vertex, with the possible exception of two vertices (the start and final vertices of the walk), has even degree.

- Using above, it is easy to see that there is a polynomial time algorithm for Euler path.
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**Rudrata Cycle:**

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Given a graph, find a cycle that visits each vertex exactly once.

In the literature this problem is known as the Hamilton cycle problem.
Cuts and Bisections
Minimum Cut

- A cut is a set of edges whose removal leaves a graph disconnected.
- Minimum cut: given a graph and a budget $b$, find a cut with at most $b$ edges.
Minimum Cut

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![Minimum Cut Diagram](image-url)
Minimum Cut

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• **Minimum cut**: given a graph and a budget $b$, find a cut with at most $b$ edges.
Minimum Cut

This problem can be solved in polynomial time by $n - 1$ max-flow computations:

• give each edge a capacity of 1,
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Linear Programming

\[
\begin{align*}
\text{max } & \quad x_1 + 6x_2 + 13x_3 \\
\text{subject to } & \quad x_1 \leq 200 \\
& \quad x_2 \leq 300 \\
& \quad x_1 + x_2 + x_3 \leq 400 \\
& \quad x_2 + 3x_3 \leq 600 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
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\text{max } & \quad 2x_1 + 5x_2 \\
2x_1 - x_2 & \leq 4 \\
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• We want to find a nonnegative integer $n$-vector $x$ such that $Ax \leq b$ and $c \cdot x \geq g$.

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But there is a redundancy here:

- the last constraint $c \cdot x \geq g$ is itself a linear inequality and
- can be absorbed into $A x \leq b$.
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Integer Linear Programming

So, we define ILP to be following search problem:

Given $A$ and $b$, find a nonnegative integer vector $x$ satisfying the inequalities $Ax \leq b$. 
3D-Matching
Bipartite Matching

BOYS
- Al
- Bob
- Chet
- Dan

GIRLS
- Alice
- Beatrice
- Carol
- Danielle
Bipartite Matching
Three-Dimensional Matching

• 3D matching:

- There are $n$ boys and $n$ girls, but also $n$ pets, and the compatibilities among them are specified by a set of triples, each containing a boy, a girl, and a pet.
- Intuitively, a triple $(b, g, p)$ means that boy $b$, girl $g$, and pet $p$ get along well together.
- We want to find $n$ disjoint triples and thereby create $n$ harmonious households.
Three-Dimensional Matching

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<table>
<thead>
<tr>
<th>Boy</th>
<th>Girl</th>
<th>Pet</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bob</td>
<td>Alice</td>
<td>Canary</td>
</tr>
<tr>
<td>Al</td>
<td>Carol</td>
<td>Armadillo</td>
</tr>
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![Diagram showing relationships between boys, girls, and pets]
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Independent Set, Vertex Cover, and Clique

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• **Clique**: Given a graph and an integer $g$, find $g$ vertices such that all possible edges between them are present.
Longest Path
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• Longest path: Given a graph $G$ with nonnegative edge weights and two distinguished vertices $s$ and $t$, along with a goal $g$.

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  - e.g., by writing $\text{IIIIIIIIII}$ for 12.
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  - Suppose now that each item’s value is equal to its weight (all given in binary), the goal $g$ is the same as the capacity $W$.
  - This special case is tantamount to finding a subset of a given set of integers that adds up to exactly $W$. Q: Could it be polynomial?
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Referred Materials

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