



Mathematical Foundation of Computer Sciences I

Regular Languages and Finite Automata

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Instructor and Textbook

Instructor



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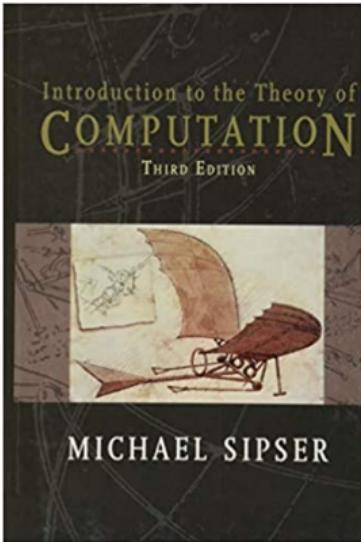
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Textbook



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[Sip12] *Introduction to the Theory of Computation*,
Michael Sipser, 2012



Regular Languages and DFA

Definition (DFA)

A deterministic finite automaton (DFA) is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

- ① Q is a finite set called the **states**,
- ② Σ is a finite set called the **alphabet**,
- ③ $\delta : Q \times \Sigma \rightarrow Q$ is the **transition function**,
- ④ $q_0 \in Q$ is the **start state**, and
- ⑤ $F \subseteq Q$ is the set of **accept states**.

Formal Definition of Computation

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a finite automaton and let $w = w_1 w_2 \dots w_n$ be a string with $w_i \in \Sigma$ for all $i \in [n]$. Then M accepts w if a sequence of states r_0, r_1, \dots, r_n in Q exists with:

- ① $r_0 = q_0$,
- ② $\delta(r_i, w_{i+1}) = r_{i+1}$ for $i = 0, \dots, n - 1$, and
- ③ $r_n \in F$.

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- ① $r_0 = q_0$,
- ② $\delta(r_i, w_{i+1}) = r_{i+1}$ for $i = 0, \dots, n - 1$, and
- ③ $r_n \in F$.

We say that M recognizes A if

$$A = \{w \mid M \text{ accepts } w\}$$



Definition (Regular languages)

A language is called **regular** if some finite automaton recognizes it.

Examples of Regular Languages



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$$\{(ab)^n \mid \forall n \geq 0\}$$

Examples of Regular Languages



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$\{a^n b^n \mid \forall n \geq 0\}$

Examples of Regular Languages



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$\{(ab)^n \mid \forall n \geq 0\}$

$\{a^n b^n \mid \forall n \geq 0\}$

$\{ab, a^2b^2, \dots a^n b^n\}$



Definition

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- **Union:** $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.
- **Concatenation:** $A \circ B = \{xy \mid x \in A \text{ and } y \in B\}$.
- **Kleene star:** $A^* = \{x_1x_2\dots x_k \mid k \geq 0 \text{ and each } x_i \in A\}$.



Theorem

The class of regular languages is closed under the union operation.

In other words, if A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.

For $i \in [2]$ let $M_i = (Q_i, \Sigma_i, \delta_i, q_{0_i}, F_i)$ recognize A_i . We can assume without loss of generality $\Sigma_1 = \Sigma_2$:

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- Let $a \in \Sigma_2 - \Sigma_1$.

For $i \in [2]$ let $M_i = (Q_i, \Sigma_i, \delta_i, q_{0_i}, F_i)$ recognize A_i . We can assume without loss of generality $\Sigma_1 = \Sigma_2$:

- Let $a \in \Sigma_2 - \Sigma_1$.
- We add $\delta_1(r, a) = r_{trap}$, where r_{trap} is a new state with $\delta_1(r_{trap}, w) = r_{trap}$ for every w .

Proof

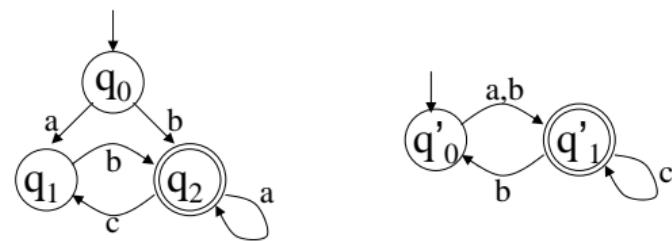
We construct $M = (Q, \Sigma, \delta, q_0, F)$ to recognize $A_1 \cup A_2$:

- ① $Q = Q_1 \times Q_2 = \{(r_1, r_2) \mid r_1 \in Q_1 \text{ and } r_2 \in Q_2\}$.
- ② $\Sigma = \Sigma_1 = \Sigma_2$.
- ③ For each $(r_1, r_2) \in Q$ and $a \in \Sigma$ we let

$$\delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a))$$

- ④ $q_0 = (q_1, q_2)$.
- ⑤ $F = (F_1 \times Q_2) \cup (Q_1 \times F_2) = \{(r_1, r_2) \mid r_1 \in F_1 \text{ or } r_2 \in F_2\}$.

A Sample



Theorem

The class of regular languages is closed under the concatenation operation.

In other words, if A_1 and A_2 are regular languages, so is $A_1 \circ A_2$.



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We prove the above theorem by nondeterministic finite automata.

Nondeterministic Finite Automata



Definition (NFA)

A nondeterministic finite automaton (NFA) is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

- ① Q is a finite set called the **states**,
- ② Σ is a finite set called the **alphabet**,
- ③ $\delta : Q \times \Sigma_\epsilon \rightarrow \mathcal{P}(Q)$ is the **transition function**, where $\Sigma_\epsilon = \Sigma \cup \{\epsilon\}$
- ④ $q_0 \in Q$ is the **start state**, and
- ⑤ $F \subseteq Q$ is the set of **accept states**.

Formal Definition of Computation

Let $N = (Q, \Sigma, \delta, q_0, F)$ be a nondeterministic finite automaton and let $w = w_1 w_2 \dots w_m$ be a string with $w_i \in \Sigma_\varepsilon$ for all $i \in [m]$. Then N accepts w if a sequence of states r_0, r_1, \dots, r_m in Q exists with:

- ① $r_0 = q_0$,
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Formal Definition of Computation

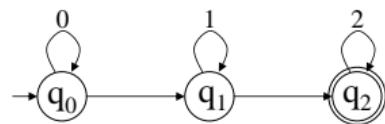
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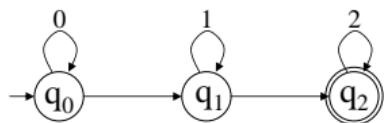
We say that N recognizes A if

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Examples of NFA

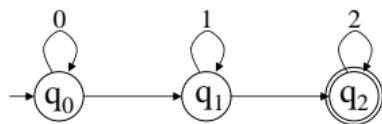


Examples of NFA

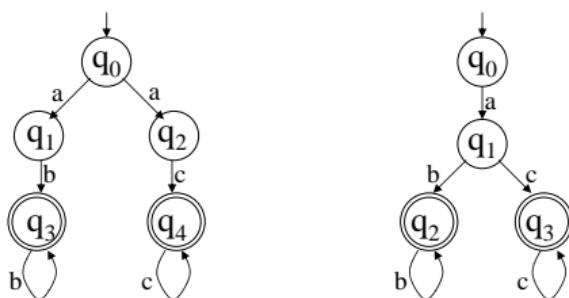


Accepts $\{0^*1^*2^*\}$

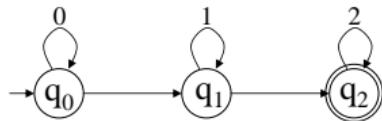
Examples of NFA



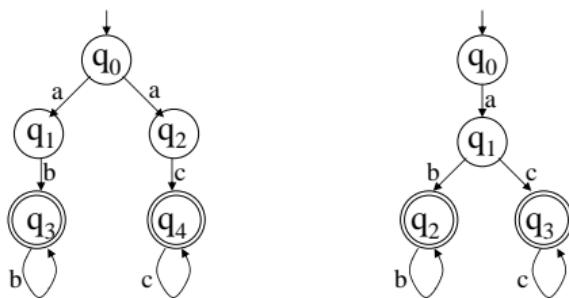
Accepts $\{0^*1^*2^*\}$



Examples of NFA



Accepts $\{0^*1^*2^*\}$



Accepts $\{ab^+, ac^+\}$



Theorem

Every NFA has an equivalent DFA, i.e., they recognize the same language.

Proof



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Proof.

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Let $N = (Q, \Sigma, \delta, q_0, F)$ be the NFA recognizing some language A . We construct a DFA $M = (Q', \Sigma, \delta', q'_0, F')$ recognizing the same A .

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First assume N has no “ ε ” arrows.

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- ① $Q' = \mathcal{P}(Q)$.
- ② Let $R \in Q'$ and $a \in \Sigma$. Then we define

$$\delta'(R, a) = \{q \in Q \mid q \in \delta(r, a) \text{ for some } r \in R\}$$

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- ③ $q'_0 = \{q_0\}$.

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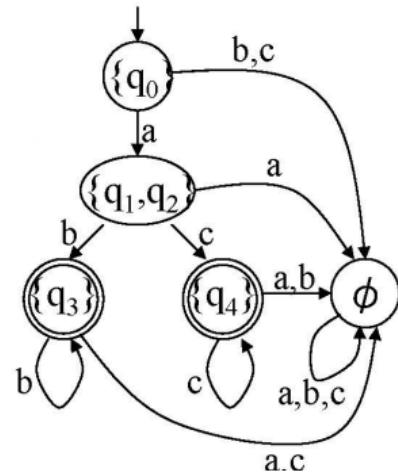
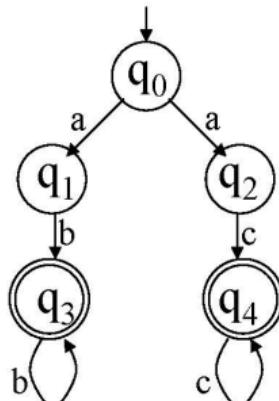
First assume N has no “ ε ” arrows.

- ① $Q' = \mathcal{P}(Q)$.
- ② Let $R \in Q'$ and $a \in \Sigma$. Then we define

$$\delta'(R, a) = \{q \in Q \mid q \in \delta(r, a) \text{ for some } r \in R\}$$

- ③ $q'_0 = \{q_0\}$.
- ④ $F' = \{R \in Q' \mid R \cap F \neq \emptyset\}$.

Determinization



Proof (cont'd)

Proof.

Now we allow " ε " arrows.

For every $R \in Q'$, i.e., $R \subseteq Q$, let

$$E(R) = \{q \in Q \mid \begin{array}{l} q \text{ can be reached from } R \\ \text{by traveling along 0 and more } \varepsilon \text{ arrows} \end{array}\}$$

① $Q' = \mathcal{P}(Q)$.

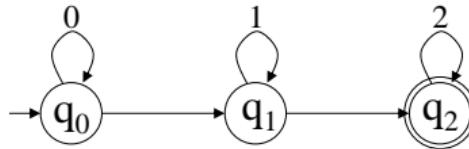
② Let $R \in Q'$ and $a \in \Sigma$. Then we define

$$\delta'(R, a) = \{q \in Q \mid q \in E(\delta(r, a)) \text{ for some } r \in R\}$$

③ $q'_0 = E(\{q_0\})$.

④ $F' = \{R \in Q' \mid R \cap F \neq \emptyset\}$.

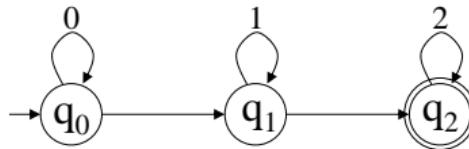
Example of ε -Transition Removal



Put a new transition \xrightarrow{a} where $\xrightarrow{\varepsilon^* a \varepsilon^*}$

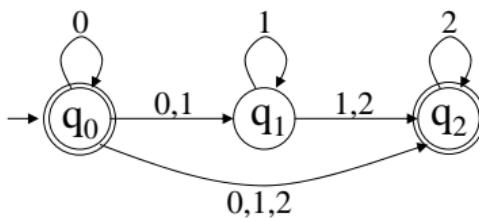
If $q_0 \xrightarrow{\varepsilon^*} q_f$ for $q_f \in F$, add q_0 to F

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Corollary

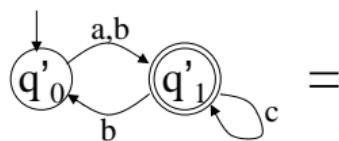
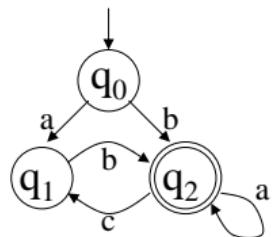
A language is regular if and only if some nondeterministic finite automaton recognizes it.

Second Proof of the Closure under Union

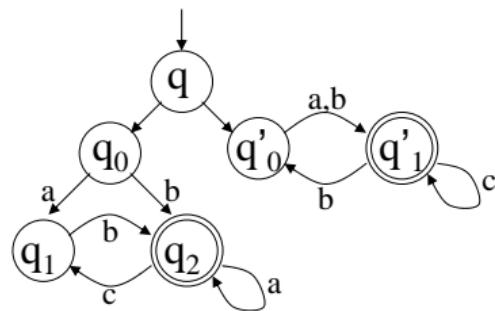
For $i \in [2]$ let $N_i = (Q_i, \Sigma, \delta_i, q_i, F_i)$ recognize A_i . We construct an $N = (Q, \Sigma, \delta, q_0, F)$ to recognize $A_1 \cup A_2$:

- ① $Q = \{q_0\} \cup Q_1 \cup Q_2$.
- ② q_0 is the start state.
- ③ $F = F_1 \cup F_2$.
- ④ For any $q \in Q$ and any $a \in \Sigma_\varepsilon$

$$\delta(q, a) = \begin{cases} \delta_1(q, a) & q \in Q_1 \\ \delta_2(q, a) & q \in Q_2 \\ \{q_1, q_2\} & q = q_0 \text{ and } a = \varepsilon \\ \emptyset & q = q_0 \text{ and } a \neq \varepsilon \end{cases}$$



=



Theorem

The class of regular languages is closed under the concatenation operation.

For $i \in [2]$ let $N_i = (Q_i, \Sigma_i, \delta_i, q_i, F_i)$ recognize A_i . We construct an $N = (Q, \Sigma, \delta, q_1, F_2)$ to recognize $A_1 \circ A_2$:

- ① $Q = Q_1 \cup Q_2$.
- ② The start state q_1 is the same as the start state of N_1 .
- ③ The accept states F_2 are the same as the accept states of N_2 .
- ④ For any $q \in Q$ and any $a \in \Sigma_\varepsilon$

$$\delta(q, a) = \begin{cases} \delta_1(q, a) & q \in Q_1 - F_1 \\ \delta_1(q, a) & q \in F_1 \text{ and } a \neq \varepsilon \\ \delta_1(q, a) \cup \{q_2\} & q \in F_1 \text{ and } a = \varepsilon \\ \delta_2(q, a) & q \in Q_2 \end{cases}$$

Theorem

The class of regular languages is closed under the star operation.

Proof

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1 . We construct an $N = (Q, \Sigma, \delta, q_0, F)$ to recognize A_1^* :

- ① $Q = \{q_0\} \cup Q_1$.
- ② The start state q_0 is the new start state.
- ③ $F = \{q_0\} \cup F_1$.
- ④ For any $q \in Q$ and any $a \in \Sigma_\varepsilon$

$$\delta(q, a) = \begin{cases} \delta_1(q, a) & q \in Q_1 - F_1 \\ \delta_1(q, a) & q \in F_1 \text{ and } a \neq \varepsilon \\ \delta_1(q, a) \cup \{q_1\} & q \in F_1 \text{ and } a = \varepsilon \\ \{q_1\} & q = q_0 \text{ and } a = \varepsilon \\ \emptyset & q = q_0 \text{ and } a \neq \varepsilon \end{cases}$$

Regular Expression

Definition

We say that R is a **regular expression** if R is

- ① a for some $a \in \Sigma$,
- ② ε ,
- ③ \emptyset ,
- ④ $(R_1 \cup R_2)$, where R_1 and R_2 are regular expressions,
- ⑤ $(R_1 \circ R_2)$, where R_1 and R_2 are regular expressions,
- ⑥ R_1^* , where R_1 is a regular expression.

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- ⑥ R_1^* , where R_1 is a regular expression.

We often write $R_1 R_2$ instead of $(R_1 \circ R_2)$ if no confusion arises.

Language Defined by Regular Expressions

regular expression R	language $L(R)$
a	$\{a\}$
ε	$\{\varepsilon\}$
\emptyset	\emptyset
$R_1 \cup R_2$	$L(R_1) \cup L(R_2)$
$R_1 \circ R_2$	$L(R_1) \circ L(R_2)$
R_1^*	$L(R_1)^*$



Theorem

A language is regular if and only if some regular expression describes it.

The Languages Defined by Regular Expressions Are Regular

- ① $R = a$: Let $N = (\{q_1, q_2\}, \Sigma, \delta, q_1, \{q_2\})$, where $\delta(q_1, a) = \{q_2\}$ and $\delta(r, b) = \emptyset$, for all $r \neq q_1$ or $b \neq a$.

The Languages Defined by Regular Expressions Are Regular



- ① $R = a$: Let $N = (\{q_1, q_2\}, \Sigma, \delta, q_1, \{q_2\})$, where $\delta(q_1, a) = \{q_2\}$ and $\delta(r, b) = \emptyset$, for all $r \neq q_1$ or $b \neq a$.
- ② $R = \epsilon$: Let $N = (\{q_1\}, \Sigma, \delta, q_1, \{q_1\})$, where $\delta(r, b) = \emptyset$, for all r and b .
- ③ $R = \emptyset$: Let $N = (\{q_1\}, \Sigma, \delta, q_1, \emptyset)$, where $\delta(r, b) = \emptyset$, for all r and b .
- ④ $R = R_1 \cup R_2$: $L(R) = L(R_1) \cup L(R_2)$.
- ⑤ $R = R_1 \circ R_2$: $L(R) = L(R_1) \circ L(R_2)$.
- ⑥ $R = R_1^*$: $L(R) = L(R_1)^*$.

We need **generalized nondeterministic finite automata (GNFA)**- nondeterministic finite automata where in the transition arrows may have any regular expressions as labels.

- ① The start state has transition arrows going to every other state but no arrows coming in from any other state.
- ② There is only a single accept state, and it has arrows coming in from every other state but no arrows going to any other state. Furthermore, the accept state is not the same as the start state.
- ③ Except for the start and accept states, one arrow goes from every state to every other state and also from each state to itself.



Definition

A GNFA is a 5-tuple $(Q, \Sigma, \delta, q_{start}, q_{accept})$, where

- Q is a finite set of states,
- Σ is a finite alphabet,
- $\delta : (Q - \{q_{accept}\}) \times (Q - \{q_{start}\}) \rightarrow R$ is the transition function, where R is the set of regular expressions,
- q_{start} is the start state, and
- q_{accept} is the accept state.

A GNFA accepts a string $w \in \Sigma^*$ if $w = w_1 w_2 \dots w_k$, where each $w_i \in \Sigma^*$ and a sequence of states q_0, q_1, \dots, q_k exists such that

- $q_0 = q_{start}$ is the start state,
- $q_k = q_{accept}$ is the accept state, and
- for each $i \in [k]$, we have $w_i \in L(R_i)$, where $R_i = \delta(q_{i-1}, q_i)$.

Regular languages can be defined by regular expressions

Let M be the DFA for language A .

- We convert M to a GNFA G by adding a new start state and a new accept state and additional transition arrows as necessary.
 - ① The start state has transition arrows going to every other state but no arrows coming in from any other state.
 - ② There is only a single accept state, and it has arrows coming in from every other state but no arrows going to any other state. Furthermore, the accept state is not the same as the start state.
 - ③ Except for the start and accept states, one arrow goes from every state to every other state and also from each state to itself.
- Then we use a procedure `convert` on G to return an equivalent regular expression.

- ① Let k be the number of states of G .
- ② If $k = 2$, then return the regular expression R labelling the arrow from q_{start} to q_{accept} .
- ③ If $k > 2$, we select any state $q_{rip} \in Q - \{q_{start}, q_{accept}\}$ and let $G' = (Q', \Sigma, \delta', q_{start}, q_{accept})$ be the GNFA, where

$$Q' = Q - \{q_{rip}\}$$

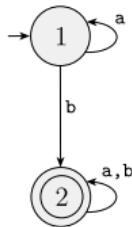
and for any $q_i \in Q' - \{q_{accept}\}$ and $q_j \in Q' - \{q_{start}\}$, let

$$\delta'(q_i, q_j) = (R_1)(R_2)^*(R_3) \cup (R_4)$$

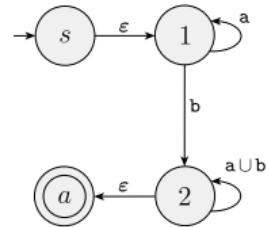
for $R_1 = \delta(q_i, q_{rip})$, $R_2 = \delta(q_{rip}, q_{rip})$, $R_3 = \delta(q_{rip}, q_j)$, and $R_4 = \delta(q_i, q_j)$.

- ④ compute $\text{convert}(G')$ and return this value.

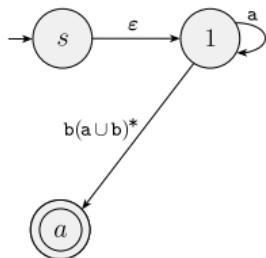
An Example



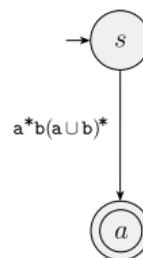
(a)



(b)



(c)



(d)

Non-Regular Languages

Languages need counting



$$C = \{w \in \{0, 1\}^* \mid w \text{ has an equal number of 0s and 1s}\}$$

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$$D = 0^+ (1^+ 0^+)^+ \cup 1^+ (0^+ 1^+)^+$$



Lemma (Pumping Lemma)

If A is a regular language, then there is a number p (i.e., the *pumping length*) where if s is any string in A of length at least p , then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

- ① for each $i \geq 0$, we have $xy^i z \in A$,
- ② $|y| > 0$, and
- ③ $|xy| \leq p$.

Any string xyz in A can be pumped along y .

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA recognizing A , and $p := |Q|$.

Let $s = s_1 s_2 \dots s_n$ be a string in A with $n \geq p$. Let r_1, \dots, r_{n+1} be the sequence of states that A enters while processing s , i.e.,

$$r_{i+1} = \delta(r_i, s_i)$$

for $i \in [n]$.

Among the first $p+1$ states in the sequence, two must be the same, say r_j and r_ℓ with $j < \ell \leq p+1$. We define

$$x = s_1 \dots s_{j-1}, y = s_j \dots s_{\ell-1}, \text{ and } z = s_\ell \dots s_n$$

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Remark: A simpler proof that only considering $s = 0^p 1^p$.

Example



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for all $i \geq 0$. Thus $xy \in 0^+$ and the contradiction follows easily.

Quiz



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Proof.

Choose p be the pumping length and consider $s = 0^p 1^{p!+p}$. By the Pumping Lemma, $s = xyz$ with $|xy| < p$ and $xy^i z \in L$.

Assume $y = 0^k$ where $k < p$, then

Then $0^{p+(i-1)k} 1^{p!+p} \in L$. Contradiction when $i = \frac{p!}{k} + 1$.

($p!$ is needed since $\frac{p!}{k}$ is a natural number.)

Other Computations

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In the automata terminology, we should guarantee

$$L(\mathcal{M}) \subseteq L(\varphi)$$

An Algorithmic Problem of FA

Given two automata M and N ,

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and,

$$L(M^c) \cup L(N) = \Sigma^*$$

New Operations

intersection

complement

emptiness

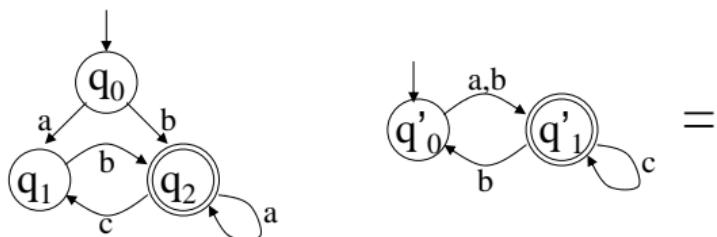
universality

Intersection of Automata

$$A = (S, \Sigma, \delta, q_0, F), B = (S', \Sigma, \delta', q'_0, F')$$

An Automaton that accepts $L(A) \cap L(B)$

$$(S \times S', \Sigma, \delta \times \delta', (q_0, q'_0), F \times F')$$

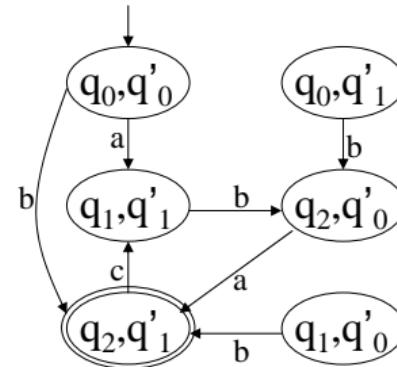
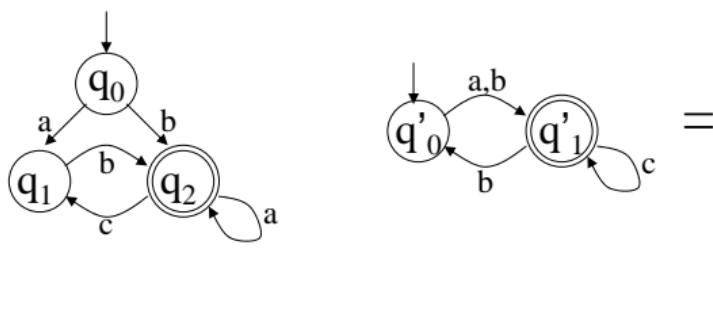


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Complement of Automata



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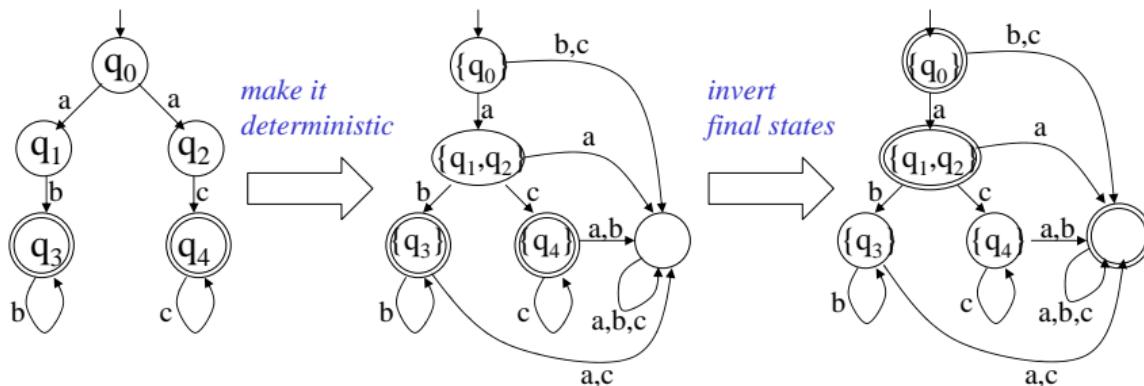
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- if A is deterministic, $A^c = (S, \Sigma, \delta, q_0, S - F)$.
- if A is non-deterministic, make A deterministic first

Assume that A is without ϵ -transition. Then

$$(P(S), \Sigma, \{(X, a, \{y \mid x \xrightarrow{a} y \text{ for } x \in X\})\}, \{q_0\}, \{X \mid X \cap F = \emptyset\})$$

Example of Complement





Emptiness?