Computability Theory IX

Undecidability

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Assignment 3 was announced!

The deadline is Dec. 12.

Undecidability

Decidability and Undecidability

A predicate $M(\mathbf{x})$ is decidable if its characteristic function $c_M(\mathbf{x})$ given by

$$c_M(\mathbf{x}) = \begin{cases} 1, & \text{if } M(\mathbf{x}) \text{ holds,} \\ 0, & \text{if } M(\mathbf{x}) \text{ does not hold.} \end{cases}$$

is computable.

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is computable.

The predicate $M(\mathbf{x})$ is undecidable if it is not decidable.

Undecidability Result

Theorem

The problem '$x \in W_x$' is undecidable.

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Proof
The characteristic function of this problem is given by

$$c(x) = \begin{cases} 1, & \text{if } x \in W_x, \\ 0, & \text{if } x \notin W_x. \end{cases}$$

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Suppose $c(x)$ was computable. Then the function $g(x)$ defined below would also be computable.

$$g(x) = \begin{cases} 0, & \text{if } c(x) = 0, \\ \text{undefined}, & \text{if } c(x) = 1. \end{cases}$$

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Suppose $c(x)$ was computable. Then the function $g(x)$ defined below would also be computable.

$$g(x) = \begin{cases} 0, & \text{if } c(x) = 0, \\ \text{undefined}, & \text{if } c(x) = 1. \end{cases}$$

Let m be an index for g. Then

$$m \in W_m \text{ iff } c(m) = 0 \text{ iff } m \notin W_m.$$

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Corollary
There is a computable function h such that both '$x \in Dom(h)$' and '$x \in Ran(h)$' are undecidable.

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Corollary
There is a computable function h such that both '$x \in Dom(h)$' and '$x \in Ran(h)$' are undecidable.

Proof
Let

$$h(x) = \begin{cases} x, & \text{if } x \in W_x, \\ \text{undefined}, & \text{if } x \notin W_x. \end{cases}$$

Clearly $x \in Dom(h)$ iff $x \in W_x$ iff $x \in Ran(h)$.

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If $y \in W_x$ were decidable then $x \in W_x$ would be decidable.

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If $y \in W_x$ were decidable then $x \in W_x$ would be decidable.

This is known as Halting Problem.

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This is known as Halting Problem.

In this proof we have reduced the problem '$x \in W_x$' to the problem '$y \in W_x$'. The reduction shows that the latter is at least as hard as the former.

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Proof
Consider the function f defined by

$$f(x,y) = \begin{cases} 0, & \text{if } x \in W_x, \\ \text{undefined}, & \text{if } x \notin W_x. \end{cases}$$

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By s-m-n theorem there is some total computable function $k(x)$ such that $\phi_{k(x)}(y) \simeq f(x,y)$.

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By s-m-n theorem there is some total computable function $k(x)$ such that $\phi_{k(x)}(y) \simeq f(x,y)$.

It is clear that $\phi_{k(x)} = \mathbf{0}$ iff $x \in W_x$.

Undecidability Result

Corollary

The problem '$\phi_x \simeq \phi_y$' is undecidable.

Undecidability Result

Theorem

Let c be any number. The followings are undecidable.

(a) Acceptance Problem: '$c \in W_x$',

(b) Printing Problem: '$c \in E_x$'.

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By s-m-n theorem there is some total computable function $k(x)$ such that $\phi_{k(x)}(y) \simeq f(x, y)$.

It is clear that $c \in W_{k(x)}$ iff $x \in W_x$ iff $c \in E_{k(x)}$.

More on Undecidability

Exercise I

$x \in E_x$

Exercise II

$W_x = W_y$

Exercise III

$\phi_x(y) = 0$

Exercise IV

E_x is infinite.

Exercise V

'$\phi_x = g$', where g is any fixed computable function.