Design and Analysis of Algorithms (XVI)

Beyond NP: PSPACE

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Geography game

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Remark. Some problems (especially involving 2-player games and AI) defy classification according to **P**,**EXPTIME**, **NP**, and **NP**-complete.

PSPACE Complexity Class

PSPACE

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Observation. **P** $\subseteq$ **PSPACE**.

PSPACE

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Proof. Consider arbitrary problem $Y \in$ **NP**.

- Since $Y \leq_P$ 3-SAT, there exists algorithm that solves Y in poly-time plus polynomial number of calls to 3-SAT black box.
- Can implement black box in poly-space.

Quiz

Show that **Co-NP** $\subseteq$ **PSPACE**

Quantified Satisfiability

Quantified satisfiability

QSAT. Let $\Phi(\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n})$ be a boolean CNF formula. Is the following propositional formula true?

$$\exists x_1\ \forall x_2\ \exists x_3\ \forall x_4 \ldots\ \forall x_{n-1}\ \exists x_n\ \Phi\left(\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}\right)$$

Intuition. Amy picks truth value for x_1, then Bob for x_2, then Amy for x_3, and so on.
Can Amy satisfy Φ no matter what Bob does?

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No. If Amy sets x_1 false; Bob sets x_2 false; Amy loses;
If Amy sets x_1 true; Bob sets x_2 true; Amy loses.

Quantified satisfiability is in PSPACE

Theorem

Q-SAT $\in$ **PSPACE**.

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- Only need one bit of information from each subproblem.
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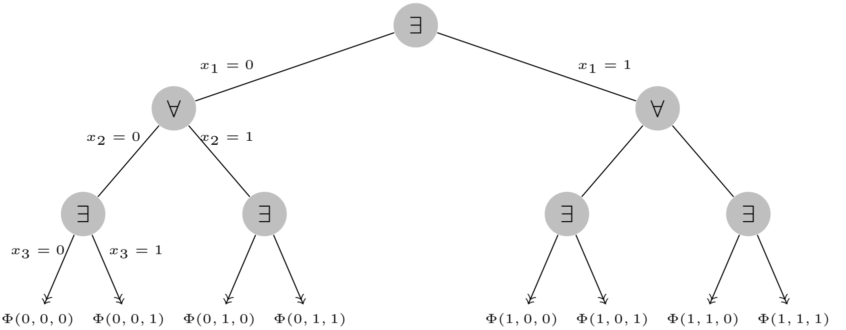
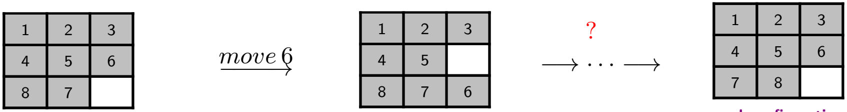


Planning Problem

15-puzzle

8-puzzle, 15-puzzle.[Noyes Chapman 1874]

- Board: 3-by-3 grid of tiles labeled 1–8.
- Legal move: slide neighboring tile into blank (white) square.
- Find sequence of legal moves to transform initial configuration into goal configuration.



initial configuration

goal configuration

Planning problem

Conditions. Set $C = \{C_1, \ldots, C_n\}$.

Initial configuration. Subset $c_0 \subseteq C$ of conditions initially satisfied.

Goal configuration. Subset $c^* \subseteq C$ of conditions we seek to satisfy.

Operators. Set $O = \{O_1, \ldots, O_k\}$.

- To invoke operator O_i, must satisfy certain prereq conditions.
- After invoking O_i certain conditions become true, and certain conditions become false.

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Examples.

- 15-puzzle.
- Rubik's cube.
- Logistical operations to move people, equipment, and materials.

Planning problem: 8-puzzle

Planning example. Can we solve the 8-puzzle?

Conditions. $C_{ij}, 1 \leq i, j \leq 9$.

Initial state. $c_0 = \{C_{11}, C_{22}, \ldots, C_{66}, C_{78}, C_{87}, C_{99}\}$.

Goal state. $c^* = \{C_{11}, C_{22}, \ldots, C_{66}, C_{77}, C_{88}, C_{99}\}$.

Operators.

- Precondition to apply $O_i = \{C_{11}, C_{22}, \ldots, C_{66}, C_{78}, C_{87}, C_{99}\}$.
- After invoking O_i, conditions C_{89} and C_{97} become $true$.
- After invoking O_i, conditions C_{78} and C_{99} become $false$.

1	2	3
4	5	6
8	7	9

$\downarrow O_i$

1	2	3
4	5	6
8	9	7

Solution. No solution to 8-puzzle or 15-puzzle!

Diversion: Why is 8-puzzle unsolvable?

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8-puzzle invariant. Any legal move preserves the parity of the number of pairs of pieces in reverse order (inversions).

3	1	2
4	5	6
8	7	

3 inversions
1-3, 2-3, 7-8

$\longrightarrow$

3	1	2
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8		7

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1-3, 2-3, 7-8

$\longrightarrow$

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4		6
8	5	7

5 inversions
1-3, 2-3, 7-8, 5-8, 5-6

1	2	3
4	5	6
7	8	

0 inversions

$\not\rightarrow$

1	2	3
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8	7	

1 inversion: 7-8

Planning problem: binary counter

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Operators. $O_1, \ldots, O_n$.

- To invoke operator O_i, must satisfy $C_1, \ldots, C_{i-1}$.
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Observation. Any solution requires at least $2^n - 1$ steps.

Planning problem is in EXPTIME

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Configuration graph G.

- Include node for each of 2^n possible configurations.
- Include an edge from configuration c' to configuration $c"$ if one of the operators can convert from c' to $c"$.

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Note. Configuration graph can have 2^n nodes, and shortest path can be of length $= 2^n - 1$.

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Proof.

- Suppose there is a path from c_1 to c_2 of length L.
- Path from c_1 to midpoint and from c_2 to midpoint are each $\leq L/2$.
- Enumerate all possible midpoints.
- Apply recursively. Depth of recursion $= \log_2 L$.

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```
boolean hasPath(c1, c2, L)
  if (L ≤ 1) return correct answer
  for each configuration c'
      boolean x = hasPath(c1, c', L/2)
      boolean y = hasPath(c2, c', L/2)
      if (x and y) return true
  return false
```

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Theorem (Stockmeyer–Meyer 1973)

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Proof. Previous algorithm solves QSAT in exponential time; and QSAT is **PSPACE**-complete.

PSPACE-complete

$$\textbf{P} \subseteq \textbf{NP} \subseteq \textbf{PSPACE} \subseteq \textbf{EXPTIME}$$

PSPACE-complete problems

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More PSPACE-complete problems.

- Competitive facility location.
- Natural generalizations of games.
  - Othello, Hex, Geography, Rush-Hour, Instant Insanity
  - Shanghai, go-moku, Sokoban
- Given a memory restricted Turing machine, does it terminate in at most k steps?
- Do two regular expressions describe different languages?
- Is it possible to move and rotate complicated object with attachments through an irregularly shaped corridor?
- Is a deadlock state possible within a system of communicating processors?

Competitive facility location

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Input. Graph $G = (V, E)$ with positive edge weights, and target B.

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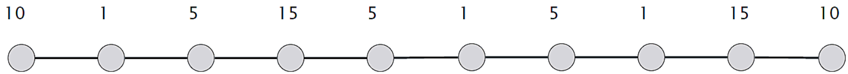

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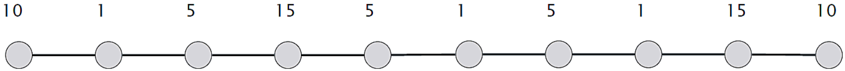

yes if $B = 20$;
no if $B = 25$

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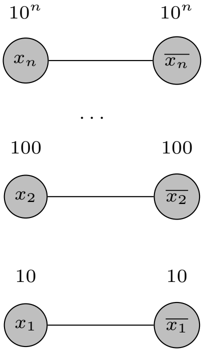
Proof.

- To solve in poly-space, use recursion like Q-SAT, but at each step there are up to n choices instead of 2.
- To show that it's complete, we show that Q-SAT polynomial reduces to it. Given an instance of Q-SAT, we construct an instance of Competitive facility location so that player 2 can force a win iff Q-SAT formula is $true$.

Competitive facility location

Construction. Given instance $\Phi(x_1, \ldots, x_n) = C_1 \wedge C_1 \wedge \ldots C_k$ of Q-SAT.

- Include a node for each literal and its negation and connect them.
  (at most one of x_i and its negation can be chosen)
- Choose $c \geq k+2$, and put weight c_i on literal x^i and its negation;
  set $B = c^{n-1} + c^{n-3} + \ldots + c^4 + c^2 + 1$.
  (ensures variables are selected in order $x_n, x_{n-1}, \ldots, x_1$)
- As is, player 2 will lose by 1 unit: $c^{n-1} + c^{n-3} + \ldots + c^4 + c^2$.



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- Given player 2 one last move on which she can try to win.
- For each clause C_j, add node with value 1 and an edge to each of its literals.
- Player 2 can make last move iff truth assignment defined alternately by the players failed to satisfy some clause.