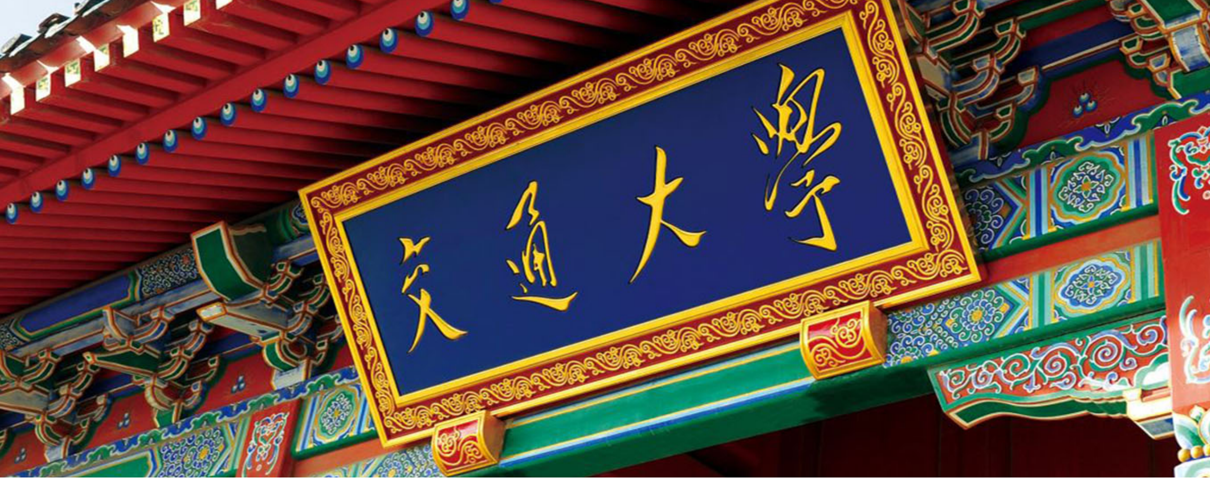

Algorithm Design and Analysis III

Algorithms with Numbers II

Guoqiang Li
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SHANGHAI JIAO TONG UNIVERSITY

Primality

Fermat's Little Theorem

Theorem

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$$a^{p-1} \equiv 1 \pmod{p}$$

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multiplying all numbers in each representation, then gives $(p-1)! \equiv a^{(p-1)}.(p-1)! \pmod{p}$, and thus

$$1 \equiv a^{(p-1)} \pmod{p}$$

A (Problematic) Algorithm for Testing Primality


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Positive integer N;

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- e.g. $341 = 11 \cdot 31$, and $2^{340} \equiv 1 \pmod{341}$

Our best hope: for composite N, most values of a will fail the test.

Rather than fixing an arbitrary value of a, we should choose it randomly from $\{1, \ldots, N-1\}$.

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There are composite numbers N such that for every $a < N$ relatively prime to N,

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Example:

$$561 = 3 \cdot 11 \cdot 17$$

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Assume some $b < N$ satisfies $b^{N-1} \equiv 1 (\mod N)$, then

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For $b \neq b'$, we have

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The one-to-one function $b \mapsto a \cdot b(\mod N)$ shows that at least as many elements fail the test as pass it.

Primality Testing without Carmichael Numbers

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- $Pr(\texttt{PRIMALITY}$ returns yes when N is prime$) = 1$
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Primality Testing with Low Error Probability


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Generating Random Primes

Lagrange's Prime Number Theorem

Let $\pi(x)$ be the number of primes $\leq x$. Then $\pi(x) \approx x/ln(x)$, or more precisely,

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Such abundance makes it simple to generate a random n-bit prime:

- Pick a random n-bit number N.
- Run a primality test on N.
- If it passes the test, output N; else repeat the process.

Generating Random Primes

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- Exercise 1.34!

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Monte Carlo Algorithm (MC):

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- Examples: Quicksort, Hashing

Cryptography

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Even Ida, an intruder, will break the rules of communications positively.

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IOW, knowing $e(x)$ tells her little or nothing about what x might be.

Private VS. Public Schemes

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Public-key schemes allow Alice to send Bob a message without having met him before.

Bob is able to implement a digital lock, to which only he has the key. Now by making this digital lock public, he gives Alice a way to send him a secure message.

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- The function e_r is a bijection, and it is its own inverse:

$$e_r(e_r(x)) = (x \oplus r) \oplus r = x \oplus 0 = x$$

Why Secure?

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This will ensure that if Eve intercepts the encoded message $y = e_r(x)$, she gets no information about x.

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- 128-bit fixed size.
- repeatedly use
- no techniques to break are better than brute force

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Eve is welcome to see as many encrypted messages, but she will not be able to decode them, under certain assumptions.

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Bob retain the value d as his secret key, with which he can decode all messages that come to him by simply raising them to the d-th power modulo N.

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$x^{1+k(p-1)(q-1)} - x$ is divisible by p (since $x^{p-1} \equiv 1 \mod p$) and likewise by q. Since p and q are primes, this expression must be divisible by $N = pq$.

RSA protocols

Bob chooses his public and secret keys:

- He starts by picking two large (n-bit) random primes p and q.
- His public key is (N, e) where $N = pq$ and e is a $2n$-bit number relatively prime to $(p-1)(q-1)$.
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Alice wishes to send message x to Bob

- She looks up his public key (N, e) and sends him $y = (x^e \mod N)$.
- He decodes the message by computing $y^d \mod N$.

Security Assumption of RSA

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How might Eve try to guess x

she could try to factor N to retrieve p and q, and then figure out d by inverting e modulo $(p-1)(q-1)$, but we believe factoring to be hard.

Digital Signature

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A signing algorithm that, given a message and a private key, produces a signature.

Digital Signature

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In a digital signature scheme, there are two algorithms, signing and verifying.

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A signature verifying algorithm that, given a message, public key and a signature, either accepts or rejects the message's claim to authenticity.

Is Communication Safe?

Is a communication safe in the internet when cryptography is unbreakable?

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- No!

The NSPK Protocol

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$$
\begin{aligned}
A \longrightarrow B: &\quad \{A, N_A\}_{+K_B} \\
B \longrightarrow A: &\quad \{N_A, N_B\}_{+K_A} \\
A \longrightarrow B: &\quad \{N_B\}_{+K_B}
\end{aligned}
$$

An Attack

$$
\begin{array}{rcll}
A & \longrightarrow & I: & \{A, N_A\}_{+K_I} \\
I(A) & \longrightarrow & B: & \{A, N_A\}_{+K_B} \\
B & \longrightarrow & I(A): & \{N_A, N_B\}_{+K_A} \\
I & \longrightarrow & A: & \{N_A, N_B\}_{+K_A} \\
A & \longrightarrow & I: & \{N_B\}_{+K_I} \\
I(A) & \longrightarrow & B: & \{N_B\}_{+K_B}
\end{array}
$$

The Fixed NSPK Protocol

$$
\begin{aligned}
A \longrightarrow B &: \quad \{A, N_A\}_{+K_B} \\
B \longrightarrow A &: \quad \{B, N_A, N_B\}_{+K_A} \\
A \longrightarrow B &: \quad \{N_B\}_{+K_B}
\end{aligned}
$$

$$
\begin{aligned}
A &\longrightarrow I : && \{A, N_A\}_{+K_I} \\
I(A) &\longrightarrow B : && \{A, N_A\}_{+K_B} \\
B &\longrightarrow I(A) : && \{B, N_A, N_B\}_{+K_A} \\
I &\not\longrightarrow A : && \{I, N_A, N_B\}_{+K_A}
\end{aligned}
$$