

Design and Analysis of Algorithms (XX)

Steiner tree and TSP

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The problem has a wide range of applications, all the way from finding minimum length interconnection of terminals in VLSI design to constructing phylogeny trees in computational biology.

We will present constant factor algorithms for metric Steiner tree, and the rest of the problem can be reduced to this case.

Metric Steiner Problem



STEINER TREE

Given an undirected graph G = (V, E) with nonnegative edge costs and whose vertices are partitioned into two sets, required and Steiner, find a minimum cost tree in G that contains all the required vertices and any subset of the Steiner vertices.

Metric Steiner Problem



STEINER TREE

Given an undirected graph G = (V, E) with nonnegative edge costs and whose vertices are partitioned into two sets, required and Steiner, find a minimum cost tree in G that contains all the required vertices and any subset of the Steiner vertices.

With a restriction to instances in which the edge costs satisfy the triangle inequality, i.e., G is a complete undirected graph, and for any three vertices u, v, and w,

 $cost(u,v) \leq cost(u,w) + cost(v,w)$

named the metric Steiner tree problem.

An Example



If we have a Steiner graph, with the cost 1 to the points connected by an edge, and cost 2 otherwise.



An Example



The optimal one:





Theorem

There is an approximation factor preserving reduction from the Steiner tree problem to the metric Steiner tree problem.

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Proof.

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Proof.



Proof.

Firstly, we will transform, in polynomial time, an instance *I* of the Steiner tree problem, consisting of graph G = (V, E), to an instance *I'* of the metric Steiner tree problem.

• Let G' be the complete undirected graph on vertex set V. Define the cost of edge (u, v) in G' to be the cost of a shortest u - v path in G. G' is called the metric closure of G.



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- The partition of V into required and Steiner vertices in I is the same as in I.
- For any edge $(u, v) \in E$, its cost in G' is no more than its cost in G.



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- The partition of V into required and Steiner vertices in I is the same as in I.
- For any edge $(u, v) \in E$, its cost in G' is no more than its cost in G.
- Therefore, the cost of an optimal solution in *I*' does not exceed the cost of an optimal solution in *I*.



Proof.



Proof.

Next, given a Steiner tree T' in I', we will show how to obtain, in polynomial time a Steiner tree T in I of at most the same cost.

• The cost of an edge (u, v) in G' corresponds to the cost of a path in G.



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Proof.

- The cost of an edge (u, v) in G' corresponds to the cost of a path in G.
- Replace each edge of T' by the corresponding path to obtain a subgraph of G.
- Clearly, in this subgraph, all the required vertices are connected. However, this subgraph may, in general, contain cycles.
- If so, remove edges to obtain tree *T*. This completes the approximation factor preserving reduction.

MST-based Algorithm



Algorithm

Let R denote the set of required vertices. It is easy to verify that a minimum spanning tree on R is a feasible solution for this problem.

An Example



If we have a Steiner graph, with the cost 1 to the points connected by an edge, and cost 2 otherwise.



A Counterexample



The cost of MST may not be optimal:



2-approximation Algorithm



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2-approximation Algorithm



However, MST-based algorithm is a good approximation algorithm.

2-approximation Algorithm



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The cost of an MST on R is within 2. OPT.



Consider a Steiner tree of cost OPT. By doubling its edges we obtain an Eulerian graph connecting all vertices of R and, possibly, some Steiner vertices.



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Find an Euler tour of this graph, for example by traversing the edges in DFS (depth first search) order:







The cost of this Euler tour is 2-OPT. Next obtain a Hamilton cycle on the vertices of R by traversing the Euler tour and short-cutting Steiner vertices and previously visited vertices of R:





Because of triangle inequality, the shortcuts do not increase the cost of the tour.



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If we delete one edge of this Hamilton cycle, we obtain a path that spans R and has cost at most $2 \cdot \text{OPT}$.

This path is also a spanning tree on R. Hence, the MST on R has cost at most 2 OPT.

Tightness of the Analysis



Consider a graph with n required vertices and one Steiner vertex. An edge between the Steiner vertex and a required vertex has cost 1, and an edge between two required vertices has cost 2:


Metric TSP

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- Interestingly, TSP cannot be approximated within any polynomial bounded ratio.
- For any polynomial time computable function α(n), TSP cannot be approximated within a factor of α(n), unless P = NP.



Given a graph G = (V, E), construct the instance of the TSP:



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- The set of nodes is the same as V.
- The distance between cities u and v is 1 if $\{u, v\}$ is an edge of G and $1 + \alpha$ otherwise, for some $\alpha > 1$ to be determined.
- The budget of the TSP instance is |V|.



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If G has a HAMILTON CYCLE, then the same cycle is also a tour within the budget of the TSP instance.

If *G* has no HAMILTON CYCLE, then there is no solution: the cheapest possible TSP tour has cost at least $n + \alpha$.



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That is, transform a graph G on n vertices to an edge-weighted complete graph G' on n vertices such that



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- if G does not have a Hamilton cycle, then an optimal TSP tour in G is of $\cos t > \alpha(n) \cdot n$.

Assign a weight of 1 to edges of G, and a weight of $\alpha(n) \cdot n$ to nonedges, to obtain G'.

Metric TSP



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Metric TSP



Notice that in order to obtain such a strong nonapproximability result, we had to assign edge costs that violate triangle inequality.

If we restrict ourselves to graphs in which edge costs satisfy triangle inequality, i.e., consider metric TSP, the problem remains NP-complete, but it is no longer hard to approximate.



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Therefore, TSP cost \geq cost of this path \geq MST cost





If we can use each edge twice, then by following the shape of the MsT we end up with a tour that visits all the cities, some of them more than once.





To fix the problem, the tour should simply skip any city it is about to revisit, and instead move directly to the next new city in its list.



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By the triangle inequality, these bypasses can only make the overall tour shorter.



A Simple Factor 2 Algorithm



Consider the following algorithm:

- **1** Find an MST, T of G
- 2 Double every edge of the MST to obtain an Eulerian graph.
- **3** Find an Eulerian tour, T, on this graph.
- Output the tour that visits vertices of G in the order of their first appearance in T. Let C be this tour.

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The above algorithm is a factor 2 approximation algorithm for metric TSP.



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Because of triangle inequality, after the short-cutting (step 4) step, $\cot(\mathcal{C}) \leq \cot(\mathcal{T})$.



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Because of triangle inequality, after the short-cutting (step 4) step, $cost(C) \leq cost(T)$.

Combining these inequalities we get that

 $cost(\mathcal{C}) \leq 2 \cdot OPT.$



A tight example for this algorithm is given by a complete graph on n vertices with edges of cost 1 and 2.



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Suppose that the MST found by the algorithm is the spanning star created by edges of cost 1. Moreover, suppose that the Euler tour constructed in Step 3 visits vertices in order shown below for n = 6:



Then the tour obtained after short-cutting contains n - 2 edges of cost 2 and has a total cost of 2n - 2. Asymptotically, this is twice the cost of the optimal TSP tour.

An Example



	С	MV	PA	SC	SV	
\mathbf{C}	0	7	12	7	4	
MV		0	8	9	4	
PA			0	14	10	
SC				0	5	
SV					0	

An Example





Improving the factor to 3/2



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Recall that a graph has an Euler tour iff all its vertices have even degrees.

Thus, we only need to be concerned about the vertices of odd degree in the MST.

Let V' denote the set of vertices of odd degree in the MST. Now, if we add to the MST a minimum cost perfect matching on V', every vertex will have an even degree, and we get an Eulerian graph.



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3 Find an Euler tour, T, of this graph.



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- **3** Find an Euler tour, T, of this graph.
- **O** Output the tour that visits vertices of G in order of their first appearance in \mathcal{T} . Let \mathcal{C} be this tour.

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Now, τ' is the union of two perfect matchings on V', each consisting of alternate edges of τ .

Thus, the cheaper of these matchings has $\cot \le \cot(\tau')/2 \le OPT/2$. Hence the optimal matching also has cost at most OPT/2.





Theorem

The above algorithm achieves an approximation guarantee of 3/2 for metric TSP.





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Proof.



Analysis (cont'd)



Theorem

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Proof.

The cost of the Euler tour,

$$\operatorname{cost}(\mathcal{T}) \le \operatorname{cost}(\mathcal{T}) + \operatorname{cost}(\mathcal{M}) \le \operatorname{OPT} + \frac{1}{2}\operatorname{OPT} = \frac{3}{2}\operatorname{OPT}$$

Using the triangle inequality, $cost(C) \leq cost(T)$, and the theorem follows.

A Tight Example



A tight example for this algorithm is given by the following graph on n vertices, with n odd.



Thick edges represent the MST found in step 1. This MST has only two odd vertices, and by adding the edge joining them we obtain a traveling salesman tour of cost $(n - 1) + \lfloor n/2 \rfloor$. In contrast, the optimal tour has cost n.

A Better Algorithm?



A Better Algorithm?



Finding a better approximation algorithm for metric TSP is currently one of the outstanding open problems in this area. Many researchers have conjectured that an approximation factor of 4/3 may be achievable.

Referred Materials

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Referred Materials



• Content of this lecture comes from Chapter 3 in [Vaz04].