Algorithm Design and Analysis (IX)

More Extensions on Ford-Fulkerson Algorithm

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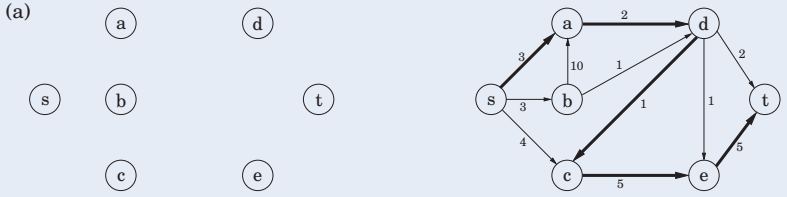
Review of Ford-Fulkerson Algorithm

Ford–Fulkerson Algorithm

Ford–Fulkerson augmenting path algorithm.

- Start with $f(e) = 0$ for each edge $e \in E$.
- Find an $s \rightsquigarrow t$ path P in the residual network G_f.
- Augment flow along path P.
- Repeat until you get stuck.

A Flow Example



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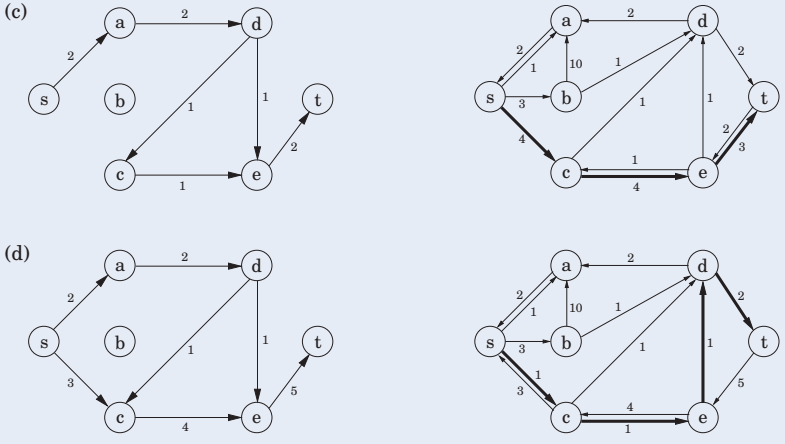

Ford–Fulkerson Algorithm

FORD–FULKERSON(G)

for *each edge* $e \in E$ **do**
  $f(e) \leftarrow 0$
end
$G_f \leftarrow$ residual network of G with respect to flow f;
while *there exists an* $s \rightsquigarrow t$ *path* P *in* G_f **do**
  $f \leftarrow$ AUGMENT(f,P);
  UPDATE(G_f);
end
RETURN f;

Theorem

The running time of Ford–Fulkerson algorithm is $O(|V| \cdot |E| \cdot C)$.

Capacity-Scaling Algorithm

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Overview. Choosing augmented paths with large bottleneck capacity.

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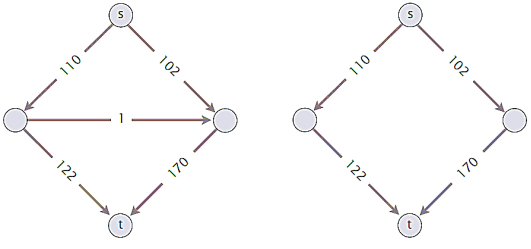
- Maintain scaling parameter Δ.
- Let $G_f(\Delta)$ be the part of the residual network containing only those edges with capacity $\geq \Delta$.
- Any augmenting path in $G_f(\Delta)$ has bottleneck capacity $\geq \Delta$.



G_f $G_f(\Delta)$, $\Delta = 100$

Capacity-Scaling Algorithm

CAPACITY-SCALING(G)

for *each edge* $e \in E$ **do**
  $f(e) \leftarrow 0$
end
$\Delta \leftarrow$ largest power of $2 \leq C$;
while $\Delta \geq 1$ **do**
  $G_f(\Delta) \leftarrow \Delta$-residual network of G with respect to flow f;
  while *there exists an* $s \rightsquigarrow t$ *path* P *in* $G_f(\Delta)$ **do**
    $f \leftarrow$ AUGMENT(f, P) ;
    UPDATE($G_\Delta(f)$) ;
  end
  $\Delta = \Delta/2$;
end
RETURN f;

Proof of Correctness

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Proof. Same as for generic Ford–Fulkerson.

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Proof.

- By integrality invariant, when $\Delta = 1 \Rightarrow G_f(\Delta) = G_f$
- Upon termination of $\Delta = 1$ phase, there are no augmenting paths.
- Result follows augmenting path theorem.

Analysis of Running Time

Lemma 1

There are $1 + \lfloor \log_2 C \rfloor$ scaling phases.

Lemma 2

There are $\leq 2|E|$ augmentations per scaling phase.

Lemma 3

Let f be the flow at the end of a Δ-scaling phase. Then

$$\mathrm{val}(f^*) \leq \mathrm{val}(f) + |E| \cdot \Delta$$

Theorem

The capacity-scaling algorithm takes $O(|E|^2 \cdot \log C)$ time.

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Proof.

- Lemma 1+ Lemma 2 $\Rightarrow O(|E| \cdot \log C)$ augmentations.
- Finding an augmenting path takes $O(|E|)$ time.

Lemma 1

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Proof.

Initially $C/2 < \Delta \leq C$; Δ decreases by a factor of 2 in each iteration.

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Proof.

- Let f be the flow at the beginning of a Δ-scaling phase.
- Lemma 3 $\Rightarrow \mathrm{val}(f^*) \leq \mathrm{val}(f) + |E| \cdot (2\Delta)$.
- Each augmentation in a Δ-phase increases $\mathrm{val}(f)$ by at least Δ.

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Proof.

- We show there exists a cut (A, B) such that $cap(A, B) \leq \text{val}(f) + |E| \cdot \Delta$.
- Choose A to be the set of nodes reachable from s in $G_f(\Delta)$.
- By definition of $A : s \in A$.
- By definition of flow $f : t \notin A$.

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$$\begin{aligned}
\mathrm{val}(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) && \geq \sum_{e \text{ out of } A} (c(e) - \Delta) - \sum_{e \text{ in to } A} \Delta \\
&\geq \sum_{e \text{ out of } A} c(e) - \sum_{e \text{ out of } A} \Delta - \sum_{e \text{ in to } A} \Delta && \geq \mathrm{cap}(A, B) - |E| \cdot \Delta
\end{aligned}$$

Shortest Augmenting Paths

Edmonds–Karp's Algorithm

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end
$G_f \leftarrow$ residual network of G with respect to flow f;
while *there exists an* $s \rightsquigarrow t$ *path in* G_f **do**
  $P \leftarrow$ BFS(G_f);
  $f \leftarrow$ AUGMENT(f, P);
  UPDATE(G_f);
end
RETURN f;

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The length of a shortest augmenting path never decreases.

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  - at most $|V| - 1$ different lengths.

Edmonds–Karp's Algorithm: Analysis

Definition

Given a digraph $G = (V, E)$ with source s, its level graph is defined by:

- $\ell(v)$ = number of edges in shortest $s \rightsquigarrow v$ path.
- $L_G = (V, E_G)$ is the subgraph of G that contains only those edges $(v, w) \in E$ with $\ell(w) = \ell(v) + 1$.

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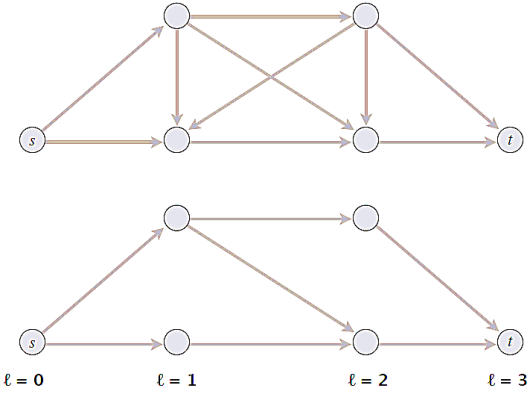


Quiz 5

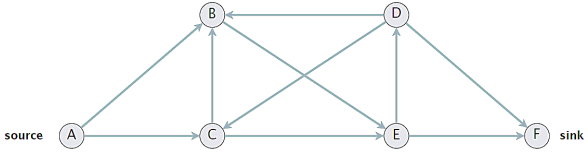
Which edges are in the level graph of the following digraph?

A. $D \to F$

B. $E \to F$

C. Both A and B.

D. Neither A nor B.



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Key property. P is a shortest $s \rightsquigarrow v$ path in G iff P is an $s \rightsquigarrow v$ path in L_G.

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Proof.

- Let f and f' be flow before and after a shortest-path augmentation.
- Let L_G and $L_{G'}$ be level graphs of G_f and $G_{f'}$.
- Only back edges added to $G_{f'}$
  (any $s \rightsquigarrow t$ path that uses a back edge is longer than previous length)

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- At least one (bottleneck) edge is deleted from L_G per augmentation.
- No new edge added to L_G until shortest path length strictly increases.

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- Dynamic trees $\Rightarrow O(|E||V|\log|V|)$ [Sleator–Tarjan 1983]

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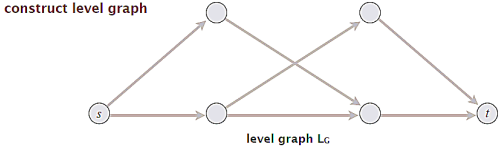

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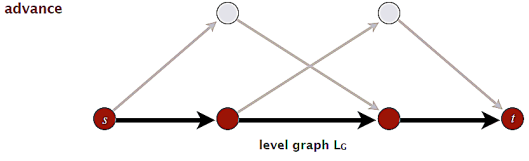

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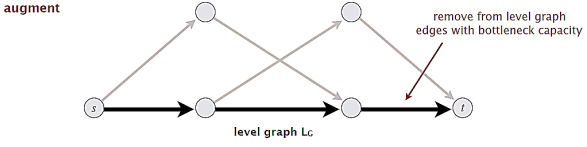

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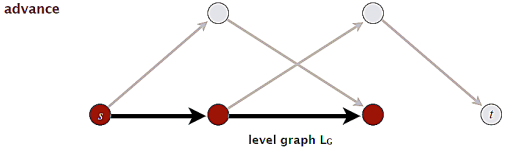

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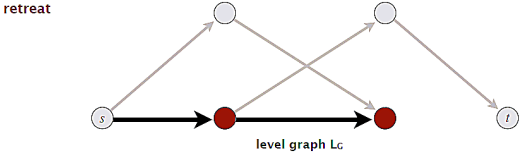

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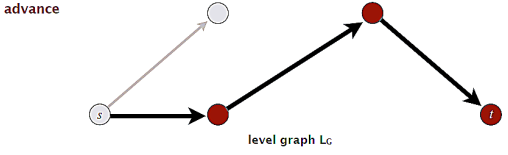

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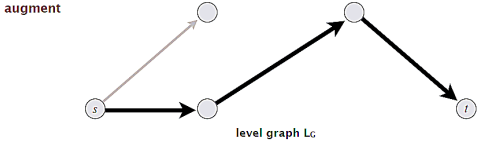

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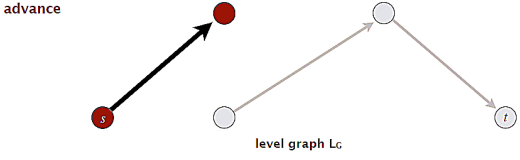

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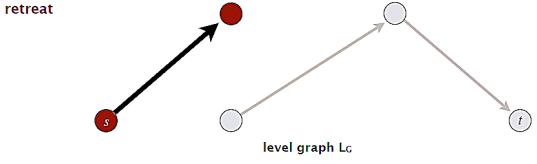

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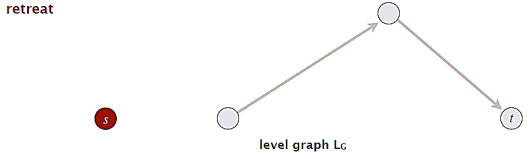

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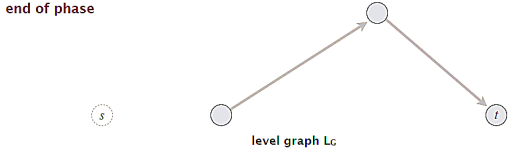


Dinitz' Algorithm

INITIALIZE(G, f)

$L_G \leftarrow$ level-graph of G_f
$P \leftarrow \varnothing$
GOTO ADVANCE(s) ;

RETREAT(v)

if $v = s$ **then** Stop;
else
  Delete v from L_G;
  Remove last edge (u, v) from P;
end
GOTO ADVANCE(u) ;

ADVANCE(v)

if $v = t$ **then**
  AUGMENT(P) ;
  Remove saturated edges from L_G;
  $P \leftarrow \varnothing$;
  GOTO ADVANCE(s) ;
end
if *there exists edge* $(v, w) \in L_G$ **then**
  Add edge (v, w) to P;
  GOTO ADVANCE(w) ;
end
else
  GOTO RETREAT(v) ;
end

Quiz 6

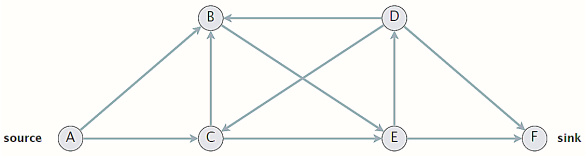
How to compute the level graph L_G efficiently?

A. Depth-first search.

B. Breadth-first search.

C. Both A and B.

D. Neither A nor B.



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Dinitz' Algorithm: Analysis

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  (because a retreat deletes one node from L_G)
- At most $|E||V|$ advances per phase. $\longleftarrow O(|E||V|)$ per phase

Dinitz' Algorithm: Analysis

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A phase can be implemented to run in $O(|E||V|)$ time.

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- Initialization happens once per phase. using BFS $\longleftarrow O(|E| + |V|)$
- At most $|E|$ augmentations per phase. $\longleftarrow O(|E|)$ per phase
  (because an augmentation deletes at least one edge from L_G)
- At most $|V|$ retreats per phase. $\longleftarrow O(|E| + |V|)$ per phase
  (because a retreat deletes one node from L_G)
- At most $|E||V|$ advances per phase. $\longleftarrow O(|E||V|)$ per phase
  (because at most $|V|$ advances before retreat or augmentation)

Dinitz' Algorithm: Analysis

Theorem (Dinitz 1970)

Dinitz' algorithm runs in $O(|E||V|^2)$ time.

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Theorem (Dinitz 1970)

Dinitz' algorithm runs in $O(|E||V|^2)$ time.

Proof.

- By Lemma, $O(|E||V|)$ time per phase.
- At most $|V| - 1$ phases (as in shortest-augmenting-path analysis).

Summary

year	method	# augmentations	running time
1955	augmenting path	$\|V\|C$	$O(\|E\|\|V\|C)$
1972	fattest path	$\|E\| \log(\|E\|C)$	$O\left(\|E\|^2 \log n \log(\|E\|C)\right)$
1972	capacity scaling	$\|E\| \log C$	$O\left(\|E\|^2 \log C\right)$
1985	improved capacity scaling	$\|E\| \log C$	$O(\|E\|\|V\| \log C)$
1970	shortest augmenting path	$\|E\|\|V\|$	$O\left(\|E\|^2\|V\|\right)$
1970	level graph	$\|E\|\|V\|$	$O\left(\|E\|\|V\|^2\right)$
1983	dynamic trees	$\|E\|\|V\|$	$O(\|E\|\|V\| \log \|V\|)$

augmenting-path algorithms with integer capacities between 1 and C

Theory Highlights

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year	method	worst case	discovered by
1951	simplex	$O\left(\lvert E\rvert\lvert V\rvert^2 C\right)$	Dantzig
1955	augmenting paths	$O(\lvert E\rvert\lvert V\rvert C)$	Ford–Fulkerson
1970	shortest augmenting paths	$O\left(\lvert E\rvert\lvert V\rvert^2\right)$	Edmonds–Karp, Dinitz
1974	blocking flows	$O\left(\lvert V\rvert^3\right)$	Karzanov
1983	dynamic trees	$O(\lvert E\rvert\lvert V\rvert \log n)$	Sleator–Tarjan
1985	improved capacity scaling	$O(\lvert E\rvert\lvert V\rvert \log C)$	Gabow
1988	push–relabel	$O\left(\lvert E\rvert\lvert V\rvert \log\left(\lvert V\rvert^2/\lvert E\rvert\right)\right)$	Goldberg–Tarjan
1998	binary blocking flows	$O\left(\lvert E\rvert^{3/2} \log\left(n^2/\lvert E\rvert\right) \log C\right)$	Goldberg–Rao
2013	compact networks	$O(\lvert E\rvert\lvert V\rvert)$	Orlin
2014	interior-point methods	$\tilde{O}\left(\lvert E\rvert\lvert E\rvert^{1/2} \log C\right)$	Lee–Sidford
2016	electrical flows	$\tilde{O}\left(\lvert E\rvert^{10/7} C^{1/7}\right)$	Madry
20xx		???	

augmenting-path algorithms with integer capacities between 1 and C

Maximum-Flow: Practice

Push–relabel algorithm (Section 7.4) of [KT05]. [Goldberg–Tarjan 1988]

Increases flow one edge at a time instead of one augmenting path at a time.

Maximum-Flow: Practice

Caveat. Worst-case running time is generally not useful for predicting or comparing max-flow algorithm performance in practice.

Best in practice. Push–relabel method with gap relabeling: $O(|E|^{3/2})$ in practice.

Referred Materials

Referred Materials

- Content of this lecture comes from Section 7.3 in [KT05].
- Suggest to read Chapter 26 in [CLRS09].