Algorithm Design and Analysis (XIX)

An Introduction to Approximation Algorithms

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Approximation Algorithms

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NP-Hard and Optimization Problems

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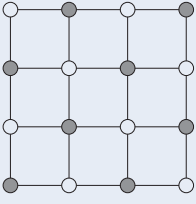
Approximability of these problems becomes a compelling subject of scientific inquiry in computer science and mathematics.

VERTEX COVER

Given an undirected graph $G = (V, E)$, and a cost function on vertices $c : V \to \mathbb{Q}^+$, find a minimum cost vertex cover, i.e., a set $V' \subseteq V$ such that every edge has at least one endpoint incident at V'.

The special case, in which all vertices are of unit cost, will be called the cardinality vertex cover problem.



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A feasible solution that achieves the optimal objective function value is called an optimal solution.

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By "close" we mean within a guaranteed factor of the optimal.

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How do we establish the approximation guarantee? The answer provides a key step in the design of approximation algorithms.

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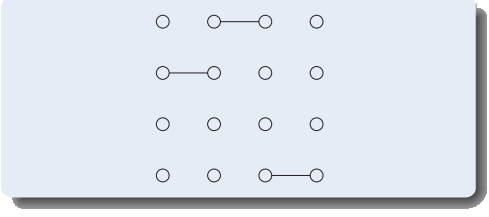

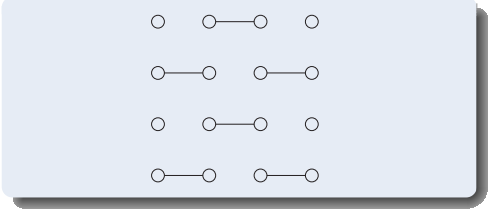

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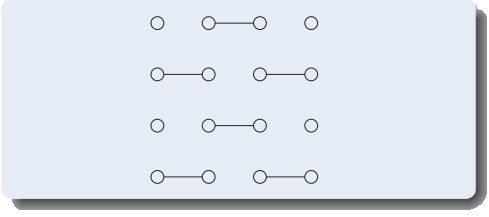
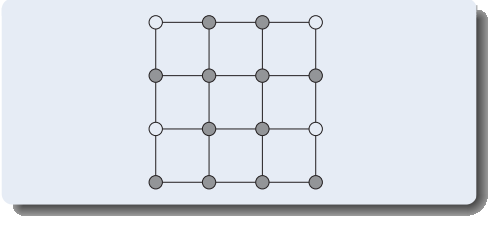
A maximal matching can clearly be computed in polynomial time by simply greedily picking edges and removing endpoints of picked edges. More sophisticated means lead to polynomial time algorithms for finding a maximum matching as well.

Approximation for CARDINALITY VC

Algorithm

Find a maximal matching in G and output the set of matched vertices.





Approximation Factor

The Algorithm is a factor 2 approximation algorithm for the cardinality vertex cover problem.

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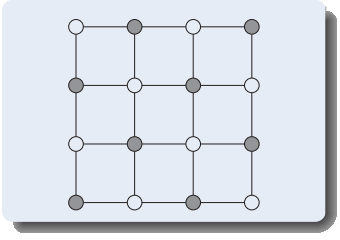
Proof.

- No edge can be left uncovered by the set of vertices picked.
- Let M be the matching picked. As argued above,

$$|M| \leq OPT$$

- The approximation factor is at most $2 \cdot OPT$.



Lower Bounding OPT

The approximation algorithm for vertex cover was very much related to, and followed naturally from, the lower bounding scheme. This is in fact typical in the design of approximation algorithms.

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Is there some other lower bounding method that can lead to an improved approximation guarantee for VERTEX COVER?

A Better Analysis?

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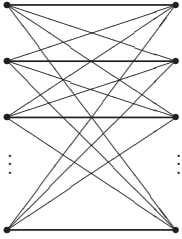
Consider the infinite family of instances given by the complete bipartite graphs $K_{n,n}$.



When run on $K_{n,n}$, Algorithm will pick all $2n$ vertices, whereas picking one side of the bipartition gives a cover of size n.

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Tight examples for an approximation algorithm give critical insight into the functioning of the algorithm.

They have often led to ideas for obtaining algorithms with improved guarantees.

A Better Guarantee?

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The lower bound, of size of a maximal matching, is half the size of an optimal vertex cover for the following infinite family of instances. Consider the complete graph K_n, where n is odd. The size of any maximal matching is $(n-1)/2$, whereas the size of an optimal cover is $n-1$.

A Better Algorithm?

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Still Open!

On the hardness of approximating minimum vertex cover

By Irit Dinur and Samuel Safra*

Abstract

We prove the Minimum Vertex Cover problem to be NP-hard to approximate to within a factor of 1.3606, extending on previous PCP and hardness of approximation technique. To that end, one needs to develop a new proof framework, and to borrow and extend ideas from several fields.

Set cover

SET COVER

Given a universe U of n elements, a collection of subsets of U, $\mathcal{S} = \{S_1, \ldots, S_k\}$, and a cost function $c : \mathcal{S} \to \mathbb{Q}^+$, find a minimum cost sub-collection of $\mathcal{S}$ that covers all elements of U.
The special case, in which all subsets are of unit cost, will be called the cardinality set cover problem.

Define the frequency f of an element to be the number of sets it is in.

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Clearly, neither dominates the other in all instances.

The special case of set cover with $f = 2$ is essentially the vertex cover problem, for which we gave a factor 2 approximation algorithm.

Set cover

Cardinality Set Cover

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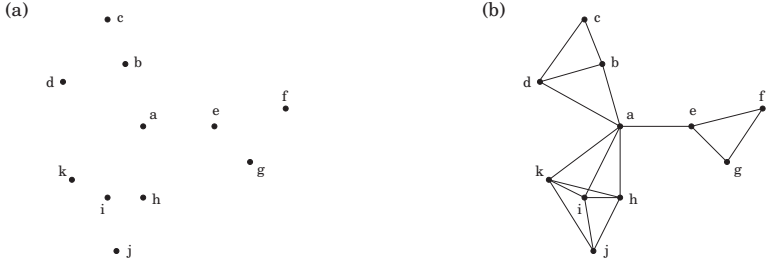
Q: What is the minimum number of schools needed?



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- For each town x, let S_x be the set of towns within 30 miles of it.
- A school at x will essentially "cover" these other towns.
- The question is then, how many sets S_x must be picked in order to cover all the towns in the county?

Set Cover Problem

Set Cover

- Input: A set of elements B, sets $S_1, \ldots, S_m \subseteq B$
- Output: A selection of the S_i whose union is B.
- Cost: Number of sets picked.

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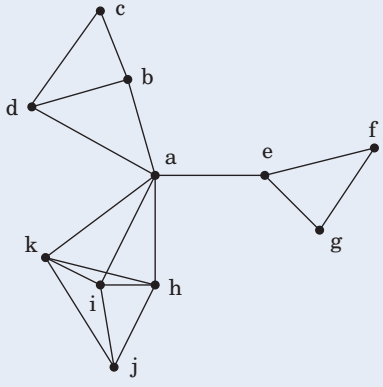

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which by repeated application implies

$$n_t \leq n_0(1 - \frac{1}{OPT})^t$$

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At $t = \ln n \cdot OPT$, therefore, n_t is strictly less than $ne^{-\ln n} = 1$, which means no elements remain to be covered.

Set cover

Generalized Set Cover

The Greedy Algorithm

Iteratively pick the most cost-effective set and remove the covered elements, until all elements are covered.

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When a set S is picked, we can think of its cost being distributed equally among the new elements covered, to set their prices.

Greedy Algorithm

1. $C \leftarrow \emptyset$.
2. While $C \neq U$ do
   - Find the most cost-effective set in the current iteration, say S.
   - Let $\alpha = \frac{c(S)}{|S-C|}$, i.e., the cost-effectiveness of S.
   - Pick S, and for each $e \in S - C$, set $price(e) = \alpha$.
   - $C \leftarrow C \cup S$.
3. Output the picked sets.

The Lemma

Lemma

Number the elements of U in the order in which they were covered by the algorithm, resolving ties arbitrarily. Let $e_1, \ldots, e_n$ be this numbering.
For each $k \in \{1, \ldots, n\}$,

$$price(e_k) \leq OPT/(n - k + 1)$$

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In the iteration in which element e_k was covered, $\overline{C}$ contained at least $n-k+1$ elements.

$$price(e_k) \leq \frac{OPT}{|\overline{C}|} \leq \frac{OPT}{n-k+1}$$

The Approximation Factor

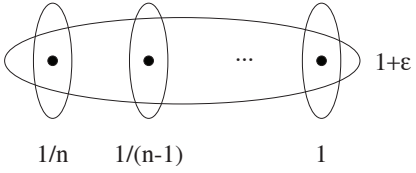
Theorem

The greedy algorithm is an H_n factor approximation algorithm for the minimum set cover problem, where

$$H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$$

A Tight Example

The following is a tight example



When run on this instance the greedy algorithm outputs the cover consisting of the n singleton sets, since in each iteration some singleton is the most cost-effective set. Thus, the algorithm outputs a cover of cost

$$= \frac{1}{n} + \frac{1}{n-1} + \ldots + 1 = H_n$$

On the other hand, the optimal cover has a cost of $1 + \varepsilon$.

Layering

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The idea in layering is to decompose the given weight function on vertices into convenient functions, called degree-weighted, on a nested sequence of subgraphs of G.

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Since U covers all the edges, $\sum_{v \in U} deg(v) \geq |E|$.

Therefore, $\omega(U) \geq c|E|$. Since $\sum_{v \in V} deg(v) = 2|E|$, $\omega(V) = 2c|E|$. The lemma follows.

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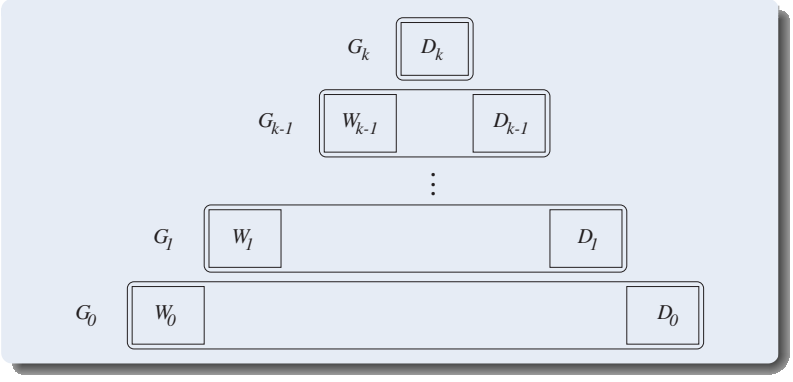
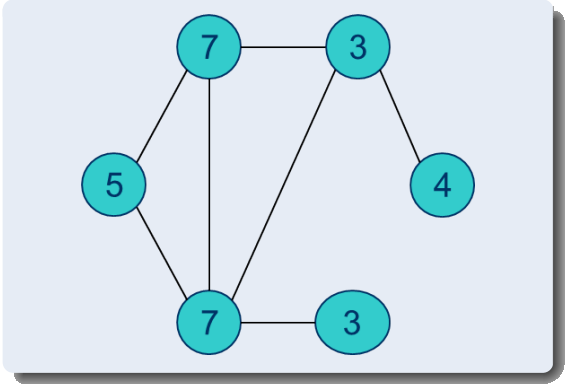
5. Let $W_i = \{v \in G_i \mid w(v) = 0\}$, $C = C \cup W_i$.
6. Let G_{i+1} be the graph induced by $V_i - (D_i \cup W_i)$. Increase i by 1 and goto step 2 until G_i is empty graph.

The Layer Algorithm



An Example



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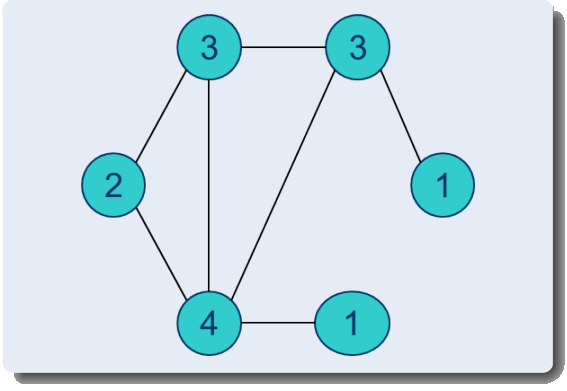

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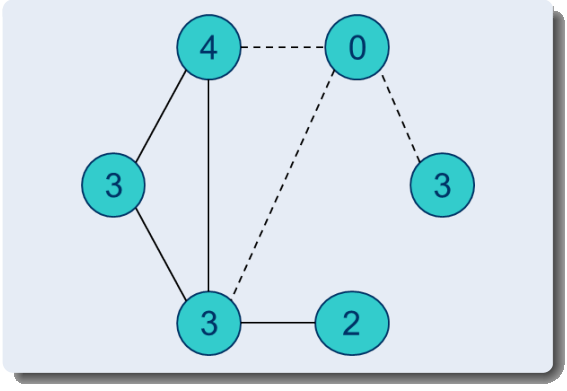

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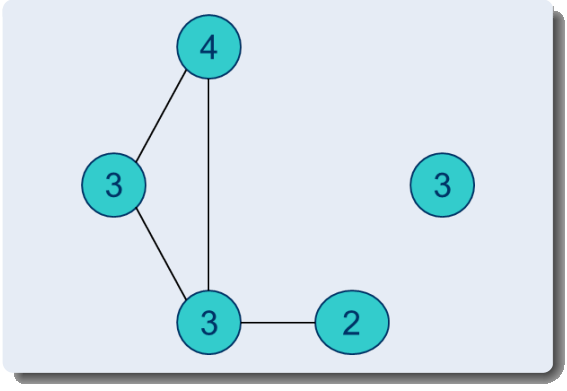

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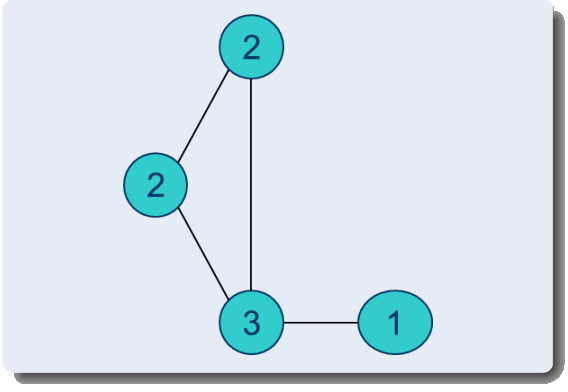

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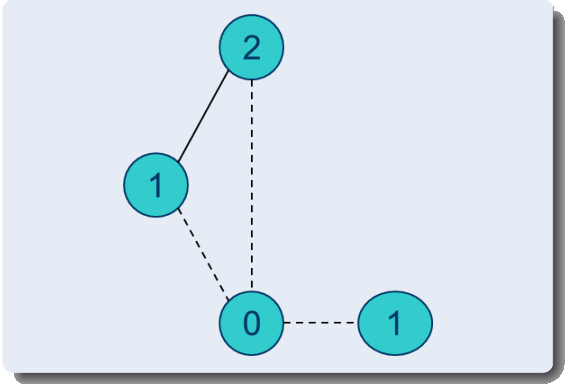

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Otherwise, there must be some $(u, v) \in E$ with $u \in D_i$ and $v \in D_j$.

Assume $i \leq j$, then (u, v) is in G_i contradicting the fact that u is of degree zero.

Analysis (cont'd)

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For $v \in V - C$, if $v \in D_j$, then

$$\omega(v) \geq \sum_{i < j} t_i(v)$$

Analysis (cont'd)

In each layer i, $C^* \cap G_i$ is a vertex cover for G_i.

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Thus by previous lemma, $t_i(C \cap G_i) \leq 2 \cdot t_i(C^* \cap G_i)$.

Analysis (cont'd)

In each layer i, $C^* \cap G_i$ is a vertex cover for G_i.

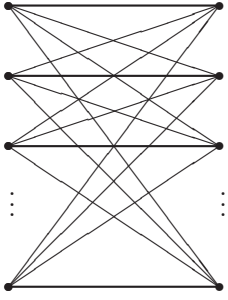
Thus by previous lemma, $t_i(C \cap G_i) \leq 2 \cdot t_i(C^* \cap G_i)$.

Therefore,

$$\omega(C) = \sum_{i=0}^{k-1} t_i(C \cap G_i) \leq 2 \sum_{0}^{k-1} t_i(C^* \cap G_i) \leq 2 \cdot \omega(C^*)$$

A Tight Example

A tight example is provided by the family of complete bipartite graphs, $K_{n,n}$, with all vertices of unit weight. The layering algorithm will pick all $2n$ vertices of $K_{n,n}$ in the cover, whereas the optimal cover picks only one side of the bipartition.



Referred Materials

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Content of this lecture comes from Chapter 1 and 2 in [Vaz04].

Suggest to read the rest part of Chapter 1 and 2 in [Vaz04].