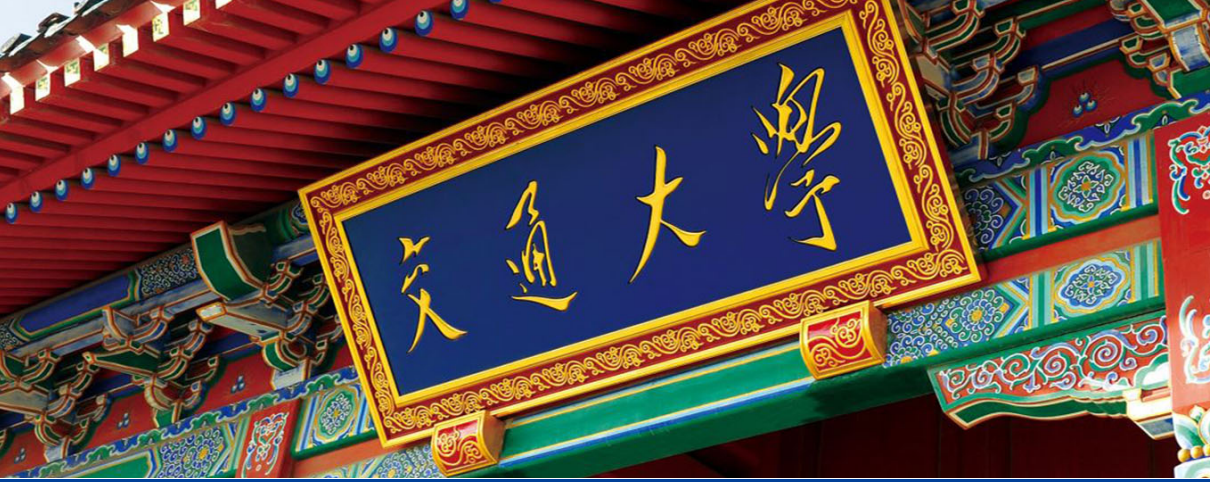

Algorithm Design and Analysis II

Algorithms with Numbers I

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Two Seemingly Similar Problems

Factoring: Given a number N, express it as a product of its prime factors.

Primality: Given a number N, determine whether it is a prime.

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Primality: Given a number N, determine whether it is a prime.

We believe that Factoring is hard and much of the electronic commerce is built on this assumption.

There are efficient algorithms for Primality, e.g., AKS test by Manindra Agrawal, Neeraj Kayal, and Nitin Saxena.

A Notable Result

The AKS primality test is a deterministic primality-proving algorithm created and published by Manindra Agrawal, Neeraj Kayal, and Nitin Saxena, computer scientists at the Indian Institute of Technology Kanpur, on August 6, 2002, The algorithm was the first to determine whether any given number is prime or composite within polynomial time. The authors received the 2006 Gödel Prize and the 2006 Fulkerson Prize for this work.

Preliminaries

How to Represent Numbers

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- 1024

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The bigger the base is, the shorter the representation is. But how much do we really gain by choosing large base?

Q: How many digits are needed to represent the number $N \geq 0$ in base b?

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Bases and Logs

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In O notation, the base is irrelevant, and thus we write the size simply as $O(\log N)$

Roles of $Log\ N$

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It is also the depth of a complete binary tree with N nodes. (More precisely: $\lfloor log\ N \rfloor$.)

It is even the sum $1 + 1/2 + 1/3 + \ldots + 1/n$, to within a constant factor.

Basics Arithmetic

Lemma

The sum of any three single-digit number is at most two digits long.

Addition

This rule holds not just in decimal but in any $b \geq 2$.

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In binary, the maximum possible sum of three single-bit numbers is 3, which is a 2-bit number.

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This simple rule gives us a way to add two numbers in any bases.

$$\begin{array}{cccccccc} 1 & & & 1 & 1 & 1 & & \\ & & & & & & & \\ & 1 & 1 & 0 & 1 & 0 & 1 \\ & 1 & 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}$$

Addition

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The total running time for the addition is of form $c_0 + c_1\, n$, where c_0 and c_1 are some constants, i.e., $O(n)$.

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In order to add two n-bit numbers, we must read them and write down the answer, and even that requires n operations.

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So the addition algorithm is optimal.

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It is often useful and necessary to handle numbers much larger than this, perhaps several thousand bits long.

To study the basic algorithms encoded in the hardware of today's computers, we shall focus on the bit complexity of the algorithm, the number of elementary operations on individual bits.

Multiplication

Multiplication

The grade-school algorithm for multiplying two number x and y is to create an array of intermediate sums.

$$
\begin{array}{cccccccc}
 & & & & 1 & 1 & 0 & 1 \\
 & & & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & 1 & 1 & 0 & 1 \\
 & & & 1 & 1 & 0 & 1 & \\
 & & 0 & 0 & 0 & 0 & & \\
+ & 1 & 1 & 0 & 1 & & & \\
\hline
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1
\end{array}
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If x and y are both n bit, then there are n intermediate rows with length of up to $2n$ bit. (Q: why?)

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\end{array}
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$$\underbrace{O(n) + \ldots + O(n)}_{n-1}$$

$$O(n^2)$$

Quiz

What is the complexity of a number times 2?

Multiplication by Al Khwarizmi

- write them next to each other.

$$11 \quad 13$$

Multiplication by Al Khwarizmi

- write them next to each other.
- halve the first number by 2, dropping the $.5$, and double the second number.

$$\begin{array}{rr} 11 & 13 \\ 5 & 26 \\ & \\ \hline \end{array}$$

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- write them next to each other.
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- keep going till the first number gets down to 1.

11	13
5	26
2	52
1	104

Multiplication by Al Khwarizmi

- write them next to each other.
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- keep going till the first number gets down to 1.
- strike out all the rows where the first number is even.

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- The left is to calculate the binary number.
- The right is to shift the row!

Multiplication á la Françis

MULTIPLY(x, y)
Two n-bit integers x and y, where $y \geq 0$;

if $y = 0$ **then** return 0;
z=MULTIPLY(x, $\lfloor y/2 \rfloor$);
if y *is even* **then**
    return $2z$;
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Another formulation:

$$x \cdot y = \begin{cases} 2(x \cdot \lfloor y/2 \rfloor) & \text{if } y \text{ is even} \\ x + 2(x \cdot \lfloor y/2 \rfloor) & \text{if } y \text{ is odd} \end{cases}$$

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Q: Can we do better?

- Yes!

Division

DIVIDE(x, y)
Two n-bit integers x and y, where $y \geq 1$;

if $x = 0$ **then** return$(0, 0)$;
(q, r)=DIVIDE$(\lfloor x/2 \rfloor, y)$;
$q = 2 \cdot q$, $r = 2 \cdot r$;
if x *is odd* **then** $r = r + 1$;
if $r \geq y$ **then** $r = r - y$, $q = q + 1$;
return(q, r);

Division


```
DIVIDE(x, y)
Two n-bit integers x and y, where y ≥ 1;
if x = 0 then return(0, 0);
(q, r)=DIVIDE(⌊x/2⌋, y);
q = 2 · q, r = 2 · r;
if x is odd then r = r + 1;
if r ≥ y then r = r − y, q = q + 1;
return(q, r);
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- Exercise 1.8!

Modular Arithmetic

What Is Modular

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x modulo N is the remainder when x is divided by N; that is, if $x = qN + r$ with $0 \leq r < N$, then x modulo N is equal to r.

x and y are congruent modulo N if they differ by a multiple of N, i.e.

$$x \equiv y \mod N \quad \Leftrightarrow \quad N \texttt{ divides } (x - y)$$

Two Interpretations

1. It limits numbers to a predefined range $\{0, 1, \ldots, N\}$ and wraps around whenever you try to leave this range - like the hand of a clock.

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1. It limits numbers to a predefined range $\{0, 1, \ldots, N\}$ and wraps around whenever you try to leave this range - like the hand of a clock.
2. Modular arithmetic deals with all the integers, but divides them into N equivalence classes, each of the form $\{i + k \cdot N \mid k \in \mathbb{Z}\}$ for some i between 0 and $N - 1$.

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- Positive integers, in the range 0 to $2^{n-1} - 1$, are stored in regular binary and have a leading bit of 0.
- Negative integers $-x$, with $1 \leq x \leq 2^{n-1}$, are stored by first constructing x in binary, then flipping all the bits, and finally adding 1. The leading bit in this case is 1.

Two's Complement

0	1	1	1	1	1	1	1	=	127
0	0	0	0	0	0	1	0	=	2
0	0	0	0	0	0	0	1	=	1
0	0	0	0	0	0	0	0	=	0
1	1	1	1	1	1	1	1	=	−1
1	1	1	1	1	1	1	0	=	−2
1	0	0	0	0	0	0	1	=	−127
1	0	0	0	0	0	0	0	=	−128

(from wiki)

Rules

Substitution rules: if $x \equiv x' \mod N$ and $y \equiv y' \mod N$, then

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$$x(y + z) \equiv xy + xz \mod N \quad \texttt{Distributivity}$$

It is legal to reduce intermediate results to their remainders modulo N at any stage.

$$2^{345} \equiv (2^5)^{69} \equiv 32^{69} \equiv 1^{69} \equiv 1 \mod 31$$

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Modular Addition

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Its running time is $O(n)$, where $n = \lceil \log N \rceil$.

Modular Multiplication

The product of x and y can be as large as $(N-1)^2$, but this is still at most $2n$ bits long since

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Multiplication thus remains a quadratic operation.

Modular Division

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In ordinary arithmetic there is just one tricky case - division by zero.

It turns out that in modular arithmetic there are potentially other such cases as well.

Whenever division is legal, however, it can be managed in cubic time, $O(n^3)$.

Modular Exponentiation

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The result is some number $\bmod N$ and is therefore a few hundred bits long. However, the raw value x^y could be much, much longer.

When x and y are just 20-bit numbers, x^y is at least

$$(2^{19})^{(2^{19})} = 2^{(19)(524288)}$$

about 10 million bits long!

Modular Exponentiation

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The resulting sequence of intermediate products,

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consists of numbers that are smaller than N, and so the individual multiplications do not take too long.

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consists of numbers that are smaller than N, and so the individual multiplications do not take too long.

But imagine if y is 500 bits long . . .

Second idea: starting with x and squaring repeatedly modulo N, we get

$$x \mod N \to x^2 \mod N \to x^4 \mod N \to x^8 \mod N \to \ldots x^{2^{\lfloor \log y \rfloor}} \mod N$$

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Each takes just $O(\log^2 N)$ time to compute, and in this case there are only $\log y$ multiplications.

To determine $x^y \mod N$, multiply together an appropriate subset of these powers, those corresponding to 1's in the binary representation of y.

For instance,

$$x^{25} = x^{11001_2} = x^{10000_2} \cdot x^{1000_2} \cdot x^{1_2} = x^{16} \cdot x^8 \cdot x^1$$

Modular Exponentiation

SHANGHAI JIAO TONG UNIVERSITY

MODEXP (x, y, N)
Two n-bit integers x and N, and an integer exponent y;

if $y = 0$ **then** return 1;
z=MODEXP $(x, \lfloor y/2 \rfloor, \mathsf{N})$;
if y *is even* **then**
  return $z^2 \mod N$;
  else return $x \cdot z^2 \mod N$;
end

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  else return $x \cdot z^2 \mod N$;
end

Another formulation:

$$x^y \mod N = \begin{cases} (x^{\lfloor y/2 \rfloor})^2 \mod N & \text{if } y \text{ is even} \\ x \cdot (x^{\lfloor y/2 \rfloor})^2 \mod N & \text{if } y \text{ is odd} \end{cases}$$

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end

The algorithm will halt after at most n recursive calls, and during each call it multiplies n-bit numbers. for a total running time of $O(n^3)$

Euclid's Algorithm for Greatest Common Divisor

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Proof:

It is enough to show the rule $gcd(x, y) = gcd(x - y, y)$. Result can be derived by repeatedly subtracting y from x.

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- if $b \leq a/2$, $a \mod b < b \leq a/2$;

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Proof:

- if $b \leq a/2$, $a \mod b < b \leq a/2$;
- if $b > a/2$, $a \mod b = a - b < a/2$.

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This means that after any two consecutive rounds, both arguments, x and y are at the very least halved in value, i.e., the length of each decreases at least one bit.

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If they are initially n-bit integers, then the base case will be reached within $2n$ recursive calls. Since each call involves a quadratic-time division, the total time is $O(n^3)$.

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$d \leq gcd(x, y)$, obviously;

$d \geq gcd(x, y)$, since $gcd(x, y)$ can divide x and y, it must also divide $ax + by = d$.

An Extension of Euclid's Algorithm

EXTENDED-EUCLID(x, y)
Two integers x and y with $x \geq y \geq 0$;

if $y = 0$ **then** return $(1, 0, x)$;
(x', y', d)=EXTENDED-EUCLID$(y, x \pmod y)$;
return $(y', x' - \lfloor x/y \rfloor y', d)$;

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Correctness of the algorithm?
DIY!

Modular Inverse

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Remark: The inverse does not always exists! for instance, 2 is not invertible modulo 6.

Lemma

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If $gcd(a, N) = 1$, then extended Euclid algorithm gives us integers x and y such that $ax + Ny = 1$, which means $ax \equiv 1 \mod N$. Thus x is a's sought inverse.

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Theorem (Modular Division Theorem)

For any $a \mod N$, a has a multiplicative inverse modulo N if and only if it is relatively prime to N.

When the inverse exists, it can be found in time $O(n^3)$ by running the extended Euclid algorithm.

This resolves the issues of modular division: when working modulo N, can divide by numbers relatively prime to N. And to actually carry out the division, multiply by the inverse.