Algorithm Design and Analysis (XII)

Simplex Algorithm

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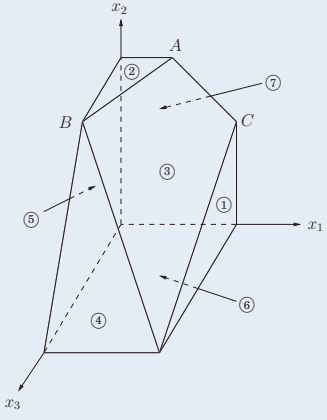
Review of Previous Lecture

$$\begin{aligned}
&\max x_1 + 6x_2 + 13x_3 \\
&x_1 \leq 200 \\
&x_2 \leq 300 \\
&x_1 + x_2 + x_3 \leq 400 \\
&x_2 + 3x_3 \leq 600 \\
&x_1, x_2, x_3 \geq 0
\end{aligned}$$

The Example



The point of final contact is the optimal vertex: $(0, 300, 100)$, with total profit $\$3100$.

Q: How would the simplex algorithm behave on this modified problem?

A possible trajectory

$$\underset{\$0}{(0,0,0)} \to \underset{\$200}{(200,0,0)} \to \underset{\$1400}{(200,200,0)} \to \underset{\$2800}{(200,0,200)} \to \underset{\$3100}{(0,300,100)}$$

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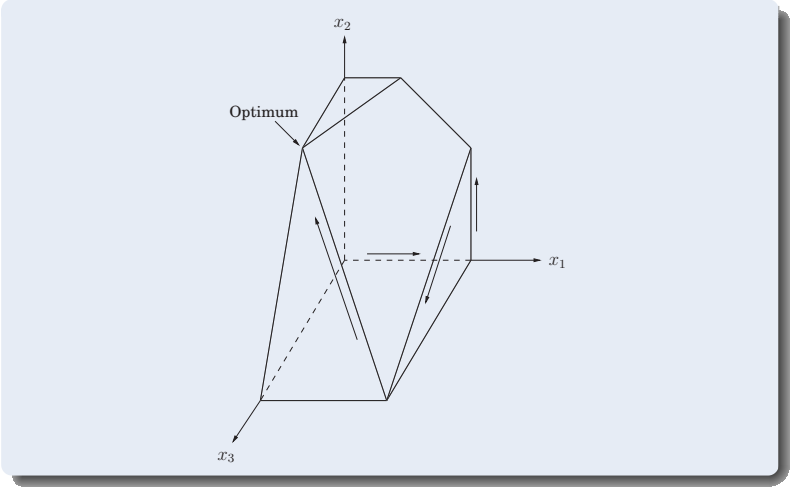

LP and Its Dual

Primal LP	Dual LP
$\max c^T \mathbf{x}$	$\min \mathbf{y}^T b$
$A\mathbf{x} \leq b$	$\mathbf{y}^T A \geq c^T$
$\mathbf{x} \geq 0$	$\mathbf{y} \geq 0$

Primal LP:

$$\begin{aligned} \max\ & c_1 x_1 + \cdots + c_n x_n \\ a_{i1} x_1 + \cdots + a_{in} x_n &\leq b_i \quad \text{for } i \in I \\ a_{i1} x_1 + \cdots + a_{in} x_n &= b_i \quad \text{for } i \in E \\ x_j &\geq 0 \quad \text{for } j \in N \end{aligned}$$

Dual LP:

$$\begin{aligned} \min\ & b_1 y_1 + \cdots + b_m y_m \\ a_{1j} y_1 + \cdots + a_{mj} y_m &\geq c_j \quad \text{for } j \in N \\ a_{1j} y_1 + \cdots + a_{mj} y_m &= c_j \quad \text{for } j \notin N \\ y_i &\geq 0 \quad \text{for } i \in I \end{aligned}$$

LP and Its Dual

$$\begin{aligned} \max\ & x_1 + 6x_2 \\ & x_1 \le 200 \\ & x_2 \le 300 \\ & x_1 + x_2 \le 400 \\ & x_1, x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} \min\ & 200y_1 + 300y_2 + 400y_3 \\ & y_1 + y_3 \ge 1 \\ & y_2 + y_3 \ge 6 \\ & y_1, y_2, y_3 \ge 0 \end{aligned}$$

Complementary Slackness

Theorem

Assume LP problem (P) has a solution x^* and its dual problem (D) has a solution y^*.

1. *If $x_j^* > 0$, then the j-th constraint in (D) is binding.*
2. *If the j-th constraint in (D) is not binding, then $x_j^* = 0$.*
3. *If $y_i^* > 0$, then the i-th constraint in (P) is binding.*
4. *If the i-th constraint in (P) is not binding, then $y_i^* = 0$.*

Standard Linear Programming

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1. It can be either a maximization or a minimization problem.
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We will now show that these various LP options can all be reduced to one another via simple transformations.

Variants of Linear Programming

To turn a maximization problem into a minimization (or vice versa), multiply the coefficients of the objective function by -1.

To turn an inequality constraint like $\sum_{i=1}^{n} a_i x_i \leq b$ into an equation, introduce a new variable s and use

$$\sum_{i=1}^{n} a_i x_i + s = b$$

$$s \geq 0$$

This s is called the slack variable for the inequality.

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This s is called the slack variable for the inequality.

To change an equality constraint into inequalities is easy: rewrite $ax = b$ as the equivalent pair of constraints $ax \leq b$ and $ax \geq b$.

Variants of Linear Programming

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- Introduce two nonnegative variables, $x^+, x^- \geq 0$.

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- Replace x, wherever it occurs in the constraints or the objective function, by $x^+ - x^-$.

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$$\begin{array}{rcl} \max x_1 + 6x_2 & & \min -x_1 - 6x_2 \\ x_1 \leq 200 & & x_1 + s_1 = 200 \\ x_2 \leq 300 & \Longrightarrow & x_2 + s_2 = 300 \\ x_1 + x_2 \leq 400 & & x_1 + x_2 + s_3 = 400 \\ x_1, x_2 \geq 0 & & x_1, x_2, s_1, s_2, s_3 \geq 0 \end{array}$$

The Simplex Algorithm

General Description

Simplex

let v be any *vertex* of the feasible region, while there is a *neighbor* v' of v with better objective value:
set $v = v'$

Vertices and Neighbors

Definition (Vertex)

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Definition (Neighbors)

Two vertices are neighbors if they have $n - 1$ defining inequalities in common.

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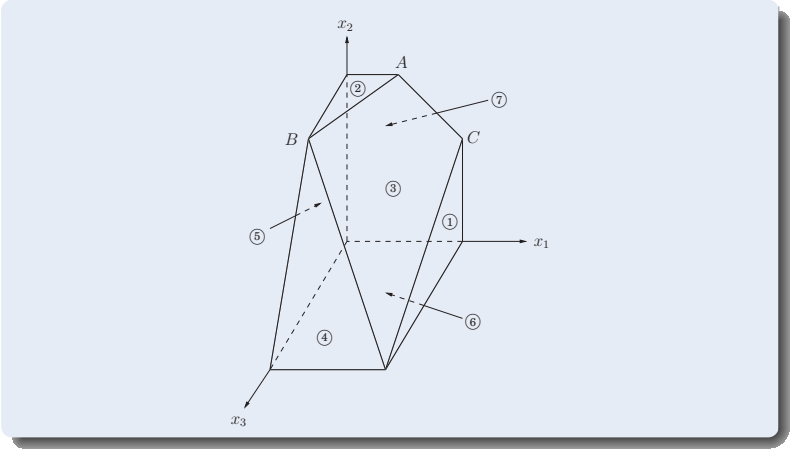

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Both tasks are easy if the vertex is at the origin. If the vertex is elsewhere, we transform the coordinate system to move it to the origin.

The Convenience for the Origin

Suppose we have some generic LP:

$$\begin{aligned} &\max c^T \mathbf{x} \\ &A\mathbf{x} \leq b \\ &\mathbf{x} \geq 0 \end{aligned}$$

where $\mathbf{x}$ is the vector of variables, $\mathbf{x} = (x_1, \ldots, x_n)$.

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Suppose the origin is feasible. Then it is certainly a vertex, since it is the unique point at which the n inequalities

$$\{x_1 \geq 0, \ldots, x_n \geq 0\}$$

are tight.

Task 1 in the Origin

Lemma

The origin is optimal if and only if all $c_i \leq 0$.

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Proof.

If all $c_i \leq 0$, then considering the constraints $x \geq 0$, we can't hope for a better objective value.

Conversely, if some $c_i > 0$, then the origin is not optimal, since we can increase the objective function by raising x_i.

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We have exactly n tight inequalities, so we are at a new vertex.

An Example

$$\max 2x_1 + 5x_2$$

$$\begin{aligned} 2x_1 - x_2 &\leq 4 \\ x_1 + 2x_2 &\leq 9 \\ -x_1 + x_2 &\leq 3 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

Vertex Elsewhere

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$$y_i = b_i - a_i \cdot x$$

The n equations of this type, one per wall, define the y_i's as linear functions of the x_i's, and this relationship can be inverted to express the x_i's as a linear function of the y_i's.

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$$\max 22 - 7/3z_1 - 1/3z_2$$

$$-1/3z_1 + 5/3z_2 \leq 6$$
$$z_1 \geq 0$$
$$z_2 \geq 0$$
$$1/3z_1 - 2/3z_2 \leq 1$$
$$1/3z_1 + 1/3z_2 \leq 4$$

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The revised "local" LP has the following three properties:

1. It includes the inequalities $\mathbf{y} \geq 0$, which are simply the transformed versions of the inequalities defining u.
2. u itself is the origin in $\mathbf{y}$-space.
3. The cost function becomes $\max c_u + \tilde{c}^T \mathbf{y}$, where c_u is the value of the objective function at u and $\tilde{c}$ is a transformed cost vector.

Loose End

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We make sure that the right-hand sides of the equations are all nonnegative: if $b_i < 0$, multiply both sides of the i-th equation by -1.

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- Let the objective, to be minimized, be $z_1 + z_2 + \ldots + z_m$.

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1. If the optimum value of $z_1 + \ldots + z_m$ is zero,then all z_i's obtained by simplex are zero, and hence from the optimum vertex of the new LP we get a starting feasible vertex of the original LP.
2. If the optimum objective turns out to be positive: We tried to minimize the sum of the z_i's, but it cannot be zero. This means that the original linear program is infeasible.

Degeneracy

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It means that if we choose any one of n sets of $n+1$ inequalities and solve the corresponding system of these linear equations in n unknowns, we'll get the same solution in all $n+1$ cases.

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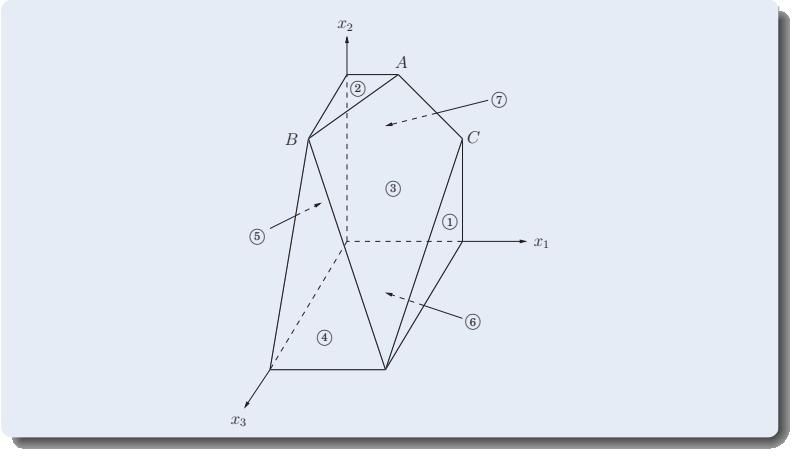

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If we modify simplex so that it detects degeneracy and continues to hop from vertex to vertex despite lack of any improvement in the cost, it may end up looping forever.

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This doesn't change the essence of the LP, but it has the effect of differentiating between the solutions of the linear systems.

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In this case simplex halts and complains.

An Example

$$\max x_1 + x_2$$
$$x_1 - x_2 \geq 0$$
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The Running Time

The Running Time of Simplex

Q: What is the running time of simplex, for a generic linear program:

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where there are n variables and $\mathbb{A}$ contains m inequality constraints?

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It is an iterative algorithm that proceeds from vertex to vertex. Let u be the current vertex.

Each of its neighbors shares $n-1$ of these inequalities, so u can have at most $n \cdot m$ neighbors.

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By Gaussian elimination this takes $O(n^3)$ time, giving total $O(mn^4)$ per iteration.

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Recall the local view from vertex u. The per-iteration overhead of rewriting the LP in terms of the current local coordinates is just $O((m+n)n)$.

The local view changes only slightly between iterations, in just one of its defining inequalities.

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Since the rest of the LP has now been rewritten in terms of the $\mathbf{y}$-coordinates, it is easy to determine how much y_i can be increased before some other inequality is violated.

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It is exponential in n.

The Running Time of Simplex

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A: At most $\binom{m+n}{n}$, i.e., the number of vertices.

It is exponential in n.

And in fact, there are examples of LPs for which simplex does indeed take an exponential number of iterations.

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However, such exponential examples do not occur in practice, and it is this fact that makes simplex so valuable and so widely used.

A Notable Result

Smoothed analysis proposed by Daniel Spielman and Shanghua Teng is a way of measuring the complexity of an algorithm. It gives a more realistic analysis of the practical performance of the algorithm. It was used to explain that the simplex algorithm runs in exponential-time in the worst-case and yet in practice it is a very efficient algorithm, which was one of the main motivations for developing smoothed analysis. The authors received the 2008 Gödel Prize and the 2009 Fulkerson Prize.

Referred Materials

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Content of this lecture comes from Section 7.6 in [DPV07].