

Algorithm Design and Analysis (XI)

Linear Programming: Introduction

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An Introduction to Linear Programming





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Linear Programming



A linear programming problem gives a set of variables, and assigns real values to them so as to

satisfy a set of linear equations and/or linear inequalities involving these variables, and
maximize or minimize a given linear objective function.



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Q: How much of each should it produce to maximize profits?

- Every box of Pyramide has a a profit of \$1.
- Every box of Nuit has a profit of \$6.
- The daily demand is limited to at most 200 boxes of Pyramide and 300 boxes of Nuit.
- The current workforce can produce a total of at most 400 boxes of chocolate per day.









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It is a convex polygon.

The Convex Polygon





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As c increases, this "profit line" moves parallel to itself, up and to the right.

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The optimum solution will be the very last feasible point that the profit line sees and must therefore be a vertex of the polygon.

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The only exceptions are cases in which there is no optimum; this can happen in two ways:

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 - For instance, $\max x_1 + x_2$
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Upon reaching a vertex that has no better neighbor, simplex declares it to be optimal and halts.





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By simple geometry. Since all the vertex's neighbors lie below the line, the rest of the feasible polygon must also lie below this line.

The Example





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Nuit and Luxe require the same packaging machinery. Luxe uses it three times as much, which imposes another constraint $x_2 + 3x_3 \le 600$.



$\max x_1 + 6x_2 + 13x_3$	
$x_1 \le 200$	1
$x_2 \leq 300$	1
$x_1 + x_2 + x_3 \le 400$	1
$x_2 + 3x_3 \le 600$	1
$x_1, x_2, x_3 \geq 0$	



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A profit of *c* corresponds to the plane $x_1 + 6x_2 + 13x_3 = c$.

As *c* increases, this profit-plane moves parallel to itself, further into the positive orthant until it no longer touches the feasible region.

The Example







The point of final contact is the optimal vertex: (0, 300, 100), with total profit \$3100.



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A possible trajectory

 $\frac{(0,0,0)}{\$0} \rightarrow \frac{(200,0,0)}{\$200} \rightarrow \frac{(200,200,0)}{\$1400} \rightarrow \frac{(200,0,200)}{\$2800} \rightarrow \frac{(0,300,100)}{\$3100}$

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Currently with 30 employees, each of whom makes 20 carpets per month and gets a monthly salary of \$2000.

With no initial surplus of carpets.



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- Overtime. Overtime pay is 80% more than regular pay. Workers can put in at most 30% overtime.
- 2 Hiring and firing, costing \$320 and \$400, respectively, per worker.
- Storing surplus production, costing \$8 per carpet per month. Currently without stored carpets on hand, and without any carpets stored at the end of year.



- w_i = number of workers during *i*-th month; $w_0 = 30$.
- x_i = number of carpets made during *i*-th month.
- o_i = number of carpets made by overtime in month *i*.
- h_i, f_i = number of workers hired and fired, respectively, at beginning of month *i*.

 s_i = number of carpets stored at end of month i; $s_0 = 0$.





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The total number of carpets made per month consists of regular production plus overtime:

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i = 1, ..., 12.

The number of workers can potentially change at the start of each month:

 $w_i = w_{i-1} + h_i - f_i$





The number of carpets stored at the end of each month is what we started with, plus the number we made, minus the demand for the month:

 $s_i = s_{i-1} + x_i - d_i$



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And overtime is limited:

 $o_i \leq 6w_i$



The objective function is to minimize the total cost:

$$\min 2000 \sum_{i} w_i + 320 \sum_{i} h_i + 400 \sum_{i} f_i + 8 \sum_{i} s_i + 180 \sum_{i} o_i$$



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In the example, most of the variables take on fairly large values, and thus rounding is unlikely to affect things too much.



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Integer Linear Programming



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In NP problems, finding the optimum integer solution of an LP is an important but very hard problem, called integer linear programming.



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Recall:

 $\begin{array}{l} \max x_1 + 6x_2 \\ x_1 \leq 200 \\ x_2 \leq 300 \\ x_1 + x_2 \leq 400 \\ x_1, x_2 \geq 0 \end{array}$





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Simplex declares the optimum solution to be $(x_1, x_2) = (100, 300)$, with objective value 1900.

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We take the first inequality and add it to six times the second inequality:

 $x_1 + 6x_2 \le 2000$



Multiplying the three inequalities by 0, 5, and 1, respectively, and adding them up yields

 $x_1 + 6x_2 \le 1900$





Let's investigate the issue by describing what we expect of these three multipliers, call them y_1, y_2, y_3 .

Multiplier		In	equa	lity	
y_1	x_1			\leq	200
y_2			x_2	\leq	300
y_3	x_1	+	x_2	\leq	400



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After the multiplication and addition steps, we get the bound:

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We want the left-hand side to look like the objective function $x_1 + 6x_2$ so that the right-hand side is an upper bound on the optimum solution.









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What we want is a bound as tight as possible, so we minimize

 $200y_1 + 300y_2 + 400y_3$

subject to the preceding inequalities. This is a new linear program!



$\min 200y_1 + 300y_2 + 400y_3 \\ y_1 + y_3 \ge 1 \\ y_2 + y_3 \ge 6 \\ y_1, y_2, y_3 \ge 0$



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- Primal: $(x_1, x_2) = (100, 300);$
- Dual: $(y_1, y_2, y_3) = (0, 5, 1)$.



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They both have value 1900 and certify each other's optimality.

Matrix-Vector Form and Its Dual



Primal LP	Dual LP	
$\max c^T \mathbf{x} \\ A \mathbf{x} \le b \\ \mathbf{x} \ge 0$	$ \min_{\mathbf{y}^T b} \mathbf{y}^T b \\ \mathbf{y}^T A \ge c^T \\ \mathbf{y} \ge 0 $	

Primal LP:

Dual LP:

$$\max c_1 x_1 + \dots + c_n x_n$$
$$a_{i1} x_1 + \dots + a_{in} x_n \le b_i \quad \text{for } i \in I$$
$$a_{i1} x_1 + \dots + a_{in} x_n = b_i \quad \text{for } i \in E$$
$$x_j \ge 0 \quad \text{for } j \in N$$

$$\min \ b_1 y_1 + \dots + b_m y_m$$

$$a_{1j} y_1 + \dots + a_{mj} y_m \ge c_j \quad \text{for } j \in N$$

$$a_{1j} y_1 + \dots + a_{mj} y_m = c_j \quad \text{for } j \notin N$$

$$y_i \ge 0 \quad \text{for } i \in I$$

Matrix-Vector Form and Its Dual



 $\max x_{1} + 6x_{2}$ $x_{1} \leq 200$ $x_{2} \leq 300$ $x_{1} + x_{2} \leq 400$ $x_{1}, x_{2} \geq 0$

 $\begin{aligned} \min 200y_1 + 300y_2 + 400y_3 \\ y_1 + y_3 &\geq 1 \\ y_2 + y_3 &\geq 6 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$

Matrix-Vector Form and Its Dual



Theorem (Duality)

If a linear program has a bounded optimum, then so does its dual, and the two optimum values coincide.



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An inequality constraint has slack if the slack variable is positive.

The complementary slackness refers to a relationship between the slackness in a primal constraint and the associated dual variable.

LP and Its Dual





 $x_1 = 100, x_2 = 300$

 $\min 200y_1 + 300y_2 + 400y_3 \\ y_1 + y_3 \ge 1 \\ y_2 + y_3 \ge 6 \\ y_1, y_2, y_3 \ge 0$

 $y_1 = 0, y_2 = 5, y_3 = 1$

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Theorem

Assume LP problem (P) has a solution x^* and its dual problem (D) has a solution y^* .

- **1** If $x_j^* > 0$, then the *j*-th constraint in (D) is binding.
- 2 If the *j*-th constraint in (D) is not binding, then $x_j^* = 0$.
- **3** If $y_i^* > 0$, then the *i*-th constraint in (*P*) is binding.
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Assignment !

A Concrete Example for Duality

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Brewery Problem



Small brewery produces ale and beer.

- Production limited by scarce resources: corn, hops, barley malt.
- Recipes for ale and beer require different proportions of resources.

Beverage	Corn(pounds)	Hops(ounces)	Malt(pounds)	Profit(\$)
Ale (barrel)	5	4	35	13
Beer (barrel)	15	4	20	23
constraint	480	160	1190	

LP and its Dual



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 $x_1^* = 12, x_2^* = 28$

Brewer: find optimal mix of beer and ale to maximize profits.

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 $x_1^* = 12, x_2^* = 28$

Brewer: find optimal mix of beer and ale to maximize profits.

 $\min 480y_1 + 160y_2 + 1190y_3$ $5y_1 + 4y_2 + 35y_3 \ge 13$ $15y_1 + 4y_2 + 20y_3 \ge 23$ $y_1, y_2, y_3 \ge 0$

 $y_1^* = 1, y_2^* = 2, y_3^* = 0$

Entrepreneur: buy individual resources from brewer at min cost.



Q. How much should brewer be willing to pay (marginal price) for additional supplies of scarce resources?



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A. At least 2 (\$1) + 5 (\$2) + 24 (\$0) = \$12 / barrel.

Referred Materials

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Referred Materials



Content of this lecture comes from Section 7.1 and 7.4 in [DPV07], Section 29.2 in [CLRS09], and Section 7.3 in [WS11].