Design and Analysis of Algorithms (XX)

Polynomial Time Approximation Scheme

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Approximation Scheme

Let Π be an $\mathbf{NP}$-hard optimization problem with objective function f_Π. We will say that algorithm $\mathcal{A}$ is an approximation scheme for Π if on input (I, ϵ), where I is an instance of Π and $\epsilon > 0$ is an error parameter, it outputs a solution s such that:

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- $f_\Pi(I, s) \leq (1 + \epsilon) \cdot \mathrm{OPT}$ if Π is a minimization problem.
- $f_\Pi(I, s) \geq (1 - \epsilon) \cdot \mathrm{OPT}$ if Π is a maximization problem.

PTAS and FPTAS

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If we require that the running time of $\mathcal{A}$ be bounded by a polynomial in the size of instance I and $1/\epsilon$, then $\mathcal{A}$ will be said to be a fully polynomial-time approximation scheme, abbreviated FPTAS.

Knapsack

Knapsack: Problem Statement

KNAPSACK

Given a set $S = \{a_1, \ldots, a_n\}$ of objects, with specified sizes and profits, $\mathrm{size}(a_i) \in \mathbb{Z}^+$ and $\mathrm{profit}(a_i) \in \mathbb{Z}^+$, and a "knapsack capacity" $B \in \mathbb{Z}^+$, find a subset of objects whose total size is bounded by B and total profit is maximized.

An Example

Objects	A	B	C	D	E
Sizes	7	2	9	3	1
Profits	3	2	3	1	2

Knapsack size: B

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An algorithm for problem Π whose running time on instance I is bounded by a polynomial in $|I_u|$ will be called a pseudo-polynomial time algorithm.

Dynamic Programming

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Knapsack with Repetition

$$K(w) = \max_{a_i : \text{size}(a_i) \leq w} \{K(w - \text{size}(a_i)) + \text{profit}(a_i)\}$$

The running time is $O(n \cdot B)$.

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Knapsack without Repetition

$$K(w, j) = \max\{K(w - \text{size}(a_j), j - 1) + \text{profit}(a_j), K(w, j - 1)\}$$

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For each $i \in \{1, \ldots, n\}$ and $p \in \{1, \ldots, nP\}$, let $S_{i,p}$ denote a subset of $\{a_1, \ldots, a_i\}$ whose total profit is exactly p and whose total size is minimized.

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The following recurrence helps compute all values $A(i,p)$ in $O(n^2P)$ time:

$$A(i+1,p) = \begin{cases} \min\{A(i,p), \mathrm{size}(a_{i+1}) + A(i, p - \mathrm{profit}(a_{i+1}))\} & \text{if } \mathrm{profit}(a_{i+1}) \le p \\ A(i,p) & \text{otherwise} \end{cases}$$

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The maximum profit achievable by objects of total size bounded by B is $\max\{p \mid A(n,p) \le B\}$.

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An Example

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	∞	∞	7	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
2	∞	2	7	∞	9	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
3	∞	2	7	∞	9	16	∞	18	∞	∞	∞	∞	∞	∞	∞
4	3	2	5	10	9	14	19	18	21	∞	∞	∞	∞	∞	∞
5	3	1	4	3	8	11	10	13	20	19	22	∞	∞	∞	∞

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In FPTAS we will ignore a certain number of least significant bits of profits of objects (depending on ϵ), so that the modified profits can be viewed as numbers bounded by a polynomial in n and $1/\epsilon$.

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4. Output S'.

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Let A denote the set output by the algorithm. Then

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Therefore,

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Therefore,

$$\begin{aligned}\text{profit}(S) &\geq K \cdot \text{profit}'(S) \geq K \cdot \text{profit}'(O) \\ &\geq \text{profit}(O) - nK = \text{OPT} - \epsilon P \geq (1-\epsilon) \cdot \text{OPT}\end{aligned}$$

Analysis

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By previous Lemma, the solution found is within $(1 - \epsilon)$ factor of OPT. Since the running time of the algorithm is

$$O(n^2 \lfloor \frac{P}{K} \rfloor) = O(n^2 \lfloor \frac{n}{\epsilon} \rfloor)$$

which is polynomial in n and $1/\epsilon$, thus it is a FPTAS for knapsack.

Bin Packing

Bin Packing: Problem Statement

BIN PACKING

Given n items with sizes $a_1, \ldots, a_n \in (0, 1]$, find a packing in unit-sized bins that minimizes the number of bins used.

An 2-approximation Algorithm

First-Fit Algorithm

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- It attempts to put the next item, a_i, in one of these bins, in this order.
- If a_i does not fit into any of these bins, it opens a new bin B_{k+1}, and puts a_i in it.

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Since the sum of the item sizes is a lower bound on OPT, $m-1 < 2\cdot\mathrm{OPT}$, i.e., $m \le 2\cdot\mathrm{OPT}$.

A Hardness Result

For any $\epsilon > 0$, there is no approximation algorithm having a guarantee of $3/2 - \epsilon$ for the bin packing problem, assuming $\mathbf{P} = \mathbf{NP}$.

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The answer to this question is “yes” iff the n items can be packed in 2 bins of size $1/2 \sum_i a_i$.

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The answer to this question is “yes” iff the n items can be packed in 2 bins of size $1/2 \sum_i a_i$.

If the answer is “yes” the $3/2 - \epsilon$ factor algorithm will have to give an optimal packing.

APTAS

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An asymptotic polynomial-time approximation scheme (APTAS) is a family of algorithm $\{A_\epsilon\}$ along with a constant c where there is an algorithm A_ϵ for each $\epsilon > 0$ such that A_ϵ returns a solution of value at most $(1+\epsilon)\mathrm{OPT} + c$ for minimization problems.

An APTAS for Bin-Packing

For any ϵ, $0 < \epsilon \leq 1/2$, there is an algorithm A_ϵ that runs in time polynomial in n and finds a packing using at most $(1+2\epsilon)\mathrm{OPT}+1$ bins.

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We will introduce the algorithm in three steps.

Instances with Large Items

Lemma

Let $\epsilon > 0$ be fixed, and let K be a fixed nonnegative integer. Consider the restriction of the bin packing problem to instances in which each item is of size at least ϵ and the number of distinct item sizes is K. There is a polynomial time algorithm that optimally solves this restricted problem.

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The total number of bins used is at most n. Therefore, the number of possible feasible packings is bounded by

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Enumerating them and picking the best packing gives the optimal answer.

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Removing the Restriction of K

Lemma

Let $\epsilon > 0$ be fixed. Consider the restriction of the bin packing problem to instances in which each item is of size at least ϵ. There is a polynomial time approximation algorithm that solves this restricted problem within a factor of $(1 + \epsilon)$.

Removing the Restriction of K

Let I denote the given instance. Sort the n items by increasing size, and partition them into $K = \lceil 1/\epsilon^2 \rceil$ groups each having at most $Q = \lfloor n\epsilon^2 \rfloor$ items. Notice that two groups may contain items of the same size.

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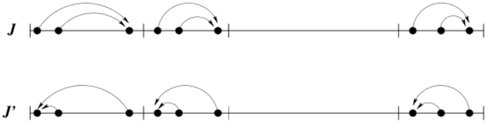

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$$\mathrm{OPT}(J) \leq \mathrm{OPT}(J') + Q \leq \mathrm{OPT}(I) + Q$$

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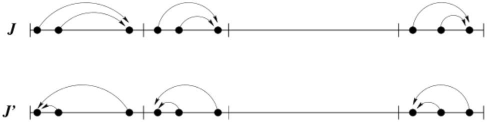
$$\mathrm{OPT}(J) \le \mathrm{OPT}(J') + Q \le \mathrm{OPT}(I) + Q$$

Since each item in I has size at least ϵ, $\mathrm{OPT}(I) \ge n\epsilon$. Therefore $Q = \lfloor n\epsilon^2 \rfloor \le \epsilon \mathrm{OPT}(I)$. Hence, $\mathrm{OPT}(J) \le (1+\epsilon)\mathrm{OPT}(I)$.



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- Let I denote the given instance, and I' denote the instance obtained by discarding items of size $< \epsilon$ from I.
- By previous lemma, we can find a packing for I' using at most $(1+\epsilon)\mathrm{OPT}(I')$ bins.
- Next, we start packing the small items (of size $< \epsilon$) in a First-Fit manner in the bins opened for packing I. Additional bins are opened if an item does not fit into any of the already open bins.

Analysis

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Therefore, the sum of the item sizes in I is at least $(M-1)(1-\epsilon)$. Since this is a lower bound on OPT, we get

Analysis

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$$M \leq \frac{\mathrm{OPT}}{(1-\epsilon)} + 1 \leq (1+2\epsilon)\mathrm{OPT} + 1$$

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Hence, for each value of ϵ, $0 < \epsilon \leq 1/2$, we have a polynomial time algorithm achieving a guarantee of $(1+2\epsilon)\mathrm{OPT} + 1$.

Summary of Algorithm

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Algorithm A_ϵ is summarized below.

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Algorithm

1. Remove items of size $< \epsilon$.
2. Round to obtain constant number of item sizes.
3. Find optimal packing.
4. Use this packing for original item sizes.
5. Pack items of size $< \epsilon$ using First-Fit.

Referred Materials

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Content of this lecture comes from Chapter 8 and 9 in [Vaz04], and Section 3.3 in [WS11].

Suggest to read Chapter 10 in [Vaz04] and Chapter 3 in [WS11].