Design and Analysis of Algorithms (IX)

More Extensions on Ford-Fulkerson Algorithm

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SHANGHAI JIAO TONG UNIVERSITY

Capacity-Scaling Algorithm

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Overview. Choosing augmented paths with large bottleneck capacity.

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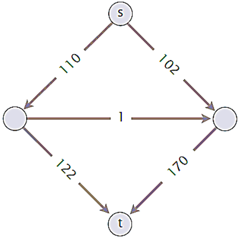
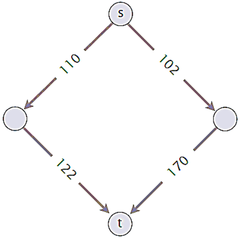
Overview. Choosing augmented paths with large bottleneck capacity.

- Maintain scaling parameter Δ.
- Let $G_f(\Delta)$ be the part of the residual network containing only those edges with capacity $\geq \Delta$.
- Any augmenting path in $G_f(\Delta)$ has bottleneck capacity $\geq \Delta$.



G_f



$G_f(\Delta)$, $\Delta = 100$

Capacity-Scaling Algorithm

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```
CAPACITY-SCALING(G)

for each edge e ∈ E do
 |  f(e) ← 0
end
Δ ← largest power of 2 ≤ C;
while Δ ≥ 1 do
 |  G_f(Δ) ← Δ-residual network of G with respect to flow f;
 |  while there exists an s ⇝ t path P in G_f(Δ) do
 |   |  f ← AUGMENT(f, P);
 |   |  UPDATE(G_Δ(f));
 |  end
 |  Δ = Δ/2;
end
RETURN f;
```

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Integrality invariant. Throughout the algorithm, every edge flow $f(e)$ and residual capacity $c_f(e)$ is an integer.

Proof. Same as for generic Ford–Fulkerson.

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Proof.

- By integrality invariant, when $\Delta = 1 \Rightarrow G_f(\Delta) = G_f$
- Upon termination of $\Delta = 1$ phase, there are no augmenting paths.
- Result follows augmenting path theorem.

Analysis of Running Time

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Lemma 1

There are $1 + \lfloor \log_2 C \rfloor$ scaling phases.

Lemma 2

There are $\leq 2|E|$ augmentations per scaling phase.

Lemma 3

Let f be the flow at the end of a Δ-scaling phase. Then

$$\mathrm{val}(f^*) \leq \mathrm{val}(f) + |E| \cdot \Delta$$

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Theorem

The capacity-scaling algorithm takes $O(|E|^2 \cdot \log C)$ *time.*

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Proof.

- Lemma 1+ Lemma 2 $\Rightarrow O(|E| \cdot \log C)$ augmentations.
- Finding an augmenting path takes $O(|E|)$ time.

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Proof.

Initially $C/2 < \Delta \leq C$; Δ decreases by a factor of 2 in each iteration.

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Proof.

- Let f be the flow at the beginning of a Δ-scaling phase.
- Lemma 3 $\Rightarrow \mathrm{val}(f^*) \leq \mathrm{val}(f) + |E| \cdot (2\Delta)$.
- Each augmentation in a Δ-phase increases $\mathrm{val}(f)$ by at least Δ.

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Proof.

- We show there exists a cut (A, B) such that $cap(A, B) \leq \text{val}(f) + |E| \cdot \Delta$.
- Choose A to be the set of nodes reachable from s in $G_f(\Delta)$.
- By definition of $A : s \in A$.
- By definition of flow $f : t \notin A$.

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$$\text{val}(f^*) \le \text{val}(f) + |E| \cdot \Delta$$

Proof.

$$\begin{aligned} \text{val}(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\ &\ge \sum_{e \text{ out of } A} (c(e) - \Delta) - \sum_{e \text{ in to } A} \Delta \\ &\ge \sum_{e \text{ out of } A} c(e) - \sum_{e \text{ out of } A} \Delta - \sum_{e \text{ in to } A} \Delta \\ &\ge \text{cap}(A, B) - |E| \cdot \Delta \end{aligned}$$

Shortest Augmenting Paths

Edmonds–Karp's Algorithm

Q. How to choose next augmenting path in Ford–Fulkerson?

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A. Pick one that uses the fewest edges.

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EDMONDS–KARP'S ALGORITHM(G)

for *each edge* $e \in E$ **do**
  $f(e) \leftarrow 0$
end
$G_f \leftarrow$ residual network of G with respect to flow f;
while *there exists an* $s \rightsquigarrow t$ *path in* G_f **do**
  $P \leftarrow$ BFS(G_f) ;
  $f \leftarrow$ AUGMENT(f, P) ;
  UPDATE(G_f) ;
end
RETURN f;

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  - at most $|E|$ augmenting paths of length $k \longleftarrow$ Lemma 1 + Lemma 2
  - at most $|V| - 1$ different lengths.

Edmonds–Karp's Algorithm: Analysis

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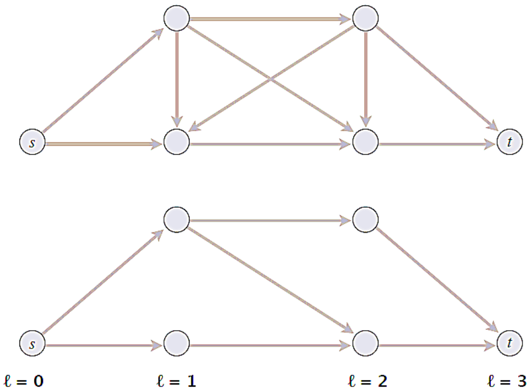
Definition

Given a digraph $G = (V, E)$ with source s, its level graph is defined by:

- $\ell(v)$ = number of edges in shortest $s \rightsquigarrow v$ path.
- $L_G = (V, E_G)$ is the subgraph of G that contains only those edges $(v, w) \in E$ with $\ell(w) = \ell(v) + 1$.

Edmonds–Karp's Algorithm: Analysis



Quiz 5

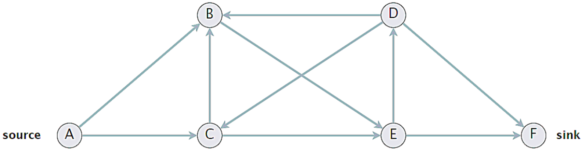
Which edges are in the level graph of the following digraph?

A. $D \to F$

B. $E \to F$

C. Both A and B.

D. Neither A nor B.



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Key property. P is a shortest $s \rightsquigarrow v$ path in G iff P is an $s \rightsquigarrow v$ path in L_G.

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Proof.

- Let f and f' be flow before and after a shortest-path augmentation.
- Let L_G and $L_{G'}$ be level graphs of G_f and $G_{f'}$.
- Only back edges added to $G_{f'}$
  (any $s \rightsquigarrow t$ path that uses a back edge is longer than previous length)

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Proof.

- At least one (bottleneck) edge is deleted from L_G per augmentation.
- No new edge added to L_G until shortest path length strictly increases.

Review of Analysis

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Note. $\Theta(|E||V|)$ augmentations necessary for some flow networks.

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- Simple idea $\Rightarrow O(|E||V|^2)$ [Dinitz 1970]

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- Try to decrease time per augmentation instead.
- Simple idea $\Rightarrow O(|E||V|^2)$ [Dinitz 1970]
- Dynamic trees $\Rightarrow O(|E||V|\log|V|)$ [Sleator–Tarjan 1983]

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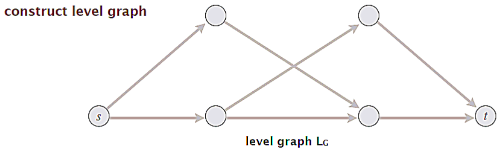

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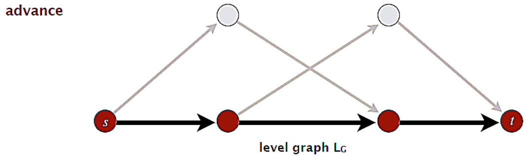

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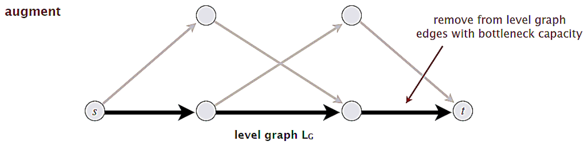

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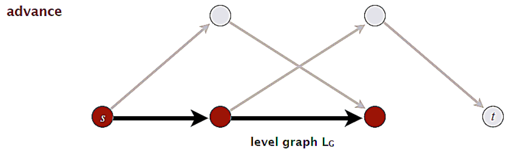

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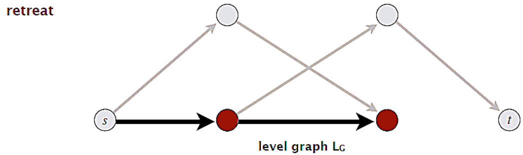

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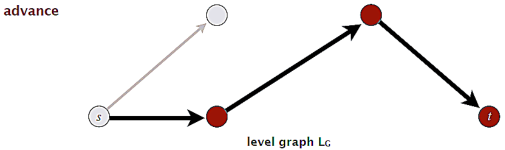

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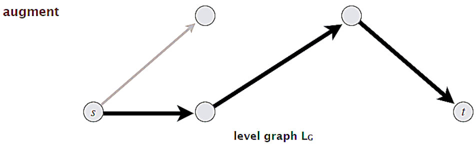

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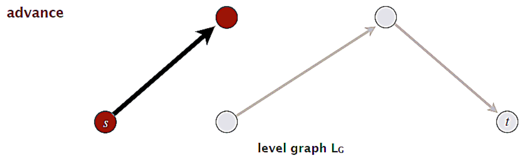

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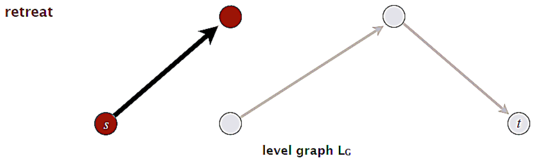

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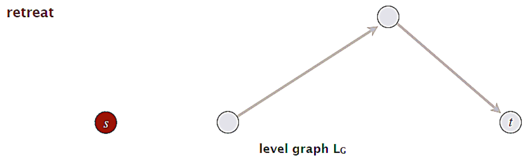

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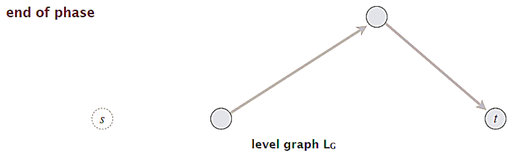


Dinitz' Algorithm

```
INITIALIZE(G, f)

L_G ← level-graph of G_f
P ← ∅
GOTO ADVANCE(s) ;


RETREAT(v)

if v = s then Stop;
else
    Delete v from L_G;
    Remove last edge (u, v)
      from P;
end
GOTO ADVANCE(u) ;
```

```
ADVANCE(v)

if v = t then
    AUGMENT(P) ;
    Remove saturated edges
      from L_G;
    P ← ∅;
    GOTO ADVANCE(s) ;
end
if there exists edge (v, w) ∈ L_G
  then
    Add edge (v, w) to P;
    GOTO ADVANCE(w) ;
end
else
    GOTO RETREAT(v) ;
end
```

Quiz 6

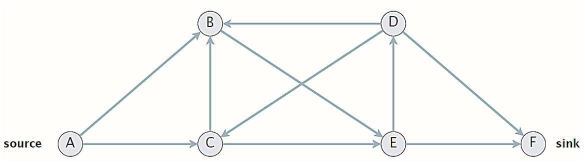
How to compute the level graph L_G efficiently?

A. Depth-first search.

B. Breadth-first search.

C. Both A and B.

D. Neither A nor B.



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A phase can be implemented to run in $O(|E||V|)$ time.

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- At most $|V|$ retreats per phase.

Dinitz' Algorithm: Analysis

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Lemma

A phase can be implemented to run in $O(|E||V|)$ time.

Proof.

- Initialization happens once per phase. using BFS
- At most $|E|$ augmentations per phase. $\longleftarrow O(|E|)$ per phase
  (because an augmentation deletes at least one edge from L_G)
- At most $|V|$ retreats per phase. $\longleftarrow O(|E| + |V|)$ per phase

Dinitz' Algorithm: Analysis

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- At most $|E|$ augmentations per phase. $\longleftarrow O(|E|)$ per phase (because an augmentation deletes at least one edge from L_G)
- At most $|V|$ retreats per phase. $\longleftarrow O(|E| + |V|)$ per phase (because a retreat deletes one node from L_G)
- At most $|E||V|$ advances per phase.

Dinitz' Algorithm: Analysis

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A phase can be implemented to run in $O(|E||V|)$ time.

Proof.

- Initialization happens once per phase. using BFS
- At most $|E|$ augmentations per phase. $\longleftarrow O(|E|)$ per phase (because an augmentation deletes at least one edge from L_G)
- At most $|V|$ retreats per phase. $\longleftarrow O(|E|+|V|)$ per phase (because a retreat deletes one node from L_G)
- At most $|E||V|$ advances per phase. $\longleftarrow O(|E||V|)$ per phase

Dinitz' Algorithm: Analysis

Lemma

A phase can be implemented to run in $O(|E||V|)$ time.

Proof.

- Initialization happens once per phase. using BFS
- At most $|E|$ augmentations per phase. $\longleftarrow O(|E|)$ per phase (because an augmentation deletes at least one edge from L_G)
- At most $|V|$ retreats per phase. $\longleftarrow O(|E| + |V|)$ per phase (because a retreat deletes one node from L_G)
- At most $|E||V|$ advances per phase. $\longleftarrow O(|E||V|)$ per phase (because at most $|V|$ advances before retreat or augmentation)

Dinitz' Algorithm: Analysis

Theorem (Dinitz 1970)

Dinitz' algorithm runs in $O(|E||V|^2)$ *time.*

Dinitz' Algorithm: Analysis

Theorem (Dinitz 1970)

Dinitz' algorithm runs in $O(|E||V|^2)$ time.

Proof.

Dinitz' Algorithm: Analysis

Theorem (Dinitz 1970)

Dinitz' algorithm runs in $O(|E||V|^2)$ time.

Proof.

- By Lemma, $O(|E||V|)$ time per phase.

Dinitz' Algorithm: Analysis

Theorem (Dinitz 1970)

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Proof.

- By Lemma, $O(|E||V|)$ time per phase.
- At most $|V| - 1$ phases

Dinitz' Algorithm: Analysis

Theorem (Dinitz 1970)

Dinitz' algorithm runs in $O(|E||V|^2)$ time.

Proof.

- By Lemma, $O(|E||V|)$ time per phase.
- At most $|V| - 1$ phases (as in shortest-augmenting-path analysis).

Summary

year	method	# augmentations	running time
1955	augmenting path	$\lvert V\rvert C$	$O(\lvert E\rvert \lvert V\rvert C)$
1972	fattest path	$\lvert E\rvert \log(\lvert E\rvert C)$	$O\left(\lvert E\rvert^2 \log n \log(\lvert E\rvert C)\right)$
1972	capacity scaling	$\lvert E\rvert \log C$	$O\left(\lvert E\rvert^2 \log C\right)$
1985	improved capacity scaling	$\lvert E\rvert \log C$	$O(\lvert E\rvert \lvert V\rvert \log C)$
1970	shortest augmenting path	$\lvert E\rvert \lvert V\rvert$	$O\left(\lvert E\rvert^2 \lvert V\rvert\right)$
1970	level graph	$\lvert E\rvert \lvert V\rvert$	$O\left(\lvert E\rvert \lvert V\rvert^2\right)$
1983	dynamic trees	$\lvert E\rvert \lvert V\rvert$	$O(\lvert E\rvert \lvert V\rvert \log \lvert V\rvert)$

augmenting-path algorithms with integer capacities between 1 and C

Theory Highlights

year	method	worst case	discovered by
1951	simplex	$O\left(\lvert E\rvert\lvert V\rvert^2 C\right)$	Dantzig
1955	augmenting paths	$O(\lvert E\rvert\lvert V\rvert C)$	Ford–Fulkerson
1970	shortest augmenting paths	$O\left(\lvert E\rvert\lvert V\rvert^2\right)$	Edmonds–Karp, Dinitz
1974	blocking flows	$O\left(\lvert V\rvert^3\right)$	Karzanov
1983	dynamic trees	$O(\lvert E\rvert\lvert V\rvert\log n)$	Sleator–Tarjan
1985	improved capacity scaling	$O(\lvert E\rvert\lvert V\rvert\log C)$	Gabow
1988	push–relabel	$O\left(\lvert E\rvert\lvert V\rvert\log\left(\lvert V\rvert^2/\lvert E\rvert\right)\right)$	Goldberg–Tarjan
1998	binary blocking flows	$O\left(\lvert E\rvert^{3/2}\log\left(n^2/\lvert E\rvert\right)\log C\right)$	Goldberg–Rao
2013	compact networks	$O(\lvert E\rvert\lvert V\rvert)$	Orlin
2014	interior-point methods	$\tilde{O}\left(\lvert E\rvert\lvert E\rvert^{1/2}\log C\right)$	Lee–Sidford
2016	electrical flows	$\tilde{O}\left(\lvert E\rvert^{10/7}C^{1/7}\right)$	Madry
20xx		???	

augmenting-path algorithms with integer capacities between 1 and C

Maximum-Flow: Practice

Push–relabel algorithm (Section 7.4) of [KT05]. [Goldberg–Tarjan 1988]

Increases flow one edge at a time instead of one augmenting path at a time.

Maximum-Flow: Practice

Caveat. Worst-case running time is generally not useful for predicting or comparing max-flow algorithm performance in practice.

Best in practice. Push–relabel method with gap relabeling: $O(|E|^{3/2})$ in practice.

Referred Materials

- Content of this lecture comes from Section 7.3 in [KT05].
- Suggest to read Chapter 26 in [CLRS09].