Algorithm Design and Analysis (XX)

Steiner tree and TSP

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School of Computer Science

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We will present constant factor algorithms for metric Steiner tree, and the rest of the problem can be reduced to this case.

Steiner Tree

Given an undirected graph $G = (V, E)$ with nonnegative edge costs and whose vertices are partitioned into two sets, required and Steiner, find a minimum cost tree in G that contains all the required vertices and any subset of the Steiner vertices.

Metric Steiner Problem

Steiner Tree

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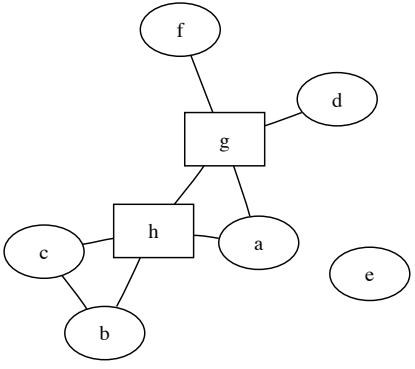
With a restriction to instances in which the edge costs satisfy the triangle inequality, i.e., G is a complete undirected graph, and for any three vertices u, v, and w,

$$cost(u, v) \leq cost(u, w) + cost(v, w)$$

named the metric Steiner tree problem.

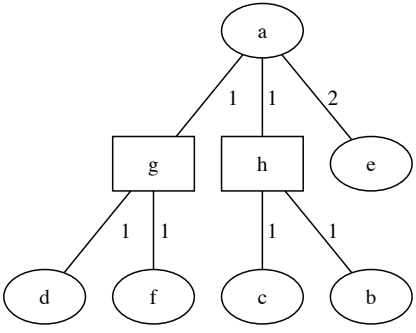
An Example

If we have a Steiner graph, with the cost 1 to the points connected by an edge, and cost 2 otherwise.



An Example

The optimal one:



Approximation Factor Preservation

Theorem

There is an approximation factor preserving reduction from the Steiner tree problem to the metric Steiner tree problem.

Approximation Factor Preservation

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- The partition of V into required and Steiner vertices in I is the same as in I.
- For any edge $(u, v) \in E$, its cost in G' is no more than its cost in G.
- Therefore, the cost of an optimal solution in I' does not exceed the cost of an optimal solution in I.

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- Replace each edge of T' by the corresponding path to obtain a subgraph of G.
- Clearly, in this subgraph, all the required vertices are connected. However, this subgraph may, in general, contain cycles.
- If so, remove edges to obtain tree T. This completes the approximation factor preserving reduction.

Algorithm

Let R denote the set of required vertices. It is easy to verify that a minimum spanning tree on R is a feasible solution for this problem.

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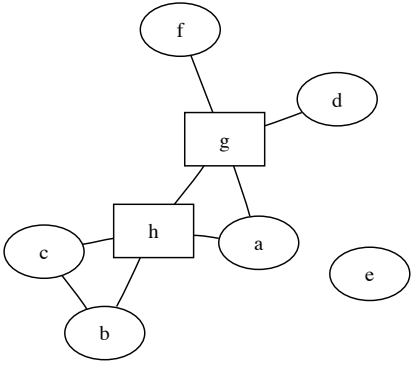
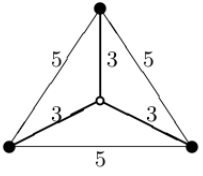


A Counterexample

The cost of MST may not be optimal:



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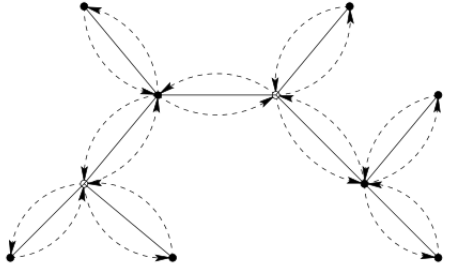
Find an Euler tour of this graph, for example by traversing the edges in DFS (depth first search) order:



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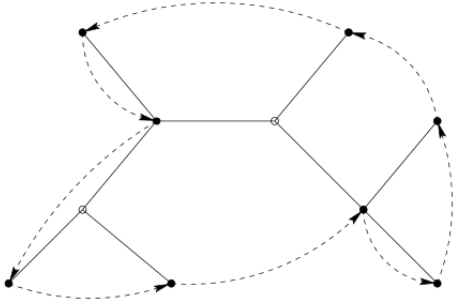
The cost of this Euler tour is $2 \cdot \mathrm{OPT}$. Next obtain a Hamilton cycle on the vertices of R by traversing the Euler tour and short-cutting Steiner vertices and previously visited vertices of R:



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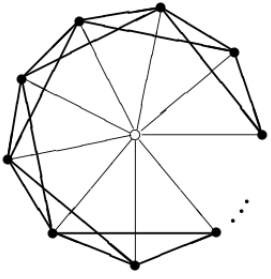
If we delete one edge of this Hamilton cycle, we obtain a path that spans R and has cost at most $2 \cdot$OPT.

This path is also a spanning tree on R. Hence, the MST on R has cost at most $2 \cdot$ OPT.

Tightness of the Analysis

Consider a graph with n required vertices and one Steiner vertex. An edge between the Steiner vertex and a required vertex has cost 1, and an edge between two required vertices has cost 2:



Metric TSP

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- Interestingly, TSP cannot be approximated within any polynomial bounded ratio.
- For any polynomial time computable function $\alpha(n)$, TSP cannot be approximated within a factor of $\alpha(n)$, unless **P** = **NP**.

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- The set of nodes is the same as V.
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HAMILTON CYCLE $\rightarrow$ TSP

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If G has a HAMILTON CYCLE, then the same cycle is also a tour within the budget of the TSP `instance`.

If G has no HAMILTON CYCLE, then there is no solution: the cheapest possible TSP tour has cost at least $n + \alpha$.

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We will describe a reduction from HAMILTON CYCLE PROBLEM (which is **NP**-completeness) to TSP problem.

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Assign a weight of 1 to edges of G, and a weight of $\alpha(n) \cdot n$ to nonedges, to obtain G'.

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If we restrict ourselves to graphs in which edge costs satisfy triangle inequality, i.e., consider metric TSP, the problem remains NP-complete, but it is no longer hard to approximate.

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TSP on Metric Space

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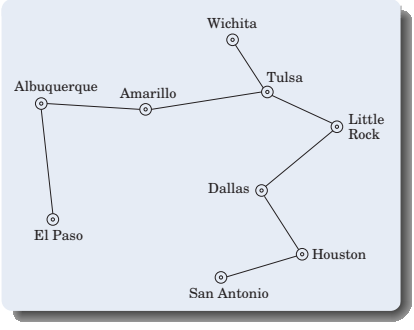
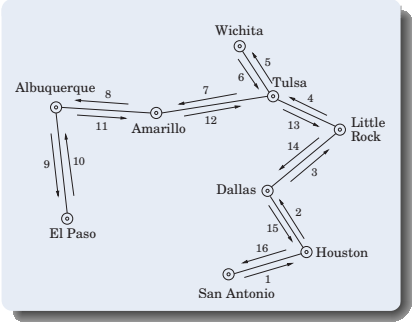
Therefore, TSP cost $\geq$ cost of this path $\geq$ MST cost



TSP on Metric Space

If we can use each edge twice, then by following the shape of the MST we end up with a tour that visits all the cities, some of them more than once.



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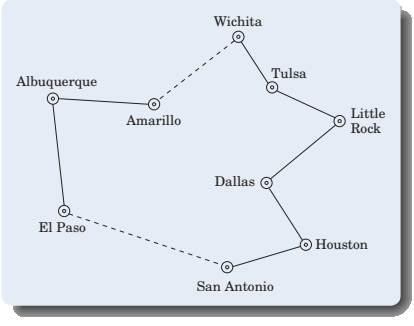
By the triangle inequality, these bypasses can only make the overall tour shorter.



Consider the following algorithm:

1. Find an MST, T of G
2. Double every edge of the MST to obtain an Eulerian graph.
3. Find an Eulerian tour, $\mathcal{T}$, on this graph.
4. Output the tour that visits vertices of G in the order of their first appearance in T. Let $\mathcal{C}$ be this tour.

A Simple Factor 2 Algorithm

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The above algorithm is a factor 2 approximation algorithm for metric TSP.

Analysis

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Combining these inequalities we get that

$$\mathrm{cost}(\mathcal{C}) \leq 2 \cdot \mathrm{OPT}.$$

A Tight Example

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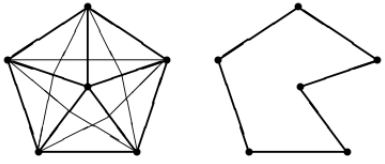
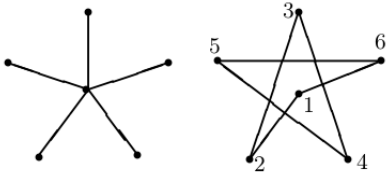


A Tight Example

Suppose that the MST found by the algorithm is the spanning star created by edges of cost 1. Moreover, suppose that the Euler tour constructed in Step 3 visits vertices in order shown below for $n = 6$:



Then the tour obtained after short-cutting contains $n - 2$ edges of cost 2 and has a total cost of $2n - 2$. Asymptotically, this is twice the cost of the optimal TSP tour.

An Example

	C	MV	PA	SC	SV
C	0	7	12	7	4
MV		0	8	9	4
PA			0	14	10
SC				0	5
SV					0

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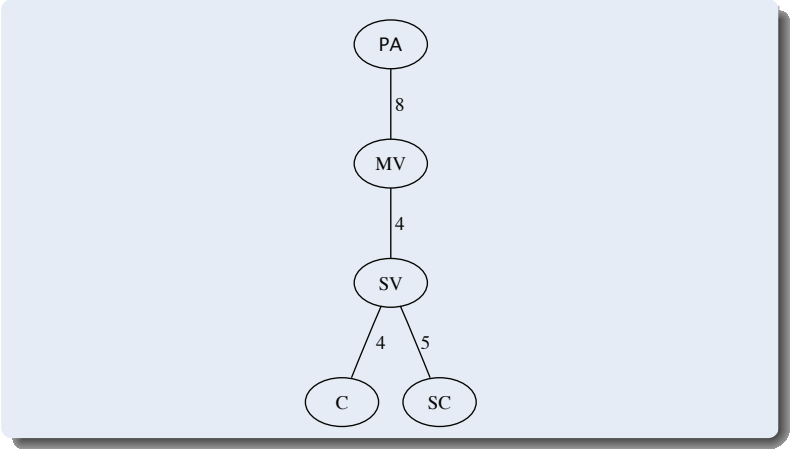

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Recall that a graph has an Euler tour iff all its vertices have even degrees.

Thus, we only need to be concerned about the vertices of odd degree in the MST.

Let V' denote the set of vertices of odd degree in the MST. Now, if we add to the MST a minimum cost perfect matching on V', every vertex will have an even degree, and we get an Eulerian graph.

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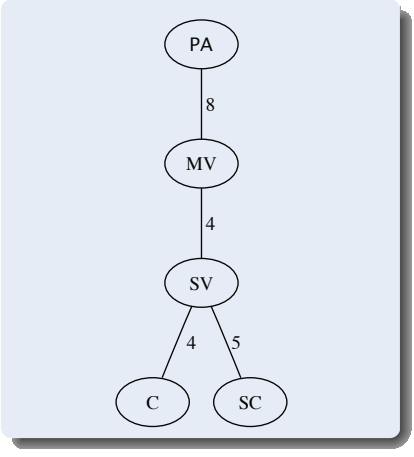

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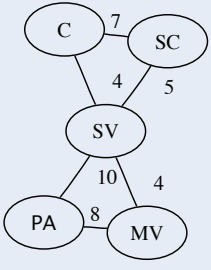

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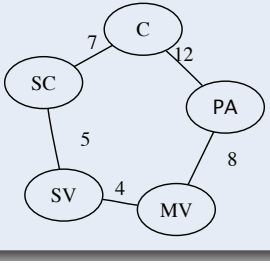

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Thus, the cheaper of these matchings has cost $\leq \mathrm{cost}(\tau')/2 \leq \mathrm{OPT}/2$. Hence the optimal matching also has cost at most $\mathrm{OPT}/2$.

Analysis (cont'd)

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The above algorithm achieves an approximation guarantee of $3/2$ *for metric TSP.*

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Proof.

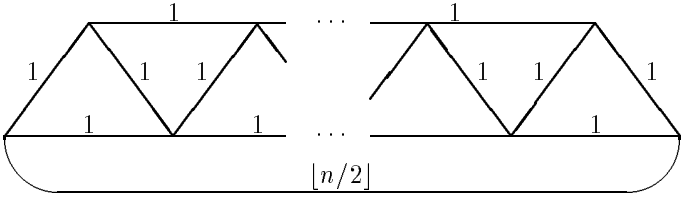
The cost of the Euler tour,

$$\text{cost}(\mathcal{T}) \leq \text{cost}(T) + \text{cost}(M) \leq \text{OPT} + \frac{1}{2}\text{OPT} = \frac{3}{2}\text{OPT}$$

Using the triangle inequality, $\text{cost}(\mathcal{C}) \leq \text{cost}(\mathcal{T})$, and the theorem follows.

A Tight Example

A tight example for this algorithm is given by the following graph on n vertices, with n odd.



Thick edges represent the MST found in step 1. This MST has only two odd vertices, and by adding the edge joining them we obtain a traveling salesman tour of cost $(n-1)+\lfloor n/2 \rfloor$. In contrast, the optimal tour has cost n.

A Better Algorithm?

A Better Algorithm?

Finding a better approximation algorithm for metric TSP is currently one of the outstanding open problems in this area. Many researchers have conjectured that an approximation factor of $4/3$ may be achievable.

Referred Materials

Referred Materials

- Content of this lecture comes from Chapter 3 in [Vaz04].