Algorithm Design and Analysis I

Prologue

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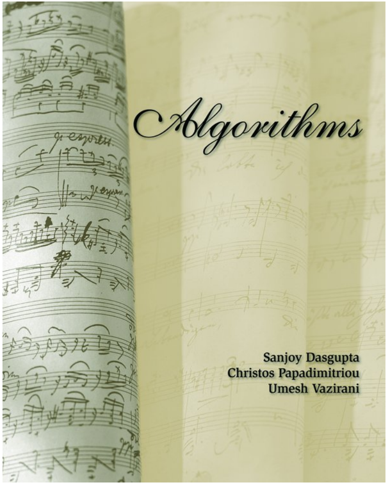
Office hour: Wed. 14:00-17:00 @ SEIEE 3-325

Reference Book

Textbook

Algorithms

- Sanjoy Dasgupta
- San Diego Christos Papadimitriou
- Umesh Vazirani
- McGraw-Hill, 2007.



Reference Book

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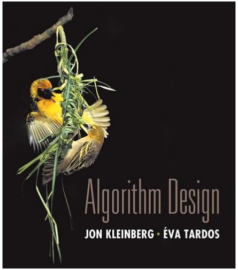
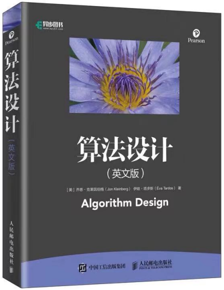
Algorithm Design

- Jon Kleinberg, Éva Tardos
- Addison-Wesley, 2005.





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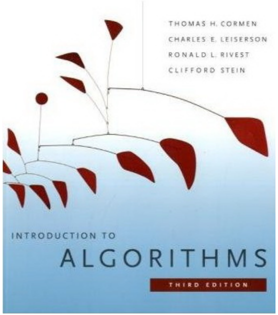
Introduction to Algorithms

- Thomas H. Cormen
- Charles E. Leiserson
- Ronald L. Rivest
- Clifford Stein
- The MIT Press (3rd edition), 2009.



Scoring Policy

0% Attendees.

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40% Homework.

- Eight assignments.
- Each one is 5pts.
- Work out individually.
- Each assignment will be evaluated by A, B, C, D, F (Excellent(5), Good(5), Fair(4), Delay(3), Fail(0))

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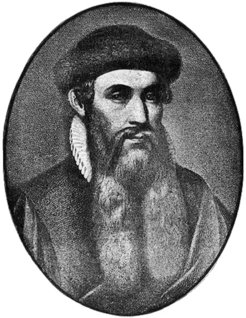
60% Final exam.

Any Questions?

Two Things Change the World

Johann Gutenberg



Johann Gutenberg (1398 - 1468)

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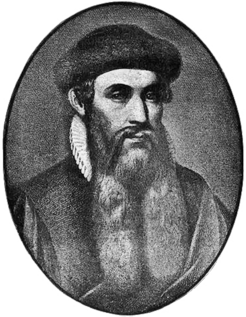

Johann Gutenberg (1398 - 1468)

In 1448 in the German city of Mainz a goldsmith named Johann Gutenberg discovered a way to print books by putting together movable metallic pieces.

Sheng BI

Bì Shēng (972-1051)

Bì Shēng was a Chinese artisan, engineer, and inventor of the world's first movable type technology, with printing being one of the Four Great Inventions of Ancient China.

Two Ideas Changed the World

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Because of the typography, literacy spread, the Dark Ages ended, the human intellect was liberated, science and technology triumphed, the Industrial Revolution happened.

Many historians say we owe all this to typography.

Others insist that the key development was not typography, but algorithms.

Decimal System

Gutenberg would write the number 1448 as $MCDXLVIII$.

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How to add two Roman numerals? What is

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The decimal system was invented in India around AD 600. Using only 10 symbols, even very large numbers were written down compactly, and arithmetic is done efficiently by elementary steps.

Al Khwarizmi

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In the 12th century, Latin translations of his work on the Indian numerals, introduced the decimal system to the Western world. (Source: Wikipedia)

Algorithms

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Al Khwarizmi laid out the basic methods for

- adding,
- multiplying,
- dividing numbers,
- extracting square roots,
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They were algorithms, a term coined to honor the wise man after the decimal system was finally adopted in Europe, many centuries later.

Chongzhi ZU

Chongzhi ZU (429 – 500)

A Chinese astronomer, inventor, mathematician, politician, and writer during the Liu Song and Southern Qi dynasties. He was most notable for calculating π as between 3.1415926 and 3.1415927, a record in precision which would not be surpassed for nearly 900 years.

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An abstract recipe, prescribing a process which may be carried out by a human, a computer or by other means.

Any well-defined computational procedure that makes some value, or set of values, as input and produces some value, of set of values, as output. An algorithm is thus a finite sequence of computational steps that transform the input into the output.

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A program is

- an implementation of an algorithm, or algorithms.
- A program does not necessarily terminate.

Fibonacci Algorithm

Leonardo Fibonacci

Leonardo Fibonacci (1170 - 1250)

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Fibonacci helped the spread of the decimal system in Europe, primarily through the publication in the early 13th century of his Book of Calculation, the Liber Abaci. (Source: Wikipedia)

Fibonacci Sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

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Formally,

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n > 1 \end{cases}$$

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Q: What is F_{100} or F_{200}?

An Exponential Algorithm


```
FIBO1(n)
a nature number n;

if n = 0 then return(0);
if n = 1 then return(1);
return(FIBO1(n − 1)+FIBO1(n − 2));
```

Three Questions about An Algorithm

1. Is it correct?
2. How much time does it take, as a function of n?
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The first question is trivial, as this algorithm is precisely Fibonacci's definition of F_n

How Much Time

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It is exponential to n.

Why Exponential Is Bad?

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$$T(200) \geq F_{200} \geq 2^{138} \approx 2.56 \times 10^{42}$$

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In 2022, the fastest is Frontier, 1.102×10^{18} per second.

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Such is the curse of exponential time.

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Now we know $\texttt{FIB1}(n)$ is correct and inefficient, so can we do better?

An Polynomial Algorithm


```
FIBO2(n)
a nature number n;

if n = 0 then return(0);
create an array f[0...n];
f[0] = 0; f[1] = 1;
for i = 2 to n do
    f[i] = f[i-1] + f[i-2];
end
return(f[n]);
```

An Analysis

The correctness of `FIBO2` is trivial.

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How long does it take?

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The correctness of `FIBO2` is trivial.

How long does it take?

The inner loop consists of a single computer step and is executed $n - 1$ times. Therefore the number of computer steps used by `FIBO2` is linear in n.

A More Careful Analysis

We count the number of basic computer steps executed by each algorithm and regard these basic steps as taking a constant amount of time.

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The n-th Fibonacci number is about $0.694n$ bits long, and this can far exceed 32 as n grows.

Arithmetic operations on arbitrarily large numbers cannot possibly be performed in a single, constant-time step.

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The addition of two n-bit numbers takes time roughly proportional to n (next lecture).

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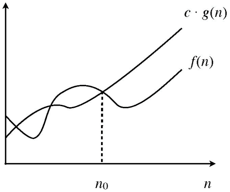
Q: Can we do better?

- Exercise 0.4

Big-O Notation

Big O notation

Upper bounds. $f(n)$ is $O(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that $0 \leq f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.



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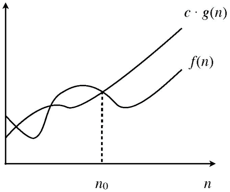

Example

Let $f(n) = 32n^2 + 17n + 1$.

- $f(n)$ is $O(n^2)$.
- $f(n)$ is neither $O(n)$ nor $O(n \log n)$.

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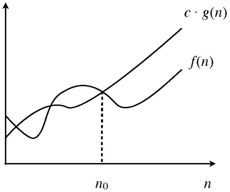

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- $f(n)$ is $O(n^2)$.
- $f(n)$ is neither $O(n)$ nor $O(n \log n)$.

Typical usage. Insertion sort makes $O(n^2)$ compares to sort n elements.

Quiz

Let $f(n) = 3n^2 + 17n \log_2 n + 1000$. Which of the following are true?

- **A** $f(n)$ is $O(n^2)$.
- **B** $f(n)$ is $O(n^3)$.
- **C** Both A and B.
- **D** Neither A nor B.

Big O notational abuses

One-way "equality". $O(g(n))$ is a set of functions, but computer scientists often write $f(n) = O(g(n))$ instead of $f(n) \in O(g(n))$.

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Example

Consider $g_1(n) = 5n^3$ and $g_2(n) = 3n^2$.

- We have $g_1(n) = O(n^3)$ and $g_2(n) = O(n^3)$.
- But, do not conclude $g_1(n) = g_2(n)$.

Big O notation: properties

Reflexivity. f is $O(f)$.

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Proof.

- $\exists c_1 > 0$ and $n_1 \geq 0$ such that $0 \leq f_1(n) \leq c_1 \cdot g_1(n)$ for all $n \geq n_1$.
- $\exists c_2 > 0$ and $n_2 \geq 0$ such that $0 \leq f_2(n) \leq c_2 \cdot g_2(n)$ for all $n \geq n_2$.
- Then, $0 \leq f_1(n) \cdot f_2(n) \leq c_1 \cdot c_2 \cdot g_1(n) \cdot g_2(n)$ for all $n \geq \max\{n_1, n_2\}$.

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- Then, $0 \leq f_1(n) \cdot f_2(n) \leq c_1 \cdot c_2 \cdot g_1(n) \cdot g_2(n)$ for all $n \geq \max\{n_1, n_2\}$.

Sums. If f_1 is $O(g_1)$ and f_2 is $O(g_2)$, then $f_1 + f_2$ is $O(\max\{g_1, g_2\})$.

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- Then, $0 \leq f_1(n) \cdot f_2(n) \leq c_1 \cdot c_2 \cdot g_1(n) \cdot g_2(n)$ for all $n \geq \max\{n_1, n_2\}$.

Sums. If f_1 is $O(g_1)$ and f_2 is $O(g_2)$, then $f_1 + f_2$ is $O(\max\{g_1, g_2\})$.

Transitivity. If f is $O(g)$ and g is $O(h)$, then f is $O(h)$.

Big O notation: properties

Reflexivity. f is $O(f)$.

Constants. If f is $O(g)$ and $c > 0$, then $c\,f$ is $O(g)$.

Products. If f_1 is $O(g_1)$ and f_2 is $O(g_2)$, then $f_1 f_2$ is $O(g_1 g_2)$.

Proof.

- $\exists c_1 > 0$ and $n_1 \geq 0$ such that $0 \leq f_1(n) \leq c_1 \cdot g_1(n)$ for all $n \geq n_1$.
- $\exists c_2 > 0$ and $n_2 \geq 0$ such that $0 \leq f_2(n) \leq c_2 \cdot g_2(n)$ for all $n \geq n_2$.
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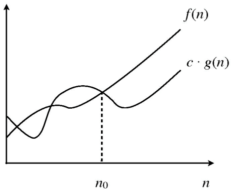
Sums. If f_1 is $O(g_1)$ and f_2 is $O(g_2)$, then $f_1 + f_2$ is $O(\max\{g_1, g_2\})$.

Transitivity. If f is $O(g)$ and g is $O(h)$, then f is $O(h)$.

Ex. $f(n) = 5n^3 + 3n^2 + n + 1234$ is $O(n^3)$.

Big Ω notation

Lower bounds. $f(n)$ is $\Omega(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that $f(n) \geq c \cdot g(n) \geq 0$ for all $n \geq n_0$.



Big Ω notation

Shanghai Jiao Tong University

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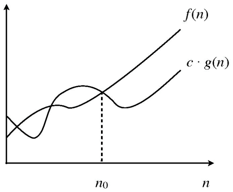

Example

Let $f(n) = 32n^2 + 17n + 1$.

- $f(n)$ is both $\Omega(n^2)$ and $\Omega(n)$.
- $f(n)$ is not $\Omega(n^3)$.

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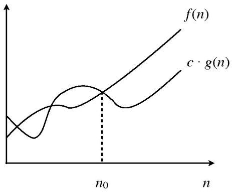

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Let $f(n) = 32n^2 + 17n + 1$.

- $f(n)$ is both $\Omega(n^2)$ and $\Omega(n)$.
- $f(n)$ is not $\Omega(n^3)$.

Typical usage. Any compare-based sorting algorithm requires $\Omega(n \log n)$ compares in the worst case.

Quiz

Which is an equivalent definition of big Omega notation?

A $f(n)$ is $\Omega(g(n))$ iff $g(n)$ is $O(f(n))$.

B $f(n)$ is $\Omega(g(n))$ iff there exist constants $c > 0$ such that

$$f(n) \geq c \cdot g(n) \geq 0$$

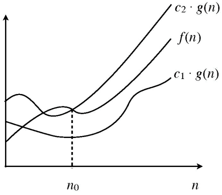
for infinitely many n.

C Both A and B.

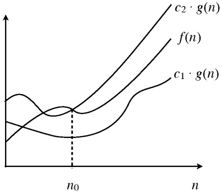
D Neither A nor B.

Big Θ notation

Tight bounds. $f(n)$ is $\Theta(g(n))$ if there exist constants $c_1 > 0, c_2 > 0$, and $n_0 \geq 0$ such that $0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ for all $n \geq n_0$.



Big Θ notation



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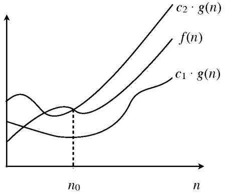
Example

Let $f(n) = 32n^2 + 17n + 1$.

- $f(n)$ is $\Theta(n^2)$.
- $f(n)$ is neither $\Theta(n^3)$ nor $\Omega(n)$.

Big Θ notation



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Example

Let $f(n) = 32n^2 + 17n + 1$.

- $f(n)$ is $\Theta(n^2)$.
- $f(n)$ is neither $\Theta(n^3)$ nor $\Omega(n)$.

Typical usage. Mergesort makes $\Theta(n \log n)$ compares to sort n elements.

Quiz

Which is an equivalent definition of big Theta notation?

A $f(n)$ is $\Theta(g(n))$ iff $f(n)$ is both $O(g(n))$ and $\Omega(g(n))$.

B $f(n)$ is $\Theta(g(n))$ iff $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$ for some constant $0 < c < +\infty$.

C Both A and B.

D Neither A nor B.

Asymptotic bounds and limits

Proposition

If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$ *for some constant* $0 < c < \infty$ *then* $f(n)$ *is* $\Theta(g(n))$.

Asymptotic bounds and limits

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Proof.

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Proof.

By definition of the limit, for any $\varepsilon > 0$, there exists n_0 such that

$$c - \varepsilon \leq \frac{f(n)}{g(n)} \leq c + \varepsilon$$

for all $n \geq n_0$.

Choose $\varepsilon = 1/2c > 0$.

Multiplying by $g(n)$ yields $1/2c \cdot g(n) \leq f(n) \leq 3/2c \cdot g(n)$ for all $n \geq n_0$.

Thus, $f(n)$ is $\Theta(g(n))$ by definition, with $c_1 = 1/2c$ and $c_2 = 3/2c$.

Asymptotic bounds and limits

Proposition

If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, *then* $f(n)$ *is* $O(g(n))$ *but not* $\Omega(g(n))$.

Proposition

If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$, *then* $f(n)$ *is* $\Omega(g(n))$ *but not* $O(g(n))$.

Asymptotic bounds for some common functions

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Polynomials. Let $f(n) = a_0 + a_1 n + \ldots + a_d n^d$ with $a_d > 0$. Then, $f(n)$ is $\Theta(n^d)$.

Asymptotic bounds for some common functions

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$$\lim_{n \to \infty} \frac{a_0 + a_1 n + \ldots + a_d n^d}{n^d} = a_d > 0$$

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Logarithms and polynomials. $\log_a n$ is $O(n^d)$ for every $a > 1$ and every $d > 0$.

$$\lim_{n \to \infty} \frac{\log_a n}{n^d} = 0$$

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Logarithms and polynomials. $\log_a n$ is $O(n^d)$ for every $a > 1$ and every $d > 0$.

$$\lim_{n\to\infty} \frac{\log_a n}{n^d} = 0$$

Exponentials and polynomials. n^d is $O(r^n)$ for every $r > 1$ and every $d > 0$.

$$\lim_{n\to\infty} \frac{n^d}{r^n} = 0$$

Asymptotic bounds for some common functions

Factorials. $n!$ is $2^{\Theta(n \log n)}$.

Asymptotic bounds for some common functions

Factorials. $n!$ is $2^{\Theta(n \log n)}$.

Stirling's formula:

$$n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$$

Big O notation with multiple variables

Upper bounds. $f(m,n)$ is $O(g(m,n))$ if there exist constants $c > 0$, $m_0 \geq 0$, and$n_0 \geq 0$ such that $f(m,n) \leq c \cdot g(m,n)$ for all $n \geq n_0$ and $m \geq m_0$.

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Example

$f(m,n) = 32mn^2 + 17mn + 32n^3$.

- $f(m,n)$ is both $O(mn^2 + n^3)$ and $O(mn^3)$.
- $f(m,n)$ is neither $O(n^3)$ nor $O(mn^2)$.

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Example

$f(m,n) = 32mn^2 + 17mn + 32n^3$.

- $f(m,n)$ is both $O(mn^2 + n^3)$ and $O(mn^3)$.
- $f(m,n)$ is neither $O(n^3)$ nor $O(mn^2)$.

Typical usage. Breadth-first search takes $O(m+n)$ time to find a shortest path from s to t in a digraph with n nodes and m edges.