

# Computability Theory V

Turing Machine

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Oct. 24, 2014

*Assignment 2 is announced! (deadline Nov. 7)*

# Turing Machine

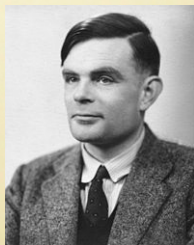
# Alan Turing

**Alan Turing** (23Jun.1912-7Jun.1954), an English student of Church, introduced a machine model for effective calculation in

“On Computable Numbers, with an Application to the Entscheidungs problem”,

Proc. of the London Mathematical Society, **42**:230-265, 1936.

Turing Machine, Halting Problem, Turing Test



British Prime Minister Gordon Brown:

“...I am pleased to have the chance to say how deeply sorry I and we all are for what happened to him ... So on behalf of the British government, and all those who live freely thanks to Alan’s work, I am very proud to say: we’re sorry, you deserved so much better.”

# Motivation

What are necessary for a machine to calculate a function?

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What are necessary for a machine to calculate a function?

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- The machine must be able to operate and manipulate numbers according to a set of predefined instructions;

and

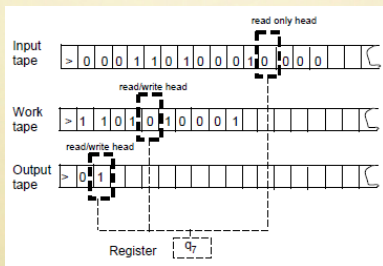
- The input number has to be stored in an accessible place;
- There should be an accessible place for the machine to store the intermediate results;
- The output number has to be put in an accessible place.

# Turing Machine

A  $k$ -tape Turing Machine  $M$  has  $k$ -tapes such that

- The first tape is the read-only **input tape**.
- The other  $k - 1$  tapes are the read/write **work tapes**.
- The  $k$ -th tape is also used as the **output tape**.

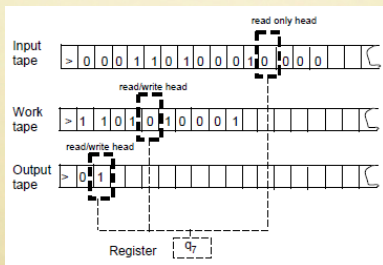
Every tape comes with a read/write **head**.



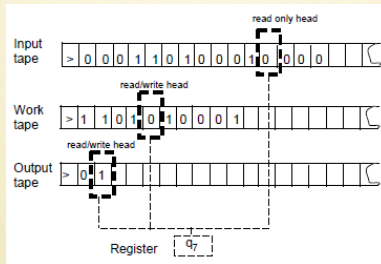
# Turing Machine

The machine is described by a tuple  $(\Gamma, Q, \delta)$  containing

- A finite set  $\Gamma$ , called **alphabet**, of symbols. It contains a blank symbol  $\square$ , a start symbol  $\triangleright$ , and the digits 0 and 1.
- A finite set  $Q$  of **states**. It contains a **start state**  $q_s$  and a **halting state**  $q_h$ .
- A **transition function**  $\delta : Q \times \Gamma^k \rightarrow Q \times \Gamma^{k-1} \times \{\leftarrow, -, \rightarrow\}^k$ , describing the rules of each computation step.



# Computation and Configuration



Configuration, initial configuration, final configuration, computation step

# An Example

$Q = \{q_s, q_h, q_1, q_2, q_3\}$ ,  $\Gamma = \{0, 1, \square, \triangleright\}$ ,  
and  $\delta$  is as follows:

| $p \in Q$ | $\sigma \in \Gamma$ | $\delta(p, \sigma)$                  |
|-----------|---------------------|--------------------------------------|
| $q_s$     | 0                   | $(q_s, 0, \rightarrow)$              |
| $q_s$     | 1                   | $(q_s, 1, \rightarrow)$              |
| $q_s$     | $\square$           | $(q_1, \square, \leftarrow)$         |
| $q_s$     | $\triangleright$    | $(q_s, \triangleright, \rightarrow)$ |
| $q_1$     | 0                   | $(q_2, \square, \rightarrow)$        |
| $q_1$     | 1                   | $(q_3, \square, \rightarrow)$        |
| $q_1$     | $\square$           | $(q_1, \square, -)$                  |
| $q_1$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |
| $q_2$     | 0                   | $(q_s, 0, \leftarrow)$               |
| $q_2$     | 1                   | $(q_s, 0, \leftarrow)$               |
| $q_2$     | $\square$           | $(q_s, 0, \leftarrow)$               |
| $q_2$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |
| $q_3$     | 0                   | $(q_s, 1, \leftarrow)$               |
| $q_3$     | 1                   | $(q_s, 1, \leftarrow)$               |
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| $q_s$     | $\triangleright$    | $(q_s, \triangleright, \rightarrow)$ |
| $q_1$     | 0                   | $(q_2, \square, \rightarrow)$        |
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Start the machine with input 010

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Start the machine with input 010

- $q_s, \underline{\triangleright}010$

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Start the machine with input **010**

- $q_s, \underline{\triangleright}010$
- $q_s, \triangleright\underline{0}10$

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Start the machine with input **010**

- $q_s, \underline{\triangleright}010$
- $q_s, \triangleright\underline{0}10$
- $q_s, \triangleright0\underline{1}0$

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- $q_s, \triangleright\underline{0}10$
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- $q_s, \triangleright01\underline{0}$

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| $q_3$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |

Start the machine with input **010**

- $q_s, \underline{\triangleright}010$
- $q_s, \triangleright\underline{0}10$
- $q_s, \triangleright0\underline{1}0$
- $q_s, \triangleright01\underline{0}$
- $q_s, \triangleright010\underline{\square}$

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| $q_1$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |
| $q_2$     | 0                   | $(q_s, 0, \leftarrow)$               |
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| $q_3$     | $\square$           | $(q_s, 1, \leftarrow)$               |
| $q_3$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |

Start the machine with input **010**

- $q_s, \underline{\triangleright}010$
- $q_s, \triangleright\underline{0}10$
- $q_s, \triangleright0\underline{1}0$
- $q_s, \triangleright01\underline{0}$
- $q_s, \triangleright010\underline{\square}$
- $q_1, \triangleright010\underline{\square}$

# An Example

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| $q_s$     | $\triangleright$    | $(q_s, \triangleright, \rightarrow)$ |
| $q_1$     | 0                   | $(q_2, \square, \rightarrow)$        |
| $q_1$     | 1                   | $(q_3, \square, \rightarrow)$        |
| $q_1$     | $\square$           | $(q_1, \square, -)$                  |
| $q_1$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |
| $q_2$     | 0                   | $(q_s, 0, \leftarrow)$               |
| $q_2$     | 1                   | $(q_s, 0, \leftarrow)$               |
| $q_2$     | $\square$           | $(q_s, 0, \leftarrow)$               |
| $q_2$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |
| $q_3$     | 0                   | $(q_s, 1, \leftarrow)$               |
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| $q_3$     | $\square$           | $(q_s, 1, \leftarrow)$               |
| $q_3$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |

Start the machine with input **010**

- $q_s, \underline{\triangleright}010$
- $q_s, \triangleright\underline{0}10$
- $q_s, \triangleright0\underline{1}0$
- $q_s, \triangleright01\underline{0}$
- $q_s, \triangleright010\underline{\square}$
- $q_1, \triangleright010\underline{\square}$
- $q_2, \triangleright01\underline{\square}\underline{\square}$

# An Example

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| $q_s$     | $\triangleright$    | $(q_s, \triangleright, \rightarrow)$ |
| $q_1$     | 0                   | $(q_2, \square, \rightarrow)$        |
| $q_1$     | 1                   | $(q_3, \square, \rightarrow)$        |
| $q_1$     | $\square$           | $(q_1, \square, -)$                  |
| $q_1$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |
| $q_2$     | 0                   | $(q_s, 0, \leftarrow)$               |
| $q_2$     | 1                   | $(q_s, 0, \leftarrow)$               |
| $q_2$     | $\square$           | $(q_s, 0, \leftarrow)$               |
| $q_2$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |
| $q_3$     | 0                   | $(q_s, 1, \leftarrow)$               |
| $q_3$     | 1                   | $(q_s, 1, \leftarrow)$               |
| $q_3$     | $\square$           | $(q_s, 1, \leftarrow)$               |
| $q_3$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |

Start the machine with input **010**

- $q_s, \underline{\triangleright}010$
- $q_s, \triangleright\underline{0}10$
- $q_s, \triangleright0\underline{1}0$
- $q_s, \triangleright01\underline{0}$
- $q_s, \triangleright010\underline{\square}$
- $q_1, \triangleright010\underline{\square}$
- $q_2, \triangleright01\underline{\square}\square$
- $q_s, \triangleright01\underline{\square}0$

# An Example

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| $q_2$     | 0                   | $(q_s, 0, \leftarrow)$               |
| $q_2$     | 1                   | $(q_s, 0, \leftarrow)$               |
| $q_2$     | $\square$           | $(q_s, 0, \leftarrow)$               |
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| $q_3$     | $\square$           | $(q_s, 1, \leftarrow)$               |
| $q_3$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |

Start the machine with input **010**

- $q_s, \underline{\triangleright}010$
- $q_s, \triangleright\underline{0}10$
- $q_s, \triangleright0\underline{1}0$
- $q_s, \triangleright01\underline{0}$
- $q_s, \triangleright010\underline{\square}$
- $q_1, \triangleright010\underline{\square}$
- $q_2, \triangleright01\underline{\square}\square$
- $q_s, \triangleright01\underline{\square}0$
- $q_1, \triangleright01\underline{\square}0$

# An Example

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| $q_s$     | 1                   | $(q_s, 1, \rightarrow)$              |
| $q_s$     | $\square$           | $(q_1, \square, \leftarrow)$         |
| $q_s$     | $\triangleright$    | $(q_s, \triangleright, \rightarrow)$ |
| $q_1$     | 0                   | $(q_2, \square, \rightarrow)$        |
| $q_1$     | 1                   | $(q_3, \square, \rightarrow)$        |
| $q_1$     | $\square$           | $(q_1, \square, -)$                  |
| $q_1$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |
| $q_2$     | 0                   | $(q_s, 0, \leftarrow)$               |
| $q_2$     | 1                   | $(q_s, 0, \leftarrow)$               |
| $q_2$     | $\square$           | $(q_s, 0, \leftarrow)$               |
| $q_2$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |
| $q_3$     | 0                   | $(q_s, 1, \leftarrow)$               |
| $q_3$     | 1                   | $(q_s, 1, \leftarrow)$               |
| $q_3$     | $\square$           | $(q_s, 1, \leftarrow)$               |
| $q_3$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |

Start the machine with input **010**

- $q_s, \underline{\triangleright}010$
- $q_s, \triangleright\underline{0}10$
- $q_s, \triangleright0\underline{1}0$
- $q_s, \triangleright01\underline{0}$
- $q_s, \triangleright010\underline{\square}$
- $q_1, \triangleright010\underline{\square}$
- $q_2, \triangleright01\underline{\square}\square$
- $q_s, \triangleright01\underline{\square}0$
- $q_1, \triangleright01\underline{\square}0$
- $q_3, \triangleright0\underline{\square}\square0$

# An Example

$Q = \{q_s, q_h, q_1, q_2, q_3\}$ ,  $\Gamma = \{0, 1, \square, \triangleright\}$ ,  
and  $\delta$  is as follows:

| $p \in Q$ | $\sigma \in \Gamma$ | $\delta(p, \sigma)$                  |
|-----------|---------------------|--------------------------------------|
| $q_s$     | 0                   | $(q_s, 0, \rightarrow)$              |
| $q_s$     | 1                   | $(q_s, 1, \rightarrow)$              |
| $q_s$     | $\square$           | $(q_1, \square, \leftarrow)$         |
| $q_s$     | $\triangleright$    | $(q_s, \triangleright, \rightarrow)$ |
| $q_1$     | 0                   | $(q_2, \square, \rightarrow)$        |
| $q_1$     | 1                   | $(q_3, \square, \rightarrow)$        |
| $q_1$     | $\square$           | $(q_1, \square, -)$                  |
| $q_1$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |
| $q_2$     | 0                   | $(q_s, 0, \leftarrow)$               |
| $q_2$     | 1                   | $(q_s, 0, \leftarrow)$               |
| $q_2$     | $\square$           | $(q_s, 0, \leftarrow)$               |
| $q_2$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |
| $q_3$     | 0                   | $(q_s, 1, \leftarrow)$               |
| $q_3$     | 1                   | $(q_s, 1, \leftarrow)$               |
| $q_3$     | $\square$           | $(q_s, 1, \leftarrow)$               |
| $q_3$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |

Start the machine with input **010**

- $q_s, \underline{\triangleright}010$
- $q_s, \triangleright\underline{0}10$
- $q_s, \triangleright0\underline{1}0$
- $q_s, \triangleright01\underline{0}$
- $q_s, \triangleright010\underline{\square}$
- $q_1, \triangleright010\underline{\square}$
- $q_2, \triangleright01\underline{\square}\square$
- $q_s, \triangleright01\underline{\square}0$
- $q_1, \triangleright01\underline{\square}0$
- $q_3, \triangleright0\underline{\square}\square0$
- $q_s, \triangleright0\underline{\square}10$

# An Example

$Q = \{q_s, q_h, q_1, q_2, q_3\}$ ,  $\Gamma = \{0, 1, \square, \triangleright\}$ ,  
and  $\delta$  is as follows:

| $p \in Q$ | $\sigma \in \Gamma$ | $\delta(p, \sigma)$                  |
|-----------|---------------------|--------------------------------------|
| $q_s$     | 0                   | $(q_s, 0, \rightarrow)$              |
| $q_s$     | 1                   | $(q_s, 1, \rightarrow)$              |
| $q_s$     | $\square$           | $(q_1, \square, \leftarrow)$         |
| $q_s$     | $\triangleright$    | $(q_s, \triangleright, \rightarrow)$ |
| $q_1$     | 0                   | $(q_2, \square, \rightarrow)$        |
| $q_1$     | 1                   | $(q_3, \square, \rightarrow)$        |
| $q_1$     | $\square$           | $(q_1, \square, -)$                  |
| $q_1$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |
| $q_2$     | 0                   | $(q_s, 0, \leftarrow)$               |
| $q_2$     | 1                   | $(q_s, 0, \leftarrow)$               |
| $q_2$     | $\square$           | $(q_s, 0, \leftarrow)$               |
| $q_2$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |
| $q_3$     | 0                   | $(q_s, 1, \leftarrow)$               |
| $q_3$     | 1                   | $(q_s, 1, \leftarrow)$               |
| $q_3$     | $\square$           | $(q_s, 1, \leftarrow)$               |
| $q_3$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |

Start the machine with input **010**

- $q_s, \underline{\triangleright}010$
- $q_s, \triangleright\underline{0}10$
- $q_s, \triangleright0\underline{1}0$
- $q_s, \triangleright01\underline{0}$
- $q_s, \triangleright010\underline{\square}$
- $q_1, \triangleright010\underline{\square}$
- $q_2, \triangleright01\underline{\square}\square$
- $q_s, \triangleright01\underline{\square}0$
- $q_1, \triangleright01\underline{\square}0$
- $q_3, \triangleright0\underline{\square}\square0$
- $q_s, \triangleright0\underline{\square}10$

# An Example

$Q = \{q_s, q_h, q_1, q_2, q_3\}$ ,  $\Gamma = \{0, 1, \square, \triangleright\}$ ,  
and  $\delta$  is as follows:

| $p \in Q$ | $\sigma \in \Gamma$ | $\delta(p, \sigma)$                  |
|-----------|---------------------|--------------------------------------|
| $q_s$     | 0                   | $(q_s, 0, \rightarrow)$              |
| $q_s$     | 1                   | $(q_s, 1, \rightarrow)$              |
| $q_s$     | $\square$           | $(q_1, \square, \leftarrow)$         |
| $q_s$     | $\triangleright$    | $(q_s, \triangleright, \rightarrow)$ |
| $q_1$     | 0                   | $(q_2, \square, \rightarrow)$        |
| $q_1$     | 1                   | $(q_3, \square, \rightarrow)$        |
| $q_1$     | $\square$           | $(q_1, \square, -)$                  |
| $q_1$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |
| $q_2$     | 0                   | $(q_s, 0, \leftarrow)$               |
| $q_2$     | 1                   | $(q_s, 0, \leftarrow)$               |
| $q_2$     | $\square$           | $(q_s, 0, \leftarrow)$               |
| $q_2$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |
| $q_3$     | 0                   | $(q_s, 1, \leftarrow)$               |
| $q_3$     | 1                   | $(q_s, 1, \leftarrow)$               |
| $q_3$     | $\square$           | $(q_s, 1, \leftarrow)$               |
| $q_3$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |

Start the machine with input **010**

- $q_s, \triangleright 0 \underline{1} 0$
- $q_s, \triangleright \underline{0} 1 0$
- $q_s, \triangleright 0 \underline{1} 0$
- $q_s, \triangleright 0 \underline{1} 0$
- $q_s, \triangleright 0 \underline{1} 0 \square$
- $q_1, \triangleright 0 \underline{1} 0 \square$
- $q_2, \triangleright 0 \underline{1} \square \square$
- $q_s, \triangleright 0 \underline{1} \square 0$
- $q_1, \triangleright 0 \underline{1} \square 0$
- $q_3, \triangleright 0 \square \square 0$
- $q_s, \triangleright 0 \square 1 0$
- $q_1, \triangleright \underline{0} \square 1 0$
- $q_2, \triangleright \square \square \underline{1} 0$

# An Example

$Q = \{q_s, q_h, q_1, q_2, q_3\}$ ,  $\Gamma = \{0, 1, \square, \triangleright\}$ ,  
and  $\delta$  is as follows:

| $p \in Q$ | $\sigma \in \Gamma$ | $\delta(p, \sigma)$                  |
|-----------|---------------------|--------------------------------------|
| $q_s$     | 0                   | $(q_s, 0, \rightarrow)$              |
| $q_s$     | 1                   | $(q_s, 1, \rightarrow)$              |
| $q_s$     | $\square$           | $(q_1, \square, \leftarrow)$         |
| $q_s$     | $\triangleright$    | $(q_s, \triangleright, \rightarrow)$ |
| $q_1$     | 0                   | $(q_2, \square, \rightarrow)$        |
| $q_1$     | 1                   | $(q_3, \square, \rightarrow)$        |
| $q_1$     | $\square$           | $(q_1, \square, -)$                  |
| $q_1$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |
| $q_2$     | 0                   | $(q_s, 0, \leftarrow)$               |
| $q_2$     | 1                   | $(q_s, 0, \leftarrow)$               |
| $q_2$     | $\square$           | $(q_s, 0, \leftarrow)$               |
| $q_2$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |
| $q_3$     | 0                   | $(q_s, 1, \leftarrow)$               |
| $q_3$     | 1                   | $(q_s, 1, \leftarrow)$               |
| $q_3$     | $\square$           | $(q_s, 1, \leftarrow)$               |
| $q_3$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |

Start the machine with input **010**

- $q_s, \underline{\triangleright}010$
- $q_s, \triangleright\underline{0}10$
- $q_s, \triangleright 0\underline{1}0$
- $q_s, \triangleright 01\underline{0}$
- $q_s, \triangleright 010\underline{\square}$
- $q_1, \triangleright 010\underline{\square}$
- $q_2, \triangleright 01\underline{\square}\square$
- $q_s, \triangleright 01\underline{\square}0$
- $q_1, \triangleright 01\underline{\square}0$
- $q_3, \triangleright 0\underline{\square}\square 0$
- $q_s, \triangleright 0\underline{\square}10$
- $q_1, \triangleright \underline{0}\square 10$
- $q_2, \triangleright \square\square 10$
- $q_0, \triangleright \square\square 10$

# An Example

$Q = \{q_s, q_h, q_1, q_2, q_3\}$ ,  $\Gamma = \{0, 1, \square, \triangleright\}$ ,  
and  $\delta$  is as follows:

| $p \in Q$ | $\sigma \in \Gamma$ | $\delta(p, \sigma)$                  |
|-----------|---------------------|--------------------------------------|
| $q_s$     | 0                   | $(q_s, 0, \rightarrow)$              |
| $q_s$     | 1                   | $(q_s, 1, \rightarrow)$              |
| $q_s$     | $\square$           | $(q_1, \square, \leftarrow)$         |
| $q_s$     | $\triangleright$    | $(q_s, \triangleright, \rightarrow)$ |
| $q_1$     | 0                   | $(q_2, \square, \rightarrow)$        |
| $q_1$     | 1                   | $(q_3, \square, \rightarrow)$        |
| $q_1$     | $\square$           | $(q_1, \square, -)$                  |
| $q_1$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |
| $q_2$     | 0                   | $(q_s, 0, \leftarrow)$               |
| $q_2$     | 1                   | $(q_s, 0, \leftarrow)$               |
| $q_2$     | $\square$           | $(q_s, 0, \leftarrow)$               |
| $q_2$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |
| $q_3$     | 0                   | $(q_s, 1, \leftarrow)$               |
| $q_3$     | 1                   | $(q_s, 1, \leftarrow)$               |
| $q_3$     | $\square$           | $(q_s, 1, \leftarrow)$               |
| $q_3$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |

Start the machine with input **010**

- $q_s, \triangleright 0 \underline{1} 0$
- $q_s, \triangleright \underline{0} 1 0$
- $q_s, \triangleright 0 \underline{1} 0$
- $q_s, \triangleright 0 \underline{1} \underline{0}$
- $q_s, \triangleright 0 \underline{1} 0 \underline{\square}$
- $q_1, \triangleright 0 \underline{1} \underline{0} \underline{\square}$
- $q_2, \triangleright 0 \underline{1} \underline{\square} \underline{\square}$
- $q_s, \triangleright 0 \underline{1} \underline{\square} 0$
- $q_1, \triangleright 0 \underline{1} \underline{\square} 0$
- $q_3, \triangleright 0 \underline{\square} \underline{\square} 0$
- $q_s, \triangleright 0 \underline{\square} 1 0$
- $q_1, \triangleright \underline{0} \underline{\square} 1 0$
- $q_2, \triangleright \underline{\square} \underline{\square} 1 0$
- $q_0, \triangleright \underline{\square} 0 \underline{1} 0$
- $q_1, \triangleright \underline{\square} 0 \underline{1} 0$

# An Example

$Q = \{q_s, q_h, q_1, q_2, q_3\}$ ,  $\Gamma = \{0, 1, \square, \triangleright\}$ ,  
and  $\delta$  is as follows:

| $p \in Q$ | $\sigma \in \Gamma$ | $\delta(p, \sigma)$                  |
|-----------|---------------------|--------------------------------------|
| $q_s$     | 0                   | $(q_s, 0, \rightarrow)$              |
| $q_s$     | 1                   | $(q_s, 1, \rightarrow)$              |
| $q_s$     | $\square$           | $(q_1, \square, \leftarrow)$         |
| $q_s$     | $\triangleright$    | $(q_s, \triangleright, \rightarrow)$ |
| $q_1$     | 0                   | $(q_2, \square, \rightarrow)$        |
| $q_1$     | 1                   | $(q_3, \square, \rightarrow)$        |
| $q_1$     | $\square$           | $(q_1, \square, -)$                  |
| $q_1$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |
| $q_2$     | 0                   | $(q_s, 0, \leftarrow)$               |
| $q_2$     | 1                   | $(q_s, 0, \leftarrow)$               |
| $q_2$     | $\square$           | $(q_s, 0, \leftarrow)$               |
| $q_2$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |
| $q_3$     | 0                   | $(q_s, 1, \leftarrow)$               |
| $q_3$     | 1                   | $(q_s, 1, \leftarrow)$               |
| $q_3$     | $\square$           | $(q_s, 1, \leftarrow)$               |
| $q_3$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |

Start the machine with input **010**

- $q_s, \triangleright 0 \underline{1} 0$
- $q_s, \triangleright 0 \underline{1} 0$
- $q_s, \triangleright 0 \underline{1} 0$
- $q_s, \triangleright 0 \underline{1} 0$
- $q_s, \triangleright 0 \underline{1} 0 \square$
- $q_1, \triangleright 0 \underline{1} 0 \square$
- $q_2, \triangleright 0 \underline{1} \square \square$
- $q_s, \triangleright 0 \underline{1} \square 0$
- $q_1, \triangleright 0 \underline{1} \square 0$
- $q_3, \triangleright 0 \square \square 0$
- $q_s, \triangleright 0 \square 1 0$
- $q_1, \triangleright 0 \underline{1} 0$
- $q_2, \triangleright \square \square \underline{1} 0$
- $q_0, \triangleright \square \underline{1} 0 1 0$
- $q_1, \triangleright \square \underline{1} 0 1 0$
- $q_h, \triangleright \square \underline{1} 0 1 0$

# The Second Example

$Q = \{q_s, q_h, q_1\}$ ,  $\Gamma = \{0, 1, \square, \triangleright\}$ , and  $\delta$  is as follows:

| $p \in Q$ | $\sigma \in \Gamma$ | $\delta(p, \sigma)$                  |
|-----------|---------------------|--------------------------------------|
| $q_s$     | 0                   | $(q_s, 0, \rightarrow)$              |
| $q_s$     | 1                   | $(q_s, 1, \rightarrow)$              |
| $q_s$     | $\square$           | $(q_1, \square, \leftarrow)$         |
| $q_s$     | $\triangleright$    | $(q_s, \triangleright, \rightarrow)$ |
| $q_1$     | 0                   | $(q_h, 1, -)$                        |
| $q_1$     | 1                   | $(q_1, 0, \leftarrow)$               |
| $q_1$     | $\triangleright$    | $(q_h, \triangleright, \rightarrow)$ |

# The Third Example

$Q = \{q_s, q_h, q_c, q_l, q_t\}$ ;  $\Gamma = \{\square, \triangleright, 0, 1\}$ ; two work tapes.

# The Third Example

$Q = \{q_s, q_h, q_c, q_l, q_t\}$ ;  $\Gamma = \{\square, \triangleright, 0, 1\}$ ; two work tapes.

$$\langle q_s, \triangleright, \triangleright, \triangleright \rangle \rightarrow \langle q_c, \triangleright, \triangleright, \rightarrow, \rightarrow, \rightarrow \rangle$$

$$\langle q_c, 0, \square, \square \rangle \rightarrow \langle q_c, 0, \square, \rightarrow, \rightarrow, - \rangle$$

$$\langle q_c, 1, \square, \square \rangle \rightarrow \langle q_c, 1, \square, \rightarrow, \rightarrow, - \rangle$$

$$\langle q_c, \square, \square, \square \rangle \rightarrow \langle q_l, \square, \square, \leftarrow, -, - \rangle$$

$$\langle q_l, 0, \square, \square \rangle \rightarrow \langle q_l, \square, \square, \leftarrow, -, - \rangle$$

$$\langle q_l, 1, \square, \square \rangle \rightarrow \langle q_l, \square, \square, \leftarrow, -, - \rangle$$

$$\langle q_l, \triangleright, \square, \square \rangle \rightarrow \langle q_t, \square, \square, \rightarrow, \leftarrow, - \rangle$$

$$\langle q_t, \square, \triangleright, \square \rangle \rightarrow \langle q_h, \triangleright, 1, -, -, - \rangle$$

$$\langle q_t, 0, 1, \square \rangle \rightarrow \langle q_h, 1, 0, -, -, - \rangle$$

$$\langle q_t, 1, 0, \square \rangle \rightarrow \langle q_h, 0, 0, -, -, - \rangle$$

$$\langle q_t, 0, 0, \square \rangle \rightarrow \langle q_t, 0, \square, \rightarrow, \leftarrow, - \rangle$$

$$\langle q_t, 1, 1, \square \rangle \rightarrow \langle q_t, 1, \square, \rightarrow, \leftarrow, - \rangle$$

# $\{0, 1, \square, \triangleright\}$ vs. Larger Alphabets

Suppose  $M$  has  $k$  tapes with the alphabet  $\Gamma$ .

# $\{0, 1, \square, \triangleright\}$ vs. Larger Alphabets

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A symbol of  $M$  is encoded by a string  $\sigma \in \{0, 1\}^*$  of length  $\log |\Gamma|$ .

# $\{0, 1, \square, \triangleright\}$ vs. Larger Alphabets

Suppose  $\mathbb{M}$  has  $k$  tapes with the alphabet  $\Gamma$ .

A symbol of  $\mathbb{M}$  is encoded by a string  $\sigma \in \{0, 1\}^*$  of length  $\log |\Gamma|$ .

States: A state  $q$  is turned into states  $q, \langle q, \sigma_1^1, \dots, \sigma_1^k \rangle$  where  $|\sigma_1^1| = \dots = |\sigma_1^k| = 1, \dots, \langle q, \sigma_{\log |\Gamma|}^1, \dots, \sigma_{\log |\Gamma|}^k \rangle$  where  $|\sigma_{\log |\Gamma|}^1| = \dots = |\sigma_{\log |\Gamma|}^k| = \log |\Gamma|$ .

# $\{0, 1, \square, \triangleright\}$ vs. Larger Alphabets

Suppose  $\mathbb{M}$  has  $k$  tapes with the alphabet  $\Gamma$ .

A symbol of  $\mathbb{M}$  is encoded by a string  $\sigma \in \{0, 1\}^*$  of length  $\log |\Gamma|$ .

States: A state  $q$  is turned into states  $q, \langle q, \sigma_1^1, \dots, \sigma_1^k \rangle$  where  $|\sigma_1^1| = \dots = |\sigma_1^k| = 1, \dots, \langle q, \sigma_{\log |\Gamma|}^1, \dots, \sigma_{\log |\Gamma|}^k \rangle$  where  $|\sigma_{\log |\Gamma|}^1| = \dots = |\sigma_{\log |\Gamma|}^k| = \log |\Gamma|$ .

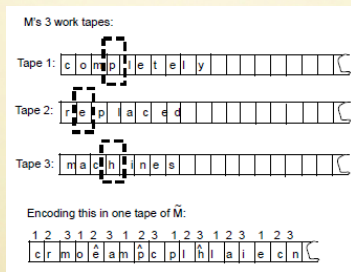
A computation step of  $\mathbb{M}$  is simulated in  $\tilde{\mathbb{M}}$  by  $\log |\Gamma|$  steps to read,  $\log |\Gamma|$  steps to write, and  $\log |\Gamma|$  steps to relocate the heads.

# One Tape vs. Many Tapes

# One Tape vs. Many Tapes

The basic idea is to interleave  $k$  tapes into one tape.

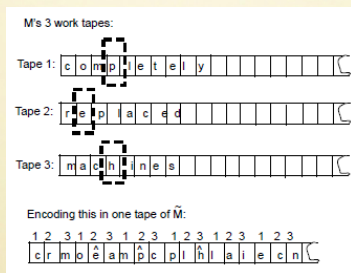
The first  $n + 1$  cells are reserved for the input.



# One Tape vs. Many Tapes

The basic idea is to interleave  $k$  tapes into one tape.

The first  $n + 1$  cells are reserved for the input.



Every symbol  $a$  of  $\mathbb{M}$  is turned into two symbols  $a, \hat{a}$  in  $\tilde{\mathbb{M}}$ , with  $\hat{a}$  used to indicate head position.

# One Tape vs. Many Tapes

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The machine  $\tilde{M}$  copies the input bits to the first imaginary tape. The head then moves left to the  $(n+2)$ -th cell.

# One Tape vs. Many Tapes

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Sweeping the tape cells from left to right. Record in the register the  $k$  symbols marked with the hat  $\hat{\_}$ .

# One Tape vs. Many Tapes

The machine  $\tilde{M}$  copies the input bits to the first imaginary tape. The head then moves left to the  $(n+2)$ -th cell.

Sweeping the tape cells from left to right. Record in the register the  $k$  symbols marked with the hat  $\hat{\_}$ .

Sweeping the tape cells from right to left to update using the transitions of  $M$ .

# One Unidirectional vs. Bidirectional Tape

The idea is that  $\tilde{M}$  makes use of the alphabet  $\Gamma \times \Gamma$ .

M's tape is infinite in both directions:

⌋ | | | c o m p l e t e l y | | | ⌋

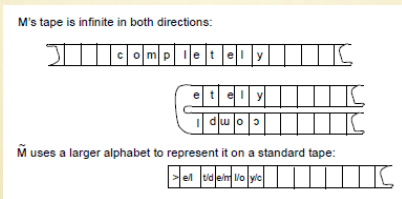
⌋ e t e l y | | | | ⌋  
⌋ i d w o r d | | | | ⌋

$\tilde{M}$  uses a larger alphabet to represent it on a standard tape:

> e|l | i d | e|l m | l | o | y | c | | | | ⌋

# One Unidirectional vs. Bidirectional Tape

The idea is that  $\tilde{M}$  makes use of the alphabet  $\Gamma \times \Gamma$ .



Every state  $q$  of  $M$  is turned into  $\bar{q}$  and  $\underline{q}$ .

## Simulation of TM by URM

# Simulating TM by URM

Suppose  $M$  is a 3-tape TM with the alphabet  $\{0, 1, \square, \triangleright\}$ .

# Simulating TM by URM

Suppose  $M$  is a 3-tape TM with the alphabet  $\{0, 1, \square, \triangleright\}$ .

The URM that simulates  $M$  can be designed as follows:

- Suppose that  $R_m$  is the right most register that is used by a program calculating  $x-1$ .
- The head positions are stored in  $R_{m+1}, R_{m+2}, R_{m+3}$ .
- The three binary strings in the tapes are stored respectively in  $R_{m+4}, R_{m+7}, R_{m+10}, \dots$ ,  
 $R_{m+5}, R_{m+8}, R_{m+11}, \dots$ ,  
 $R_{m+6}, R_{m+9}, R_{m+12}, \dots$
- The states of  $M$  are encoded by the states of the URM.
- The transition function of  $M$  can be easily simulated by the program of the URM.

# Homework

Encode the addition function by  $k$ -tape Turing machine two nature number partitioned by  $\#$  on the input tape, for example, **11010#1001**, and then try to encode the function by 1-tape Turing machine.