Design and Analysis of Algorithms (XI)

Linear Programming: Introduction

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An Introduction to Linear Programming

Linear Programming

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A linear programming problem gives a set of variables, and assigns real values to them so as to

1. satisfy a set of linear equations and/or linear inequalities involving these variables, and
2. maximize or minimize a given linear objective function.

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- and the more decadent and deluxe Pyramide Nuit.

Q: How much of each should it produce to maximize profits?

- Every box of Pyramide has a a profit of $\$1$.
- Every box of Nuit has a profit of $\$6$.
- The daily demand is limited to at most 200 boxes of Pyramide and 300 boxes of Nuit.
- The current workforce can produce a total of at most 400 boxes of chocolate per day.

LP Formulation

`Objective function`	$\max x_1 + 6x_2$
`Constraints`	$x_1 \leq 200$
	$x_2 \leq 300$
	$x_1 + x_2 \leq 400$
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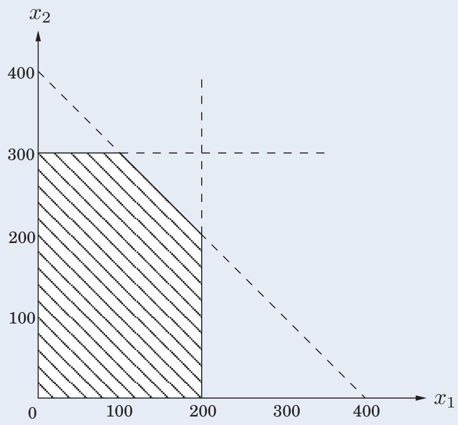
It is a convex polygon.

The Convex Polygon



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The optimum solution will be the very last feasible point that the profit line sees and must therefore be a vertex of the polygon.

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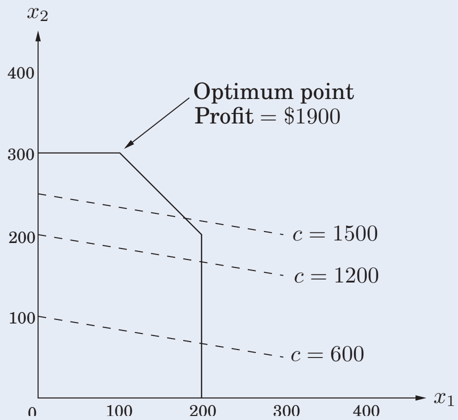

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Upon reaching a vertex that has no better neighbor, simplex declares it to be optimal and halts.

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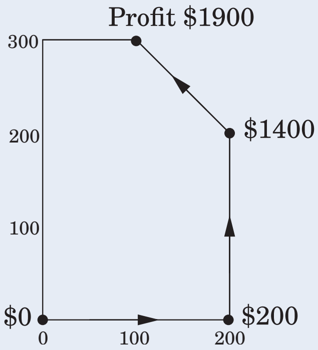
By simple geometry. Since all the vertex's neighbors lie below the line, the rest of the feasible polygon must also lie below this line.

The Example



More Products

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Nuit and Luxe require the same packaging machinery. Luxe uses it three times as much, which imposes another constraint $x_2 + 3x_3 \leq 600$.

LP

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$$\max x_1 + 6x_2 + 13x_3$$
$$x_1 \leq 200$$
$$x_2 \leq 300$$
$$x_1 + x_2 + x_3 \leq 400$$
$$x_2 + 3x_3 \leq 600$$
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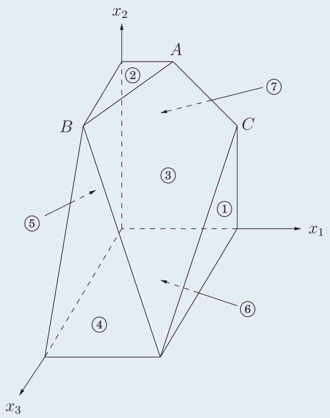
A profit of c corresponds to the plane $x_1 + 6x_2 + 13x_3 = c$.

As c increases, this profit-plane moves parallel to itself, further into the positive orthant until it no longer touches the feasible region.

The Example



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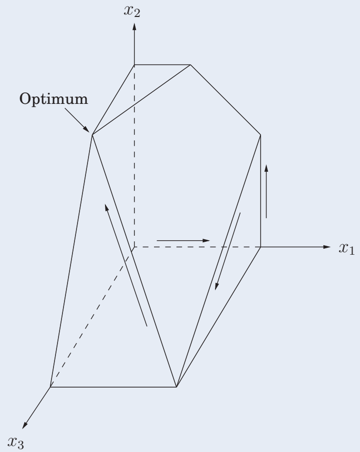
A possible trajectory

$$\frac{(0,0,0)}{\$0} \to \frac{(200,0,0)}{\$200} \to \frac{(200,200,0)}{\$1400} \to \frac{(200,0,200)}{\$2800} \to \frac{(0,300,100)}{\$3100}$$

The Example



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With no initial surplus of carpets.

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2. Hiring and firing, costing $\$320$ and $\$400$, respectively, per worker.
3. Storing surplus production, costing $\$8$ per carpet per month. Currently without stored carpets on hand, and without any carpets stored at the end of year.

$$
\begin{aligned}
w_i &= \text{number of workers during } i\text{-th month; } w_0 = 30. \\
x_i &= \text{number of carpets made during } i\text{-th month.} \\
o_i &= \text{number of carpets made by overtime in month } i. \\
h_i, f_i &= \text{number of workers hired and fired, respectively,} \\
 & \quad \text{at beginning of month } i. \\
s_i &= \text{number of carpets stored at end of month } i;\ s_0 = 0.
\end{aligned}
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$$x_i = 20w_i + o_i$$

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The number of workers can potentially change at the start of each month:

$$w_i = w_{i-1} + h_i - f_i$$

LP Formulation

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The number of carpets stored at the end of each month is what we started with, plus the number we made, minus the demand for the month:

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And overtime is limited:

$$o_i \leq 6w_i$$

LP Formulation

The objective function is to minimize the total cost:

$$\min 2000 \sum_i w_i + 320 \sum_i h_i + 400 \sum_i f_i + 8 \sum_i s_i + 180 \sum_i o_i$$

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This number would have to be rounded to either 10 or 11 in order to make sense, and the overall cost would then increase correspondingly.

In the example, most of the variables take on fairly large values, and thus rounding is unlikely to affect things too much.

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In NP problems, finding the optimum integer solution of an LP is an important but very hard problem, called integer linear programming.

Duality

Product Planning Revisit

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Recall:

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Can this answer be checked somehow?

We take the first inequality and add it to six times the second inequality:

$$x_1 + 6x_2 \leq 2000$$

Product Planning Revisit

Multiplying the three inequalities by 0, 5, and 1, respectively, and adding them up yields

$$x_1 + 6x_2 \leq 1900$$

Multipliers

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Let's investigate the issue by describing what we expect of these three multipliers, call them y_1, y_2, y_3.

Multiplier	Inequality
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We want the left-hand side to look like the objective function $x_1 + 6x_2$ so that the right-hand side is an upper bound on the optimum solution.

Multipliers

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if

$$y_1, y_2, y_3 \ge 0$$
$$y_1 + y_3 \ge 1$$
$$y_2 + y_3 \ge 6$$

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What we want is a bound as tight as possible, so we minimize

$$200y_1 + 300y_2 + 400y_3$$

subject to the preceding inequalities. This is a new linear program!

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Here is just such a pair:

- Primal: $(x_1, x_2) = (100, 300)$;
- Dual: $(y_1, y_2, y_3) = (0, 5, 1)$.

The Dual Program

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They both have value 1900 and certify each other's optimality.

Matrix-Vector Form and Its Dual

Primal LP

$$\max c^T \mathbf{x}$$
$$A\mathbf{x} \le b$$
$$\mathbf{x} \ge 0$$

Dual LP

$$\min \mathbf{y}^T b$$
$$\mathbf{y}^T A \ge c^T$$
$$\mathbf{y} \ge 0$$

Primal LP:

$$\begin{aligned} \max\ & c_1x_1 + \cdots + c_nx_n \\ a_{i1}x_1 + \cdots + a_{in}x_n &\le b_i \quad \text{for } i \in I \\ a_{i1}x_1 + \cdots + a_{in}x_n &= b_i \quad \text{for } i \in E \\ x_j &\ge 0 \quad \text{for } j \in N \end{aligned}$$

Dual LP:

$$\begin{aligned} \min\ & b_1y_1 + \cdots + b_my_m \\ a_{1j}y_1 + \cdots + a_{mj}y_m &\ge c_j \quad \text{for } j \in N \\ a_{1j}y_1 + \cdots + a_{mj}y_m &= c_j \quad \text{for } j \notin N \\ y_i &\ge 0 \quad \text{for } i \in I \end{aligned}$$

Matrix-Vector Form and Its Dual

$$\max x_1 + 6x_2$$
$$x_1 \leq 200$$
$$x_2 \leq 300$$
$$x_1 + x_2 \leq 400$$
$$x_1, x_2 \geq 0$$

$$\min 200y_1 + 300y_2 + 400y_3$$
$$y_1 + y_3 \geq 1$$
$$y_2 + y_3 \geq 6$$
$$y_1, y_2, y_3 \geq 0$$

Matrix-Vector Form and Its Dual

Theorem (Duality)

If a linear program has a bounded optimum, then so does its dual, and the two optimum values coincide.

Complementary Slackness

The number of variables in the dual is equal to that of constraints in the primal and the number of constraints in the dual is equal to that of variables in the primal.

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An inequality constraint has slack if the slack variable is positive.

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An inequality constraint has slack if the slack variable is positive.

The complementary slackness refers to a relationship between the slackness in a primal constraint and the associated dual variable.

LP and Its Dual

$$\max x_1 + 6x_2$$
$$x_1 \le 200$$
$$x_2 \le 300$$
$$x_1 + x_2 \le 400$$
$$x_1, x_2 \ge 0$$

$$x_1 = 100, x_2 = 300$$

$$\min 200y_1 + 300y_2 + 400y_3$$
$$y_1 + y_3 \ge 1$$
$$y_2 + y_3 \ge 6$$
$$y_1, y_2, y_3 \ge 0$$

$$y_1 = 0, y_2 = 5, y_3 = 1$$

Complementary Slackness

Theorem

Assume LP problem (P) has a solution x^* and its dual problem (D) has a solution y^*.

1. *If $x_j^* > 0$, then the j-th constraint in (D) is binding.*
2. *If the j-th constraint in (D) is not binding, then $x_j^* = 0$.*
3. *If $y_i^* > 0$, then the i-th constraint in (P) is binding.*
4. *If the i-th constraint in (P) is not binding, then $y_i^* = 0$.*

Complementary Slackness

Theorem

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Proof.

Complementary Slackness

Theorem

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Proof.

Assignment !

A Concrete Example for Duality

Brewery Problem

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Small brewery produces ale and beer.

- Production limited by scarce resources: corn, hops, barley malt.
- Recipes for ale and beer require different proportions of resources.

Beverage	Corn(pounds)	Hops(ounces)	Malt(pounds)	Profit($)
Ale (barrel)	5	4	35	13
Beer (barrel)	15	4	20	23
constraint	480	160	1190	

LP and its Dual

LP and its Dual

$$\begin{aligned} \max 13x_1 + 23x_2 \\ 5x_1 + 15x_2 \leq 480 \\ 4x_1 + 4x_2 \leq 160 \\ 35x_1 + 20x_2 \leq 1190 \\ x_1, x_2 \geq 0 \end{aligned}$$

LP and its Dual

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$$
\begin{aligned}
\max\ 13x_1 + 23x_2 \\
5x_1 + 15x_2 \le 480 \\
4x_1 + 4x_2 \le 160 \\
35x_1 + 20x_2 \le 1190 \\
x_1, x_2 \ge 0
\end{aligned}
$$

$$
\begin{aligned}
\min\ 480y_1 + 160y_2 + 1190y_3 \\
5y_1 + 4y_2 + 35y_3 \ge 13 \\
15y_1 + 4y_2 + 20y_3 \ge 23 \\
y_1, y_2, y_3 \ge 0
\end{aligned}
$$

LP and its Dual

$$\begin{aligned} \max\ & 13x_1 + 23x_2 \\ & 5x_1 + 15x_2 \le 480 \\ & 4x_1 + 4x_2 \le 160 \\ & 35x_1 + 20x_2 \le 1190 \\ & x_1, x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} \min\ & 480y_1 + 160y_2 + 1190y_3 \\ & 5y_1 + 4y_2 + 35y_3 \ge 13 \\ & 15y_1 + 4y_2 + 20y_3 \ge 23 \\ & y_1, y_2, y_3 \ge 0 \end{aligned}$$

$x_1^* = 12, x_2^* = 28$

Brewer: find optimal mix of beer and ale to maximize profits.

LP and its Dual

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$$\max 13x_1 + 23x_2$$
$$5x_1 + 15x_2 \leq 480$$
$$4x_1 + 4x_2 \leq 160$$
$$35x_1 + 20x_2 \leq 1190$$
$$x_1, x_2 \geq 0$$

$$\min 480y_1 + 160y_2 + 1190y_3$$
$$5y_1 + 4y_2 + 35y_3 \geq 13$$
$$15y_1 + 4y_2 + 20y_3 \geq 23$$
$$y_1, y_2, y_3 \geq 0$$

$x_1^* = 12, x_2^* = 28$

$y_1^* = 1, y_2^* = 2, y_3^* = 0$

Brewer: find optimal mix of beer and ale to maximize profits.

Entrepreneur: buy individual resources from brewer at min cost.

LP Duality: Sensitivity Analysis

Q. How much should brewer be willing to pay (marginal price) for additional supplies of scarce resources?

LP Duality: Sensitivity Analysis

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A. corn \$1, hops \$2, malt \$0.

LP Duality: Sensitivity Analysis

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Q. How much should brewer be willing to pay (marginal price) for additional supplies of scarce resources?

A. corn \$1, hops \$2, malt \$0.

Q. Suppose a new product "light beer" is proposed. It requires 2 corn, 5 hops, 24 malt. How much profit must be obtained from light beer to justify diverting resources from production of beer and ale?

LP Duality: Sensitivity Analysis

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A. corn \$1, hops \$2, malt \$0.

Q. Suppose a new product "light beer" is proposed. It requires 2 corn, 5 hops, 24 malt. How much profit must be obtained from light beer to justify diverting resources from production of beer and ale?

A. At least 2 (\$1) + 5 (\$2) + 24 (\$0) = \$12 / barrel.

Referred Materials

Referred Materials

Content of this lecture comes from Section 7.1 and 7.4 in [DPV07], Section 29.2 in [CLRS09], and Section 7.3 in [WS11].