Design and Analysis of Algorithms (XVII)

DPLL Algorithm

Guoqiang Li
School of Software

Shanghai Jiao Tong University

Why SAT Solving

The First Example

Let $S = \{s_1, \ldots, s_n\}$ be a set of radio stations, each of which has to be allocated one of k transmission frequencies, for some $k < n$. Two stations that are too close to each other cannot have the same frequency. The set of pairs having this constraint is denoted by E. Satisfying

- Every station is assigned at least one frequency.
- Every station is assigned not more than one frequency.
- Close stations are not assigned the same frequency.

Give solution to work out that whether k is enough for a given situation.

The Solution

Define a set of propositional variables

$$\{x_{ij} | i \in \{1, \ldots, n\}, j \in \{1, \ldots, k\}\}$$

Intuitively, variable x_{ij} is set to true if and only if station i is assigned the frequency j.

The Solution

SHANGHAI JIAO TONG UNIVERSITY

The Solution

Every station is assigned at least one frequency:

$$\bigwedge_{i=1}^{n} \bigvee_{j=1}^{k} x_{ij}$$

The Solution

Every station is assigned at least one frequency:

$$\bigwedge_{i=1}^{n} \bigvee_{j=1}^{k} x_{ij}$$

Every station is assigned not more than one frequency:

$$\bigwedge_{i=1}^{n} \bigwedge_{j=1}^{k-1} (x_{ij} \to \wedge_{j<t\leq k} \neg x_{it})$$

The Solution

Every station is assigned at least one frequency:

$$\bigwedge_{i=1}^{n} \bigvee_{j=1}^{k} x_{ij}$$

Every station is assigned not more than one frequency:

$$\bigwedge_{i=1}^{n} \bigwedge_{j=1}^{k-1} (x_{ij} \rightarrow \wedge_{j<t\leq k} \neg x_{it})$$

Close stations are not assigned the same frequency. For each $(i, j) \in E$,

$$\bigwedge_{t=1}^{k} x_{it} \rightarrow \neg x_{jt}$$

The Second Example

Consider the two code fragments. The fragment on the right-hand side might have been generated from the fragment on the left-hand side by an optimizing compiler. We would like to check if the two programs are equivalent.

```
if(!a && !b) h();
else
    if(!a) g();
    else f();
```

```
if(a) f();
else
    if(b) g();
    else h();
```

The Solution

$$(\text{if } x \text{ then } y \text{ else } z) \equiv (x \wedge y) \vee (\neg x \wedge z)$$

The Solution

(if x then y else z) $\equiv (x \wedge y) \vee (\neg x \wedge z)$

$$(\neg a \wedge \neg b) \wedge h \vee \neg(\neg a \wedge \neg b) \wedge (\neg a \wedge g \vee a \wedge f)$$

$$\leftrightarrow a \wedge f \vee \neg a \wedge (b \wedge g \vee \neg b \wedge h)$$

Before Beginning

SHANGHAI JIAO TONG UNIVERSITY

Q: Can a proportional formula be transformed into an equivalent CNF formula effectively?

Before Beginning

Q: Can a proportional formula be transformed into an equivalent CNF formula effectively?

A: It can, however, while potentially increasing the size of the formula exponentially.

Before Beginning

Shanghai Jiao Tong University

Q: Can a proportional formula be transformed into an equivalent CNF formula effectively?

A: It can, however, while potentially increasing the size of the formula exponentially.

The propositional formula can be transformed into an equisatisfiable CNF formula with only a linear increase in the size of the formula.

Before Beginning

Q: Can a proportional formula be transformed into an equivalent CNF formula effectively?

A: It can, however, while potentially increasing the size of the formula exponentially.

The propositional formula can be transformed into an equisatisfiable CNF formula with only a linear increase in the size of the formula.

The price to be paid is n new Boolean variables, known as Tseitin's encoding.

Tseitin's Encoding

The Exponential Way

$CNF(\phi)\{$

case

- ϕ is a literal: return ϕ
- ϕ is $\varphi_1 \wedge \varphi_2$: return $CNF(\varphi_1) \wedge CNF(\varphi_2)$
- ϕ is $\varphi_1 \vee \varphi_2$: return $Dist(CNF(\varphi_1), CNF(\varphi_2))$

$\}$

$Dist(\varphi_1, \varphi_2)\{$

case

- φ_1 is $\psi_{11} \wedge \psi_{12}$: return $Dist(\psi_{11}, \varphi_2) \wedge Dist(\psi_{12}, \varphi_2)$
- φ_2 is $\psi_{21} \wedge \psi_{22}$: return $Dist(\varphi_1, \psi_{21}) \wedge Dist(\varphi_1, \psi_{22})$

$\}$

The Exponential Way

Consider the formula $\phi = (x_1 \wedge y_1) \vee (x_2 \wedge y_2)$

The Exponential Way

Consider the formula $\phi = (x_1 \wedge y_1) \vee (x_2 \wedge y_2)$

$$CNF(\phi) = (x_1 \vee x_2) \wedge (x_1 \vee y_2) \wedge (y_1 \vee x_2) \wedge (y_1 \vee y_2)$$

The Exponential Way

Consider the formula $\phi = (x_1 \wedge y_1) \vee (x_2 \wedge y_2)$

$$CNF(\phi) = (x_1 \vee x_2) \wedge (x_1 \vee y_2) \wedge (y_1 \vee x_2) \wedge (y_1 \vee y_2)$$

Now consider: $\phi_n = (x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee \ldots \vee (x_n \wedge y_n)$

The Exponential Way

Consider the formula $\phi = (x_1 \wedge y_1) \vee (x_2 \wedge y_2)$

$$CNF(\phi) = (x_1 \vee x_2) \wedge (x_1 \vee y_2) \wedge (y_1 \vee x_2) \wedge (y_1 \vee y_2)$$

Now consider: $\phi_n = (x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee \ldots \vee (x_n \wedge y_n)$

Q: How many clauses $CNF(\phi_n)$ returns?

The Exponential Way

Consider the formula $\phi = (x_1 \wedge y_1) \vee (x_2 \wedge y_2)$

$$CNF(\phi) = (x_1 \vee x_2) \wedge (x_1 \vee y_2) \wedge (y_1 \vee x_2) \wedge (y_1 \vee y_2)$$

Now consider: $\phi_n = (x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee \ldots \vee (x_n \wedge y_n)$

Q: How many clauses $CNF(\phi_n)$ returns?

A: 2^n

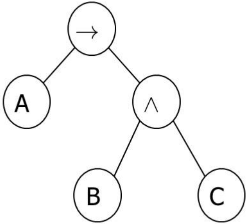
Tseitin's Encoding

Consider the formula $(A \to (B \wedge C))$

Tseitin's Encoding

Consider the formula $(A \to (B \wedge C))$

The parse tree:

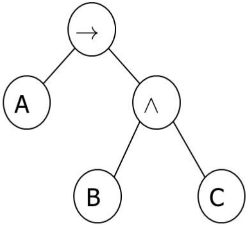


Tseitin's Encoding

SHANGHAI JIAO TONG UNIVERSITY

Consider the formula $(A \to (B \wedge C))$

The parse tree:



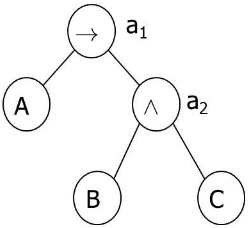
Associate a new auxiliary variable with each gate.

Add constraints that define these new variables.

Finally, enforce the root node.

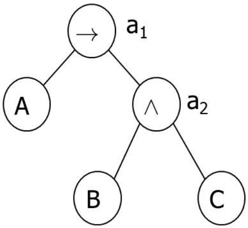
Tseitin's Encoding

$$(a_1 \leftrightarrow (A \rightarrow a_2)) \wedge (a_2 \leftrightarrow (B \wedge C)) \wedge (a_1)$$



Tseitin's Encoding

$$(a_1 \leftrightarrow (A \rightarrow a_2)) \wedge (a_2 \leftrightarrow (B \wedge C)) \wedge (a_1)$$

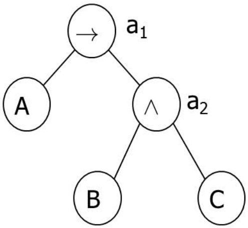


Each such constraint has a CNF representation with 3 or 4 clauses.

Tseitin's Encoding

SHANGHAI JIAO TONG UNIVERSITY

$(a_1 \leftrightarrow (A \rightarrow a_2)) \wedge (a_2 \leftrightarrow (B \wedge C)) \wedge (a_1)$



Each such constraint has a CNF representation with 3 or 4 clauses.

First: $(a_1 \vee A) \wedge (a_1 \vee \neg a_2) \wedge (\neg a_1 \vee A \vee a_2)$

Second: $(\neg a_2 \vee B) \wedge (\neg a_2 \vee C) \wedge (a_2 \vee \neg B \vee \neg C)$

Tseitin's Encoding

$$\phi_n = (x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee \ldots \vee (x_n \wedge y_n)$$

Tseitin's Encoding

$$\phi_n = (x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee \ldots \vee (x_n \wedge y_n)$$

With Tseitin's encoding we need:

- n auxiliary variables $a_1, \ldots, a_n$.
- Each adds 3 constraints.
- Top clause: $(a_1 \vee \ldots \vee a_n)$

Tseitin's Encoding

$$\phi_n = (x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee \ldots \vee (x_n \wedge y_n)$$

With Tseitin's encoding we need:

- n auxiliary variables $a_1, \ldots, a_n$.
- Each adds 3 constraints.
- Top clause: $(a_1 \vee \ldots \vee a_n)$

Hence, we have

- $3n + 1$ clauses, instead of 2^n.
- $3n$ variables rather than $2n$.

Methodologies

Two Usual Ways to Implement

Exhaustive Search (DPLL Algorithm): traversing and backtracking on a binary tree.

Two Usual Ways to Implement

Exhaustive Search (DPLL Algorithm): traversing and backtracking on a binary tree.

Stochastic Search: guessing a full assignment, and flipping values of variables according to some heuristic.

A Brief History

Shanghai Jiao Tong University

Originally, DPLL was incomplete method for SAT in FO logic

A Brief History

Originally, DPLL was incomplete method for SAT in FO logic

First paper (Davis and Putnam) in 1960: memory problems

A Brief History

Originally, DPLL was incomplete method for SAT in FO logic

First paper (Davis and Putnam) in 1960: memory problems

Second paper (Davis, Logemann and Loveland) in 1962: Depth-first-search with backtracking

A Brief History

Originally, DPLL was incomplete method for SAT in FO logic

First paper (Davis and Putnam) in 1960: memory problems

Second paper (Davis, Logemann and Loveland) in 1962: Depth-first-search with backtracking

Late 90's and early 00's improvements make DPLL efficient:

A Brief History

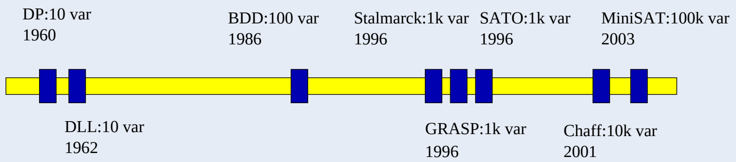
Originally, DPLL was incomplete method for SAT in FO logic

First paper (Davis and Putnam) in 1960: memory problems

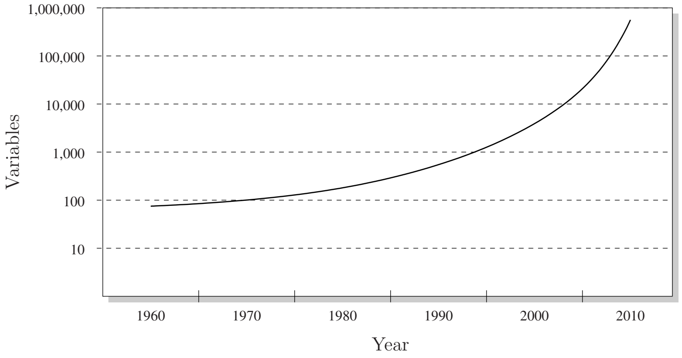
Second paper (Davis, Logemann and Loveland) in 1962: Depth-first-search with backtracking

Late 90's and early 00's improvements make DPLL efficient:

Break-through systems: GRASP, SATO, zChaff, MiniSAT, Z3



A Brief History



Backtracking

Backtracking

It is often possible to reject a solution by looking at just a small portion of it.

An Solution of SAT

For example, if an `instance` of SAT contains the clause $(x_1 \vee x_2)$, then all assignments with $x_1 = x_2 = \texttt{false}$ can be instantly eliminated.

An Solution of SAT

For example, if an `instance` of SAT contains the clause $(x_1 \vee x_2)$, then all assignments with $x_1 = x_2 = \texttt{false}$ can be instantly eliminated.

To put it differently, by quickly checking and discrediting this partial assignment, we are able to prune a quarter of the entire search space.

An Solution of SAT

For example, if an `instance` of SAT contains the clause $(x_1 \vee x_2)$, then all assignments with $x_1 = x_2 = \texttt{false}$ can be instantly eliminated.

To put it differently, by quickly checking and discrediting this partial assignment, we are able to prune a quarter of the entire search space.

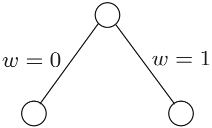
A promising direction, but can it be systematically exploited?

An Example

SHANGHAI JIAO TONG UNIVERSITY

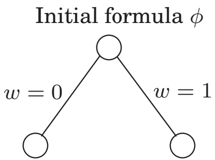
$$(w \vee x \vee y \vee z)(w \vee \overline{x})(x \vee \overline{y})(y \vee \overline{z})(z \vee \overline{w})(\overline{w} \vee \overline{z})$$

Initial formula ϕ



An Example

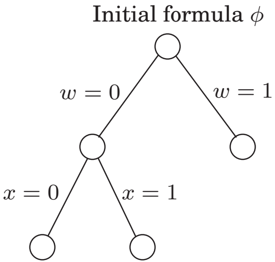
$$(w \vee x \vee y \vee z)(w \vee \overline{x})(x \vee \overline{y})(y \vee \overline{z})(z \vee \overline{w})(\overline{w} \vee \overline{z})$$



Plugging $w = 0$ and $w = 1$ into Φ, we find that no clause is immediately violated and thus neither of these two partial assignments can be eliminated outright.

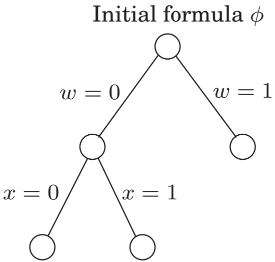
An Example

$$\Phi = (w \vee x \vee y \vee z)(w \vee \overline{x})(x \vee \overline{y})(y \vee \overline{z})(z \vee \overline{w})(\overline{w} \vee \overline{z})$$



An Example

$$\Phi = (w \vee x \vee y \vee z)(w \vee \overline{x})(x \vee \overline{y})(y \vee \overline{z})(z \vee \overline{w})(\overline{w} \vee \overline{z})$$



The partial assignment $w = 0, x = 1$ violates the clause $(w \vee \overline{x})$ and can be terminated, thereby pruning a good chunk of the search space.

An Example

$$\Phi = (w \vee x \vee y \vee z)(w \vee \overline{x})(x \vee \overline{y})(y \vee \overline{z})(z \vee \overline{w})(\overline{w} \vee \overline{z})$$

An Example

$$\Phi = (w \vee x \vee y \vee z)(w \vee \overline{x})(x \vee \overline{y})(y \vee \overline{z})(z \vee \overline{w})(\overline{w} \vee \overline{z})$$

Backtracking explores the space of assignments, only growing the tree only at nodes where there is uncertainty.

An Example

$$\Phi = (w \vee x \vee y \vee z)(w \vee \overline{x})(x \vee \overline{y})(y \vee \overline{z})(z \vee \overline{w})(\overline{w} \vee \overline{z})$$

Backtracking explores the space of assignments, only growing the tree only at nodes where there is uncertainty.

Each node of the search tree can be described either by a partial assignment or by the clauses that remain.

An Example

$$\Phi = (w \vee x \vee y \vee z)(w \vee \overline{x})(x \vee \overline{y})(y \vee \overline{z})(z \vee \overline{w})(\overline{w} \vee \overline{z})$$

Backtracking explores the space of assignments, only growing the tree only at nodes where there is uncertainty.

Each node of the search tree can be described either by a partial assignment or by the clauses that remain.

If $w = 0$ and $x = 0$ then any clause with w or x is instantly satisfied and any literal $\overline{w}$ or $\overline{x}$ is not satisfied and can be removed.

An Example

$$\Phi = (w \vee x \vee y \vee z)(w \vee \overline{x})(x \vee \overline{y})(y \vee \overline{z})(z \vee \overline{w})(\overline{w} \vee \overline{z})$$

Backtracking explores the space of assignments, only growing the tree only at nodes where there is uncertainty.

Each node of the search tree can be described either by a partial assignment or by the clauses that remain.

If $w = 0$ and $x = 0$ then any clause with w or x is instantly satisfied and any literal $\overline{w}$ or $\overline{x}$ is not satisfied and can be removed.

What's left is

$$(y \vee z)(\overline{y})(y \vee \overline{z})$$

An Example

$$\Phi = (w \vee x \vee y \vee z)(w \vee \overline{x})(x \vee \overline{y})(y \vee \overline{z})(z \vee \overline{w})(\overline{w} \vee \overline{z})$$

Backtracking explores the space of assignments, only growing the tree only at nodes where there is uncertainty.

Each node of the search tree can be described either by a partial assignment or by the clauses that remain.

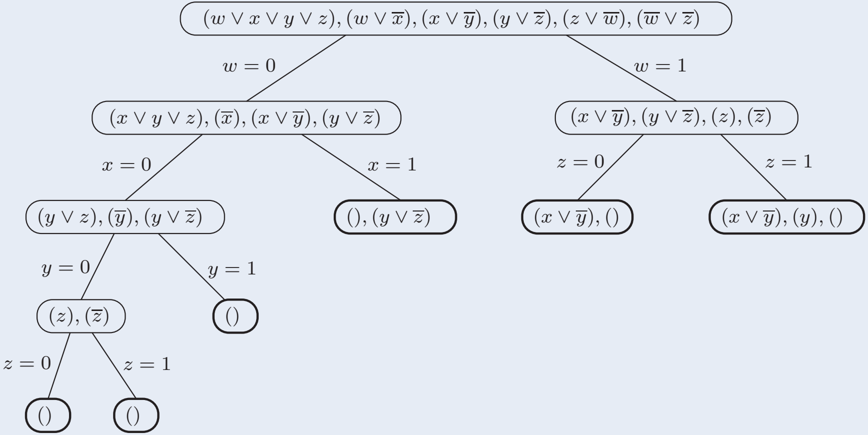
If $w = 0$ and $x = 0$ then any clause with w or x is instantly satisfied and any literal $\overline{w}$ or $\overline{x}$ is not satisfied and can be removed.

What's left is

$$(y \vee z)(\overline{y})(y \vee \overline{z})$$

Thus the nodes of the search tree, representing partial assignments, are themselves SAT subproblems.

An Example


$(w \vee x \vee y \vee z), (w \vee \overline{x}), (x \vee \overline{y}), (y \vee \overline{z}), (z \vee \overline{w}), (\overline{w} \vee \overline{z})$
$w = 0$
$w = 1$
$(x \vee y \vee z), (\overline{x}), (x \vee \overline{y}), (y \vee \overline{z})$
$(x \vee \overline{y}), (y \vee \overline{z}), (z), (\overline{z})$
$x = 0$
$x = 1$
$z = 0$
$z = 1$
$(y \vee z), (\overline{y}), (y \vee \overline{z})$
$(), (y \vee \overline{z})$
$(x \vee \overline{y}), ()$
$(x \vee \overline{y}), (y), ()$
$y = 0$
$y = 1$
$(z), (\overline{z})$
$()$
$z = 0$
$z = 1$
$()$
$()$

Basic Functions

`Decide()`: Choose the next variable and value. Return False if all variables are assigned.

`BCP()`: Apply repeatedly the unit clause rule. Return False if reached a conflict.

`Resolve-conflict()`: Backtrack until no conflict. Return False if impossible.

Algorithm

SHANGHAI JIAO TONG UNIVERSITY

```
SAT()

while true do
    if ¬ Decide () then
        return true ;
    end
    else
        while ¬ BCP () do
            if ¬ Resolve-conflict () then return false;
        end
    end
end
```

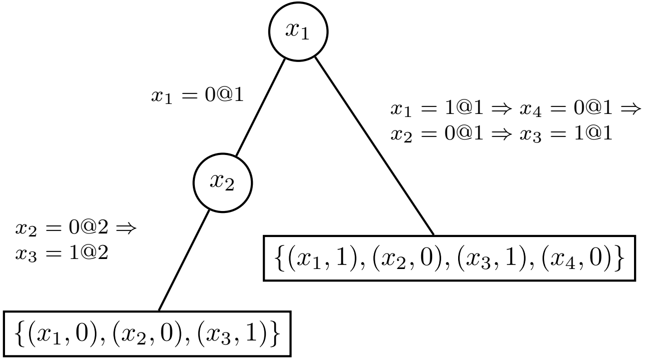
Basic Backtracking Search

Organize the search in the form of a decision tree

- Each node corresponds to a decision.
- Definition: Decision Level (DL) is the depth of the node in the decision tree.
- Notation: $x = v@d$, where $x \in \{0, 1\}$ is assigned to v at decision level d.

Backtracking Search in Action

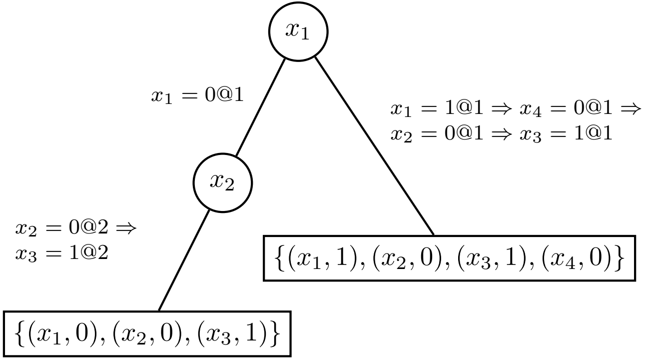
$$(x_2 \vee x_3), (\neg x_1 \vee, \neg x_4), (\neg x_2 \vee x_4)$$



Backtracking Search in Action

SHANGHAI JIAO TONG UNIVERSITY

$$(x_2 \vee x_3), (\neg x_1 \vee, \neg x_4), (\neg x_2 \vee x_4)$$

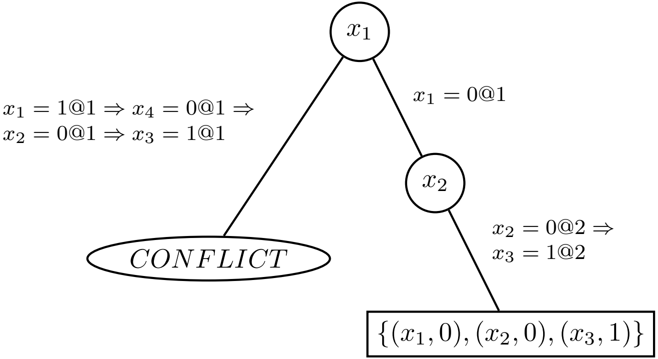


No backtrack in this example, regardless of the decision!

Backtracking Search in Action

SHANGHAI JIAO TONG UNIVERSITY

$$(x_2 \vee x_3), (\neg x_1 \vee, \neg x_4), (\neg x_2 \vee x_4), (\neg x_1 \vee x_2 \vee \neg x_3)$$



Status of a Clause

A clause can be

- Satisfied: at least one literal is satisfied
- Unsatisfied: all literals are assigned but non are satisfied
- Unit: all but one literals are assigned but none are satisfied
- Unresolved: all other cases

Status of a Clause

Shanghai Jiao Tong University

A clause can be

- Satisfied: at least one literal is satisfied
- Unsatisfied: all literals are assigned but non are satisfied
- Unit: all but one literals are assigned but none are satisfied
- Unresolved: all other cases

Example: $C = (x_1 \vee x_2 \vee x_3)$

x_1	x_2	x_3	C
1	0		Satisfied
0	0	0	Unsatisfied
0	0		Unit
	0		Unresolved

Decision Heuristics - DLIS

DLIS (Dynamic Largest Individual Sum)

Choose the assignment that increases the most the number of satisfied clauses.

Decision Heuristics - DLIS

DLIS (Dynamic Largest Individual Sum)

Choose the assignment that increases the most the number of satisfied clauses.

For a given variable x:

- C_{xp}: $\sharp$ unresolved clauses in which x appears positively
- C_{xn}: $\sharp$ unresolved clauses in which x appears negatively
- Let x be the literal for which C_{xp} is maximal
- Let y be the literal for which C_{yn} is maximal
- If $C_{xp} > C_{yn}$ choose x and assign it TRUE
- Otherwise choose y and assign it FALSE

Decision Heuristics - DLIS

Shanghai Jiao Tong University

DLIS (Dynamic Largest Individual Sum)

Choose the assignment that increases the most the number of satisfied clauses.

For a given variable x:

- C_{xp}: $\sharp$ unresolved clauses in which x appears positively
- C_{xn}: $\sharp$ unresolved clauses in which x appears negatively
- Let x be the literal for which C_{xp} is maximal
- Let y be the literal for which C_{yn} is maximal
- If $C_{xp} > C_{yn}$ choose x and assign it TRUE
- Otherwise choose y and assign it FALSE

Requires l ($\sharp$ literals) queries for each decision.

Decision Heuristics - JW

Jeroslow-Wang

Compute for every clause w and every literal l in each phase

$$J(l) = \sum_{l \in w, w \in \varphi} 2^{-|w|}$$

where $|w|$ the length.

Decision Heuristics - JW

Jeroslow-Wang

Compute for every clause w and every literal l in each phase

$$J(l) = \sum_{l \in w, w \in \varphi} 2^{-|w|}$$

where $|w|$ the length.

Choose the literal l that maximizes $J(l)$.

This gives an exponentially higher weight to literals in shorter clauses.

Next

We will see other (more advanced) decision Heuristics soon.

Next

We will see other (more advanced) decision Heuristics soon.

These heuristics are integrated with a mechanism called Learning with Conflict-Clauses, which we will learn next.

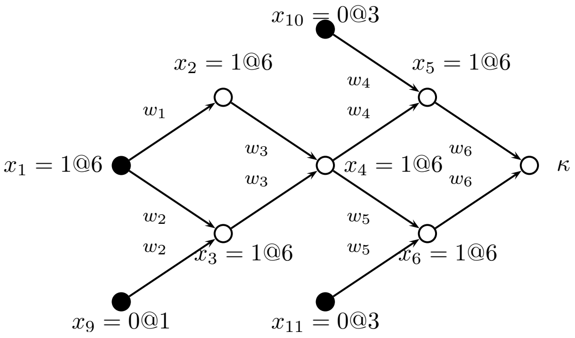
Learning New Clause

Implication Graphs and Learning

Current truth assignment $\{x_9 = 0@1, x_{10} = 0@3, x_{11} = 0@3, x_{12} = 1@2, x_{13} = 1@2\}$

Current decision assignment $\{x_1 = 1@6\}$

$$
\begin{aligned}
w_1 &= \neg x_1 \vee x_2 \\
w_2 &= \neg x_1 \vee x_3 \vee x_9 \\
w_3 &= \neg x_2 \vee \neg x_3 \vee x_4 \\
w_4 &= \neg x_4 \vee x_5 \vee x_{10} \\
w_5 &= \neg x_4 \vee x_6 \vee x_{11} \\
w_6 &= \neg x_5 \vee \neg x_6 \\
w_7 &= x_1 \vee x_7 \vee \neg x_{12} \\
w_8 &= x_1 \vee x_8 \\
w_9 &= \neg x_7 \vee \neg x_8 \vee \neg x_{13}
\end{aligned}
$$

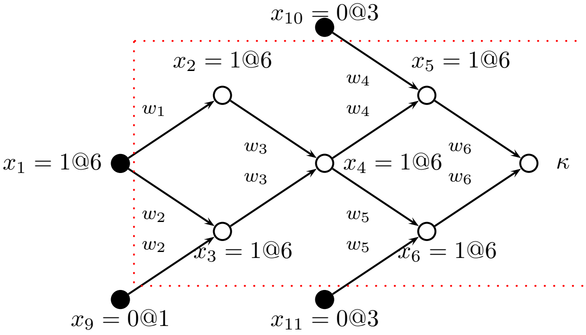


Implication Graphs and Learning

Current truth assignment $\{x_9 = 0@1, x_{10} = 0@3, x_{11} = 0@3, x_{12} = 1@2, x_{13} = 1@2\}$

Current decision assignment $\{x_1 = 1@6\}$

$$
\begin{aligned}
w_1 &= \neg x_1 \vee x_2 \\
w_2 &= \neg x_1 \vee x_3 \vee x_9 \\
w_3 &= \neg x_2 \vee \neg x_3 \vee x_4 \\
w_4 &= \neg x_4 \vee x_5 \vee x_{10} \\
w_5 &= \neg x_4 \vee x_6 \vee x_{11} \\
w_6 &= \neg x_5 \vee \neg x_6 \\
w_7 &= x_1 \vee x_7 \vee \neg x_{12} \\
w_8 &= x_1 \vee x_8 \\
w_9 &= \neg x_7 \vee \neg x_8 \vee \neg x_{13}
\end{aligned}
$$



We learn the conflict clause $w_{10} : (\neg x_1 \vee x_9 \vee x_{11} \vee x_{10})$

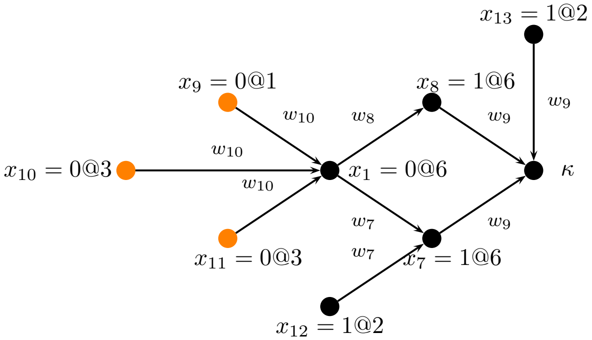
Flipped Assignment

SHANGHAI JIAO TONG UNIVERSITY

Current truth assignment $\{x_9 = 0@1, x_{10} = 0@3, x_{11} = 0@3, x_{12} = 1@2, x_{13} = 1@2\}$

Current flipped assignment $\{x_1 = 0@6\}$

$$w_1 = \neg x_1 \vee x_2$$
$$w_2 = \neg x_1 \vee x_3 \vee x_9$$
$$w_3 = \neg x_2 \vee \neg x_3 \vee x_4$$
$$w_4 = \neg x_4 \vee x_5 \vee x_{10}$$
$$w_5 = \neg x_4 \vee x_6 \vee x_{11}$$
$$w_6 = \neg x_5 \vee \neg x_6$$
$$w_7 = x_1 \vee x_7 \vee \neg x_{12}$$
$$w_8 = x_1 \vee x_8$$
$$w_9 = \neg x_7 \vee \neg x_8 \vee \neg x_{13}$$
$$w_{10} = \neg x_1 \vee x_9 \vee x_{11} \vee x_{10}$$



Flipped Assignment

Current truth assignment $\{x_9 = 0@1, x_{10} = 0@3, x_{11} = 0@3, x_{12} = 1@2, x_{13} = 1@2\}$

Current flipped assignment $\{x_1 = 0@6\}$

$w_1 = \neg x_1 \vee x_2$

$w_2 = \neg x_1 \vee x_3 \vee x_9$

$w_3 = \neg x_2 \vee \neg x_3 \vee x_4$

$w_4 = \neg x_4 \vee x_5 \vee x_{10}$

$w_5 = \neg x_4 \vee x_6 \vee x_{11}$

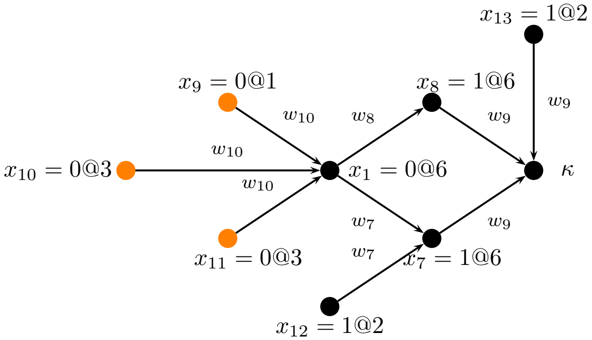
$w_6 = \neg x_5 \vee \neg x_6$

$w_7 = x_1 \vee x_7 \vee \neg x_{12}$

$w_8 = x_1 \vee x_8$

$w_9 = \neg x_7 \vee \neg x_8 \vee \neg x_{13}$

$w_{10} = \neg x_1 \vee x_9 \vee x_{11} \vee x_{10}$



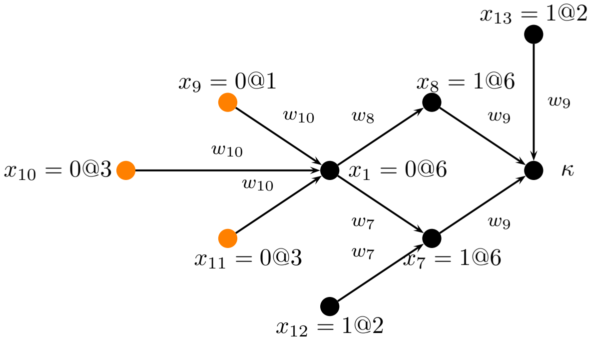
Another conflict clause: $w_{11} : (\neg x_{13} \vee \neg x_{12} \vee x_{11} \vee x_{10} \vee x_9)$

Flipped Assignment

Current truth assignment $\{x_9 = 0@1, x_{10} = 0@3, x_{11} = 0@3, x_{12} = 1@2, x_{13} = 1@2\}$

Current flipped assignment $\{x_1 = 0@6\}$

$w_1 = \neg x_1 \vee x_2$
$w_2 = \neg x_1 \vee x_3 \vee x_9$
$w_3 = \neg x_2 \vee \neg x_3 \vee x_4$
$w_4 = \neg x_4 \vee x_5 \vee x_{10}$
$w_5 = \neg x_4 \vee x_6 \vee x_{11}$
$w_6 = \neg x_5 \vee \neg x_6$
$w_7 = x_1 \vee x_7 \vee \neg x_{12}$
$w_8 = x_1 \vee x_8$
$w_9 = \neg x_7 \vee \neg x_8 \vee \neg x_{13}$
$w_{10} = \neg x_1 \vee x_9 \vee x_{11} \vee x_{10}$



Another conflict clause: $w_{11} : (\neg x_{13} \vee \neg x_{12} \vee x_{11} \vee x_{10} \vee x_9)$

Where should we backtrack to now?

Non-Chronological Backtracking

Which assignments caused the conflicts?

- $x_9 = 0@1$
- $x_{10} = 0@3$
- $x_{11} = 0@3$
- $x_{12} = 1@2$
- $x_{13} = 1@2$

These assignments are sufficient for causing a conflict.

Non-Chronological Backtracking

Which assignments caused the conflicts?

- $x_9 = 0@1$
- $x_{10} = 0@3$
- $x_{11} = 0@3$
- $x_{12} = 1@2$
- $x_{13} = 1@2$

These assignments are sufficient for causing a conflict.

Backtrack to $DL = 3$

Non-Chronological Backtracking

So the rule is: backtrack to the largest decision level in the conflict clause.

This works for both the initial conflict and the conflict after the flip.

Q: What if the flipped assignment works?

Non-Chronological Backtracking

So the rule is: backtrack to the largest decision level in the conflict clause.

This works for both the initial conflict and the conflict after the flip.

Q: What if the flipped assignment works?

A: Change the decision retroactively.

Non-Chronological Backtracking

$$
\begin{aligned}
x_1 &= 0 \\
x_2 &= 0 \\
x_3 &= 1 \\
x_4 &= 0 \\
x_5 &= 0
\end{aligned}
$$

Non-Chronological Backtracking

SHANGHAI JIAO TONG UNIVERSITY

$$
\begin{aligned}
x_1 &= 0 \\
x_2 &= 0 \\
x_3 &= 1 \\
x_4 &= 0 \\
x_5 &= 0
\end{aligned}
$$

Non-Chronological Backtracking

$x_1 = 0$

$x_2 = 0$

$x_3 = 1$

$x_4 = 0$

$x_5 = 0$ $\qquad$ $x_5 = 1$

$x_7 = 0$

$x_9 = 1$

Non-Chronological Backtracking

SHANGHAI JIAO TONG UNIVERSITY

$x_1 = 0$

$x_2 = 0$

$x_3 = 1$

$x_4 = 0$

$x_5 = 0$ $\quad$ $x_5 = 1$

$x_7 = 0$

$x_9 = 1$ $\quad$ $x_9 = 0$

Non-Chronological Backtracking

$x_1 = 0$		
$x_2 = 0$		
$x_3 = 1$		$x_3 = 0$
$x_4 = 0$		
$x_5 = 0$	$x_5 = 1$	
	$x_7 = 0$	
	$x_9 = 1$	$x_9 = 0$

Non-Chronological Backtracking

$x_1 = 0$

$x_2 = 0$

$x_3 = 0$

$x_6 = 0$

$\dots$

$\dots$

More Conflict Clauses

Shanghai Jiao Tong University

Definition

A Conflict Clause is any clause implied by the formula.

More Conflict Clauses

Definition

A Conflict Clause is any clause implied by the formula.

Let L be a set of literals labeling nodes that form a cut in the implication graph, separating the conflict node from the roots.

More Conflict Clauses

Shanghai Jiao Tong University

Definition

A Conflict Clause is any clause implied by the formula.

Let L be a set of literals labeling nodes that form a cut in the implication graph, separating the conflict node from the roots.

Claim: $\bigvee_{l \in L} \neg l$ is a Conflict Clause.

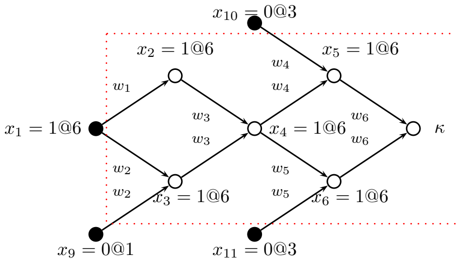
More Conflict Clauses

Definition

A Conflict Clause is any clause implied by the formula.

Let L be a set of literals labeling nodes that form a cut in the implication graph, separating the conflict node from the roots.

Claim: $\bigvee_{l \in L} \neg l$ is a Conflict Clause.



$x_{10} \vee \neg x_1 \vee x_9 \vee x_{11}$

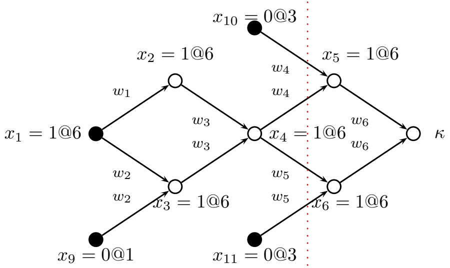
More Conflict Clauses

Definition

A Conflict Clause is any clause implied by the formula.

Let L be a set of literals labeling nodes that form a cut in the implication graph, separating the conflict node from the roots.

Claim: $\bigvee_{l \in L} \neg l$ is a Conflict Clause.



$x_{10} \vee \neg x_1 \vee x_9 \vee x_{11}$

$x_{10} \vee \neg x_4 \vee x_{11}$

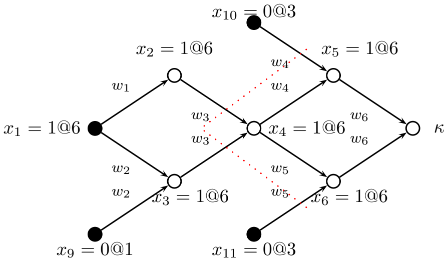
More Conflict Clauses

Definition

A Conflict Clause is any clause implied by the formula.

Let L be a set of literals labeling nodes that form a cut in the implication graph, separating the conflict node from the roots.

Claim: $\bigvee_{l \in L} \neg l$ is a Conflict Clause.



$x_{10} \vee \neg x_1 \vee x_9 \vee x_{11}$

$x_{10} \vee \neg x_4 \vee x_{11}$

$x_{10} \vee \neg x_2 \vee \neg x_3 \vee x_{11}$

Conflict Clause

How many clauses should we add?

If not all, then which ones?

- Shorter ones?
- Check their influence on the backtracking level?
- The most "influential"?

Conflict Clause

SHANGHAI JIAO TONG UNIVERSITY

Definition

An Asserting Clause is a Conflict Clause with a single literal from the current decision level. Backtracking (to the right level) makes it a Unit clause.

Asserting clauses are those that force an immediate change in the search path.

Modern solvers only consider Asserting Clauses.

Unique Implication Points (UIPs)

Definition (Unique Implication Point (UIP))

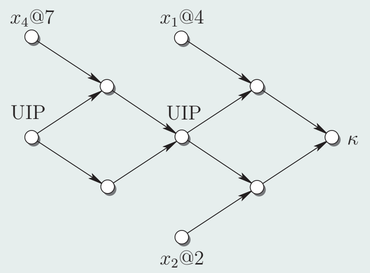
A Unique Implication Point (UIP) is an internal node in the Implication Graph that all paths from the decision to the conflict node go through it.

Unique Implication Points (UIPs)

Definition (Unique Implication Point (UIP))

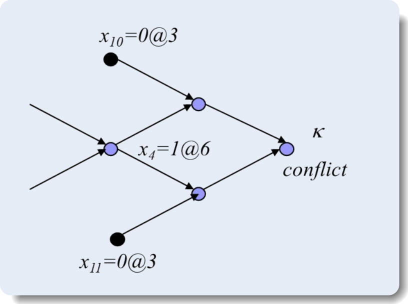
A Unique Implication Point (UIP) is an internal node in the Implication Graph that all paths from the decision to the conflict node go through it.

The First-UIP is the closest UIP to the conflict.



Alternative Backtracking

Conflict-Driven Backtracking



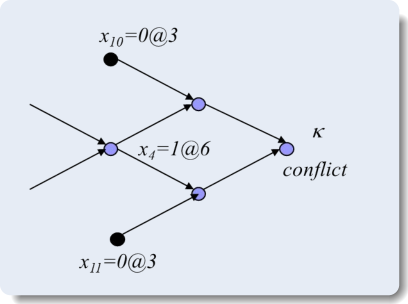
Conflict clause: $(x_{10} \vee \neg x_4 \vee \neg x_{11})$

With standard Non-Chronological Backtracking we backtracked to $DL = 6$.

Conflict-driven Backtrack: backtrack to the second highest decision level in the clause (without erasing it).

In this case, to $DL = 3$.

Conflict-Driven Backtracking



Conflict clause: $(x_{10} \vee \neg x_4 \vee \neg x_{11})$

With standard Non-Chronological Backtracking we backtracked to $DL = 6$.

Conflict-driven Backtrack: backtrack to the second highest decision level in the clause (without erasing it).

In this case, to $DL = 3$.

Q: why?

Conflict-Driven Backtracking

Shanghai Jiao Tong University

$$
\begin{aligned}
x_1 &= 0 \\
x_2 &= 0 \\
x_3 &= 1 \\
x_4 &= 0 \\
x_5 &= 0
\end{aligned}
$$

Conflict-Driven Backtracking

$$x_1 = 0$$
$$x_2 = 0$$
$$x_3 = 1$$
$$x_4 = 0$$
$$x_5 = 0$$

Conflict-Driven Backtracking

$$
\begin{aligned}
x_1 &= 0 \\
x_2 &= 0 \\
x_3 &= 1 \\
x_4 &= 0 \\
x_5 &= 0
\end{aligned}
$$

Conflict-Driven Backtracking

$$x_1 = 0$$
$$x_2 = 0$$
$$x_5 = 1$$
$$x_7 = 0$$
$$x_9 = 1$$

Conflict-Driven Backtracking

$x_1 = 0$

$x_2 = 0$

$x_5 = 1$

$x_7 = 0$

$x_9 = 1$

Conflict-Driven Backtracking

$x_1 = 0$

$x_2 = 0$

$x_5 = 1$

$x_7 = 0$

$x_9 = 1$

Conflict-Driven Backtracking

$x_1 = 0$

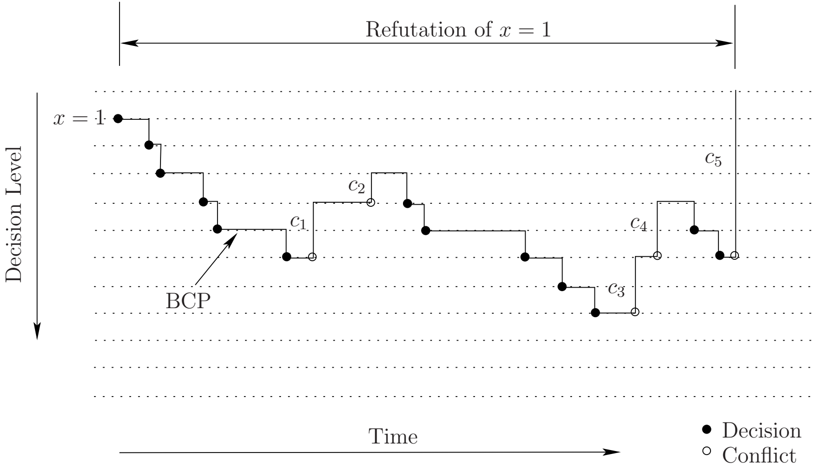
$x_2 = 0$

$x_5 = 1$

$x_9 = 0$

$x_6 = 0$

Conflict-Driven Backtracking



Conflict-Driven Backtracking

So the rule is: backtrack to the second highest decision level dl, but do not erase it.

This way the literal with the currently highest decision level will be implied in $DL = dl$.

Q: what if the conflict clause has a single literal?

For example, from $(x \vee \neg y) \wedge (x \vee y)$ and decision $x = 0$, we learn the conflict clause (x).

Resolution

Resolution

The binary resolution is a sound inference rule:

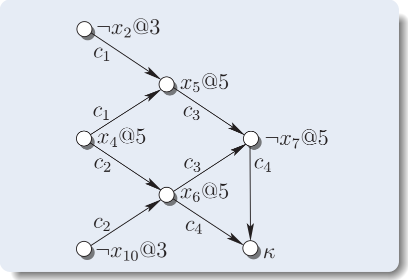
$$\frac{(a_1 \vee \ldots \vee a_n \vee \beta) \qquad (b_1 \vee \ldots \vee b_m \vee \neg\beta)}{(a_1 \vee \ldots \vee a_n \vee b_1 \vee \ldots \vee b_m)} \text{ Binary Resolution}$$

Example

$$\frac{x_1 \vee x_2 \qquad \neg x_1 \vee x_3 \vee x_4}{x_2 \vee x_3 \vee x_4}$$

Example

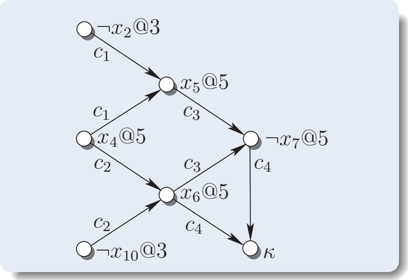
$$
\begin{aligned}
c_1 &= (\neg x_4 \vee x_2 \vee x_5)\\
c_2 &= (\neg x_4 \vee x_{10} \vee x_6)\\
c_3 &= (\neg x_5 \vee \neg x_6 \vee \neg x_7)\\
c_4 &= (\neg x_6 \vee x_7)
\end{aligned}
$$



Conflict Clause : $c_5 = (\neg x_4 \vee x_2 \vee x_{10})$

Example

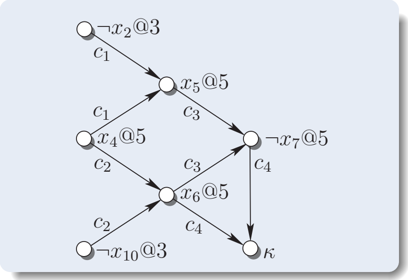
$$
\begin{aligned}
c_1 &= (\neg x_4 \vee x_2 \vee x_5) \\
c_2 &= (\neg x_4 \vee x_{10} \vee x_6) \\
c_3 &= (\neg x_5 \vee \neg x_6 \vee \neg x_7) \\
c_4 &= (\neg x_6 \vee x_7)
\end{aligned}
$$



Assume that the implication order in the BCP was x_4, x_5, x_6, x_7.

Example

$$c_1 = (\neg x_4 \vee x_2 \vee x_5)$$
$$c_2 = (\neg x_4 \vee x_{10} \vee x_6)$$
$$c_3 = (\neg x_5 \vee \neg x_6 \vee \neg x_7)$$
$$c_4 = (\neg x_6 \vee x_7)$$

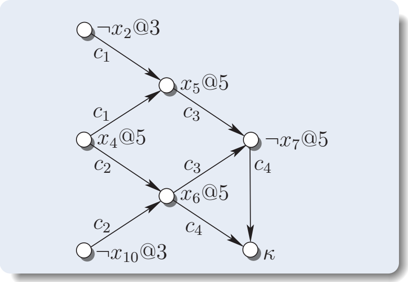


Assume that the implication order in the BCP was x_4, x_5, x_6, x_7.

name	cl	lit	var	$ante$

Example

$$c_1 = (\neg x_4 \vee x_2 \vee x_5)$$
$$c_2 = (\neg x_4 \vee x_{10} \vee x_6)$$
$$c_3 = (\neg x_5 \vee \neg x_6 \vee \neg x_7)$$
$$c_4 = (\neg x_6 \vee x_7)$$

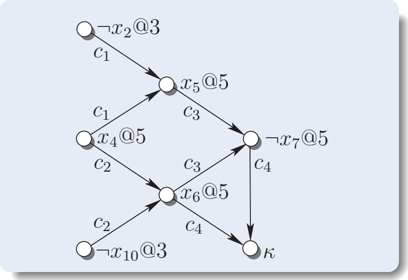


Assume that the implication order in the BCP was x_4, x_5, x_6, x_7.

name	cl	lit	var	$ante$
c_4	$(\neg x_6 \vee x_7)$	x_7	x_7	c_3

Example

$$
\begin{aligned}
c_1 &= (\neg x_4 \vee x_2 \vee x_5) \\
c_2 &= (\neg x_4 \vee x_{10} \vee x_6) \\
c_3 &= (\neg x_5 \vee \neg x_6 \vee \neg x_7) \\
c_4 &= (\neg x_6 \vee x_7)
\end{aligned}
$$

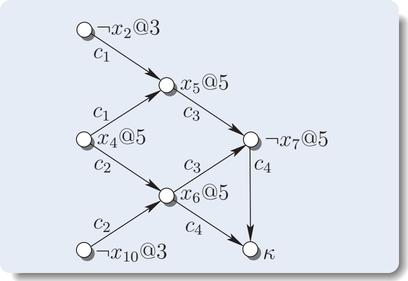


Assume that the implication order in the BCP was x_4, x_5, x_6, x_7.

name	cl	lit	var	$ante$
c_4	$(\neg x_6 \vee x_7)$	x_7	x_7	c_3
	$(\neg x_5 \vee \neg x_6)$	$\neg x_6$	x_6	c_2

Example

$$c_1 = (\neg x_4 \vee x_2 \vee x_5)$$
$$c_2 = (\neg x_4 \vee x_{10} \vee x_6)$$
$$c_3 = (\neg x_5 \vee \neg x_6 \vee \neg x_7)$$
$$c_4 = (\neg x_6 \vee x_7)$$

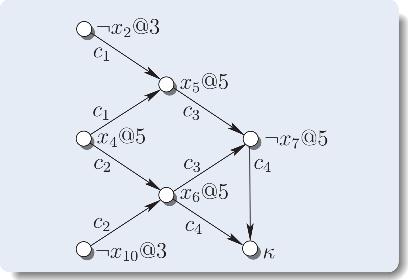


Assume that the implication order in the BCP was x_4, x_5, x_6, x_7.

name	*cl*	*lit*	*var*	*ante*
c_4	$(\neg x_6 \vee x_7)$	x_7	x_7	c_3
	$(\neg x_5 \vee \neg x_6)$	$\neg x_6$	x_6	c_2
	$(\neg x_4 \vee x_{10} \vee \neg x_5)$	$\neg x_5$	x_5	c_1

Example

$$c_1 = (\neg x_4 \vee x_2 \vee x_5)$$
$$c_2 = (\neg x_4 \vee x_{10} \vee x_6)$$
$$c_3 = (\neg x_5 \vee \neg x_6 \vee \neg x_7)$$
$$c_4 = (\neg x_6 \vee x_7)$$



Assume that the implication order in the BCP was x_4, x_5, x_6, x_7.

name	cl	lit	var	$ante$
c_4	$(\neg x_6 \vee x_7)$	x_7	x_7	c_3
	$(\neg x_5 \vee \neg x_6)$	$\neg x_6$	x_6	c_2
	$(\neg x_4 \vee x_{10} \vee \neg x_5)$	$\neg x_5$	x_5	c_1
c_5	$(\neg x_4 \vee x_2 \vee x_{10})$			

The Algorithm

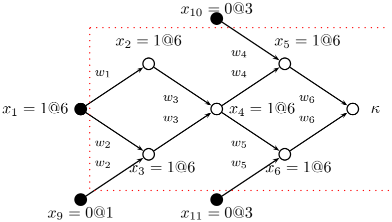

```
ANALYZE-CONFLICT()
```

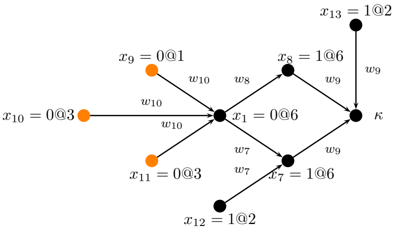
if $current\_desicion\_level = 0$ **then** `return` False;
;
while $\neg$ STOP-CRITERION-MET (cl) **do**
  $lit :=$ LAST-ASSIGNED-LITERAL (cl);
  $var :=$ VARIABLE-OF-LITERAL (lit);
  $ante :=$ **Antecedent**(lit);
  $cl :=$ RESOLVE $(cl, ante, var)$;
end
ADD-CLAUSE-TO-DATABASE (cl);

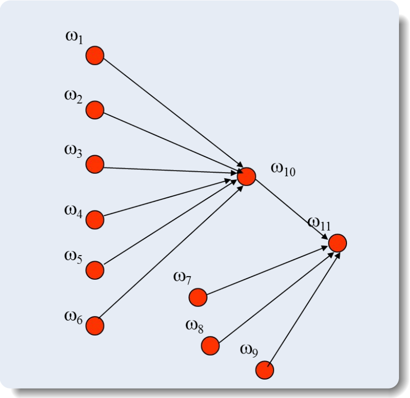
name	cl	lit	var	$ante$
c_4	$(\neg x_6 \vee x_7)$	x_7	x_7	c_3
	$(\neg x_5 \vee \neg x_6)$	$\neg x_6$	x_6	c_2
	$(\neg x_4 \vee x_{10} \vee \neg x_5)$	$\neg x_5$	x_5	c_1
c_5	$(\neg x_4 \vee x_2 \vee x_{10})$			

Resolution Graph

The resolution graph keeps track of the inference relation.



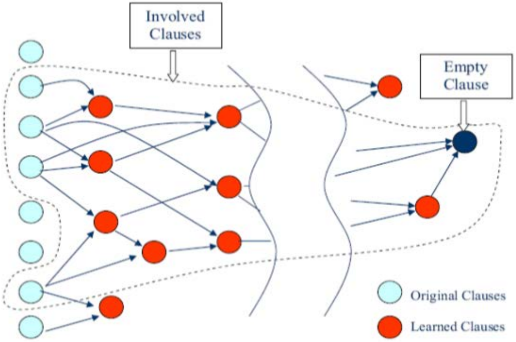




Resolution Graph

What is it good for?

Example: for computing an unsatisfiable core



from SAT'03

Decision Heuristics - VSIDS

VSIDS (Variable State Independent Decaying Sum)

Each literal has a counter initialized to 0.

When a clause is added, the counters are updated.

The unassigned variable with the highest counter is chosen.

Periodically, all the counters are divided by a constant.

Decision Heuristics - VSIDS

VSIDS (Variable State Independent Decaying Sum)

Each literal has a counter initialized to 0.

When a clause is added, the counters are updated.

The unassigned variable with the highest counter is chosen.

Periodically, all the counters are divided by a constant.

firstly implemented in **Chaff**

Decision Heuristics - VSIDS

Chaff holds a list of unassigned variables sorted by the counter value.

Updates are needed only when adding conflict clauses.

Thus, decision is made in constant time.

Decision Heuristics - VSIDS

VSIDS is a quasi-static strategy:

- static because it does not depend on current assignment
- dynamic because it gradually changes. Variables that appear in recent conflicts have higher priority.

This strategy is a conflict-driven decision strategy, which dramatically improves performance.

Decision Heuristics - Berkmin

Keep conflict clauses in a stack

Choose the first unresolved clause in the stack (If there is no such clause, use VSIDS)

Choose from this clause a variable + value accvrding to some scoring (e.g. VSIDS)

Decision Heuristics - Berkmin

Keep conflict clauses in a stack

Choose the first unresolved clause in the stack (If there is no such clause, use VSIDS)

Choose from this clause a variable + value accvrding to some scoring (e.g. VSIDS)

This gives absolute priority to conflicts.

SAT Solver

SAT solver is to be said as the ”most successful formal tools”.

SAT Solver

SAT solver is to be said as the ”most successful formal tools”.

There are a SAT Competitions every one or two years.

- http://www.satcompetition.org/

SAT Solver

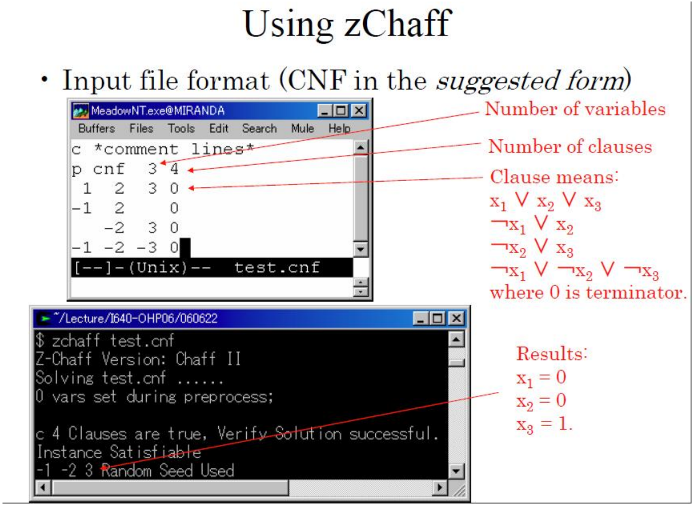
SAT solver is to be said as the ”most successful formal tools”.

There are a SAT Competitions every one or two years.

- http://www.satcompetition.org/

Zchaff(The champion of 2004) can handle 100,000 variables with millions of clauses (Experiments: 800 variables with 9,000 clauses in 0.0sec).

Zchaff


Using zChaff
• Input file format (CNF in the *suggested form*)
MeadowNT.exe@MIRANDA
Buffers Files Tools Edit Search Mule Help
c *comment lines*
p cnf 3 4
1 2 3 0
-1 2 0
-2 3 0
-1 -2 -3 0
[--]-(Unix)-- test.cnf
Number of variables
Number of clauses
Clause means:
$x_1 \vee x_2 \vee x_3$
$\neg x_1 \vee x_2$
$\neg x_2 \vee x_3$
$\neg x_1 \vee \neg x_2 \vee \neg x_3$
where 0 is terminator.
~/Lecture/I640-OHP06/060622
$ zchaff test.cnf
Z-Chaff Version: Chaff II
Solving test.cnf ......
0 vars set during preprocess;
c 4 Clauses are true, Verify Solution successful.
Instance Satisfiable
-1 -2 3 Random Seed Used
Results:
$x_1 = 0$
$x_2 = 0$
$x_3 = 1.$

More on SAT Society

SMT solver, string solver.

Referred Materials

Referred Materials

Daniel Kroening, Ofer Strichman. Decision Procedures: An Algorithmic Point of View, Springer, 2008

Suggest to read:

Marijn J. H. Heule, Oliver Kullmann. The Science of Brute Force. *Communications of the ACM*, Vol. 60(8), 70-79, 2017