Algorithm Design XXIII

Randomized Algorithms

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"algorithms which employ a degree of randomness as part of its logic"

—Wikipedia

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algorithms that flip coins

Why Randomized Algorithms

Randomized algorithms are a basis of

- Online algorithms

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- Approximation algorithms

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Cost of Randomness

- chance the answer is wrong.
- chance the algorithm takes a long time.

Pro and Con of Randomization

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- chance the algorithm takes a long time.

Benefits of Randomness

- on average, with high probability gives a faster or simpler algorithm.
- some problems require randomness.

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- Always bounded in runtime
- Correctness is random
- Examples: Karger's min-cut algorithm

Las Vegas VS. Monte Carlo

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Test the output of MC algorithm and stop only when a correct solution is found.

Las Vegas Algorithm: Quicksort

Quick Sort

`QuickSort(`x`);`

if $x == []$ **then** `return []`;
Choose pivot $t \in [n]$;
`return`
`QuickSort`$([x_i | x_i < x_t]) + [x_t] +$ `QuickSort`$([x_i | x_i \geq x_t])$;

Why Random?

If $t = 1$ always, then the complexity is $\Theta(n^2)$.

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Note that, we do not need an adversary for bad inputs.

In practice, lists that need to be sorted will consist of a mostly sorted list with a few new entries.

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Complexity Analysis

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$Z_{ij} = 1$ iff the first pivot in $\{i, i+1, \ldots, j\}$ is i or j.

- if the pivot is $< i$ or $> j$, then i and j are not compared;
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Thus, we have

$$Pr[Z_{ij}] = \frac{2}{j-i+1}$$

.

$$
\begin{aligned}
E[Time] &\leq E\left[\sum_{i<j} Z_{ij}\right] \\
&= \sum_{i<j} E\left[Z_{ij}\right] \\
&= \sum_{i} \sum_{i<j} \left(\frac{2}{j-i+1}\right) \\
&= 2 \cdot \sum_{i} \left(\frac{1}{2} + \cdots + \frac{1}{n-i+1}\right) \\
&\leq 2 \cdot n \cdot (H_n - 1) \\
&\leq 2 \cdot n \cdot \log n
\end{aligned}
$$

Monte Carlo: Min Cuts

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The global min cut is the min (s,t) cut over all $s,t \in V$.

Iterate over all choices of s and t and pick the smallest (s, t) cut, by simply running $O(n^2)$ the Ford-Fulkersons algorithms over all pairs.

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The complexity can be reduced by a factor of n by noting that each node must be in one of the two partitioned subsets.

Select any node s and compute a $\min(s, t)$ cut for all other vertices and return the smallest cut. This results in $O(n)$ the Ford-Fulkersons algorithm, resulting in complexity $O(n \cdot nm) = O(n^2 m)$.

`MinCut1(`G`);`

while $n > 1$ **do**
    Choose a random edge;
    Contract into a single vertex;
end

The First Randomized Algorithm

```
MinCut1(G);
while n > 1 do
    Choose a random edge;
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```

Contracting an edge means that removing the edge and combining the two vertices into a super-node.

Lemma

The chance the algorithm fails in step 1 is $\leq \frac{2}{n}$

Analysis

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Proof.

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$$OPT = \text{cost of true min cut} = \mid E(S, \bar{S}) \mid$$

and m is the number of edge.

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For all $u \in V$, let $d(u)$ be the degree of u.

$$OPT =\mid E(S, \bar{S}) \mid \leq \min_u d(u) \leq \frac{1}{n} \sum_{u \in V} d(u) \leq \frac{2 \cdot m}{n}$$

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Dividing both sides by m we have

$$\frac{\mid E(S, \bar{S}) \mid}{m} \leq \frac{2}{n}$$

Continue the analysis in each subsequent round, we have

$$Pr(\text{fail in } 1^{st} \text{ step}) \leq \frac{2}{n}$$

$$Pr(\text{fail in } 2^{nd} \text{ step} \mid \text{success in } 1^{st} \text{ step}) \leq \frac{2}{n-1}$$

$$\vdots$$

$$Pr(\text{fail in } i^{th} \text{ step} \mid \text{success till } (i-1)^{th} \text{ step}) \leq \frac{2}{n+1-i}$$

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$$Pr[Z_i | \bar{Z}_1 \cap \bar{Z}_2 \cap \cdots \cap \bar{Z}_{i-1}] \leq \frac{2}{n+1-i}$$

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Thus the probability of success is given by

$$\begin{aligned} Pr(Succ) &= Pr[\bar{Z}_1 \cap \bar{Z}_2 \cap \cdots \cap \bar{Z}_{n-2}] \geq (1 - \frac{2}{n})(1 - \frac{2}{n-1}) \ldots (1 - \frac{2}{3}) \\ &\geq \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdot \frac{n-5}{n-3} \cdot \ldots \cdot \frac{2}{4} \cdot \frac{1}{3} = \frac{1}{n} \cdot \frac{1}{n-1} \cdot \frac{2}{1} \cdot \frac{1}{1} \\ &= \frac{2}{n(n-1)} \geq \frac{2}{n^2} \end{aligned}$$

A Lemma

Suppose that the `MinCut1` is terminated when the number of vertices remaining in the contracted graph is exactly t. Then any specific min cut survives in the resulting contracted graph with probability at least

$$\frac{\binom{t}{2}}{\binom{n}{2}} = \Omega(\frac{t}{n})^2$$

The Second Randomized Algorithm


```
MinCut2(G, k);
i = 0;
while i > k do
    MinCut1(G);
    i++;
end
pick the optimization;
```

The Second Randomized Algorithm

`MinCut2(`G, k`);`

$i = 0$;
while $i > k$ **do**
    `MinCut1(`G`);`
    $i++$;
end
pick the optimization;

If we run `MinCut1` k times and pick the set of min cost, then

$$Pr(failure) \leq (1 - \frac{2}{n^2})^k \leq e^{\frac{-2k}{n^2}}$$

A Useful Bound

$$(1-a) \leq e^{-a} \qquad \forall a > 0$$

How to Choose k

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Set $k = \frac{n^2}{2} \log(\frac{1}{\delta})$ to get $1 - \delta$ success probability.

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Initial stages of the algorithm are very likely to be correct. In particular, the first step is wrong with probability at most $2/n$.

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As contracting more edges, failure probability goes up.

Since earlier ones are more accurate and slower, why not do less of them at the beginning, and more as the number of edges decreases?

The Third Randomized Algorithm

`MinCut3(`G`);`

Repeat twice{
  take $n - \frac{n}{\sqrt{2}}$ steps of contraction;
  recursively apply this algorithm;
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take better result;

$$T(n) = 2T(\frac{n}{\sqrt{2}}) + O(n^2) = n^2 \log n$$

$$
\begin{aligned}
pr(n) &= 1 - (\text{failure probability of one branch})^2 \\
&= 1 - (1 - \text{success in one branch})^2 \\
&\geq 1 - (1 - (\frac{\frac{n}{\sqrt{2}}}{n})^2 \cdot pr(\frac{n}{\sqrt{2}}))^2 \\
&= 1 - \left(1 - \frac{1}{2}\, pr\left(\frac{n}{\sqrt{2}}\right)\right)^2 \\
&= pr\left(\frac{n}{\sqrt{2}}\right) - \frac{1}{4}\, pr\left(\frac{n}{\sqrt{2}}\right)^2
\end{aligned}
$$

Success Probability

Let $x = \log_{\sqrt{2}} n$, by setting $f(x) = pr(n)$, we get

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$f(x) = \frac{1}{x}$ gives:

$$f(x-1) - f(x) = \frac{1}{x-1} - \frac{1}{x} = \frac{1}{x(x-1)} \approx \frac{1}{(x-1)^2} = f(x-1)^2$$

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Thus, $pr(n) = O\left(\frac{1}{\log n}\right)$.

Repeat `MinCut3` $O(\log n \log \frac{1}{\delta})$ times.

The Fourth Randomized Algorithm

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Complexity & Success Analysis: DIY!

Linearity of Expectation

Linearity of Expectation

$$E[\sum_i X_i] = \sum_i E[X_i]$$

Sailors Problem

A ship arrives at a port, and the 40 sailors on board go ashore for revelry. Later at night, the 40 sailors return to the ship and, in their state of inebriation, each chooses a random cabin to sleep in.

What is the expected number of sailors sleeping in their own cabins.

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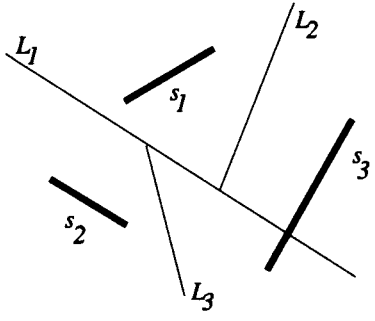
$$E[\sum_{i=1}^{40} X_i] = \sum_{i=1}^{40} E[X_i] = 1$$

Binary Planar Partition

Given a set $S = \{S_1, S_2, \ldots, S_n\}$ of non-intersecting line segments in the plane, we wish to find a binary planar partition such that every region in the partition contains at most one line segment (or a portion of one line segment).

An Example



Binary Partition Tree

A binary planar partition consists of a binary tree together with some additional information.

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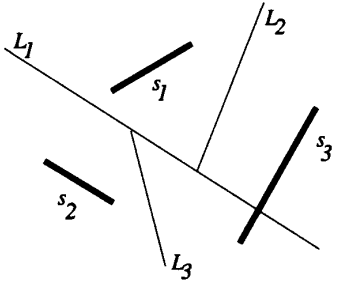
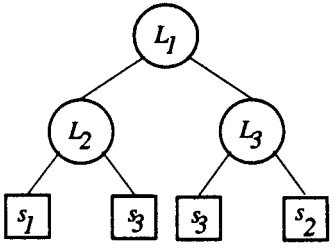
The region corresponding to the root is the entire plane. The region $r(v)$ is partitioned by $l(v)$ into two regions $r_1(v)$ and $r_2(v)$, which are the regions associated with the two children of v.

The Example





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it is not clear that a partition of size $O(n)$ always exists.

For a line segment s, let $l(s)$ denote the line obtained by extending s on both sides to infinity.

Autopartition

For a line segment s, let $l(s)$ denote the line obtained by extending s on both sides to infinity.

For the set $S = \{s_1, s_2, \ldots, s_n\}$ of line segments, a simple and natural class of partitions is the set of autopartitions, which are formed by only using lines from the set $\{l(s_1), l(s_2), \ldots, l(s_n)\}$

An Algorithm

`AutoPartition(`$\{s_1, s_2, \ldots, s_n\}$`)`

Pick a permutation π of $\{1, 2, \ldots, n\}$ uniformly at random from the $n!$ possible permutations;

while *a region contains more than one segment* **do**

  cut it with $l(s_i)$ where i is first in the ordering π such that s_i cuts that region;

end

Theorem

The expected size of the autopartition produced by `AutoPartition` *is* $O(n \log n)$.

$index(u, v) = i$ if $l(u)$ intersects $i - 1$ other segments before hitting v.

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Let $C_{uv} = 1$ if $u \dashv v$ and 0 otherwise,

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Let $C_{uv} = 1$ if $u \dashv v$ and 0 otherwise,

$$E[C_{uv}] = Pr[u \dashv v] \leq \frac{1}{index(u, v) + 1}$$

$$
\begin{aligned}
Pr(\text{partition numbers}) &= n + E[\sum_u \sum_v C_{uv}] \\
&= n + \sum_u \sum_v E[C_{uv}] \\
&= n + \sum_u \sum_{v \neq u} Pr[u \dashv v] \\
&\leq n + \sum_u \sum_{v \neq u} \frac{1}{index(u,v)+1} \\
&\leq n + \sum_u \sum_{i=1}^{n-1} \frac{2}{i+1} \\
&\leq n + 2nH_n
\end{aligned}
$$

Probability Basics and Fundamental Inequalities

Flipping an unbiased coin 1000 times. How many heads?

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Binomial Distribution

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$$X \sim B(1000, 1/2) \qquad Pr[X = x] = \binom{1000}{x} 2^{-1000}$$

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But that gives no intuition! Some tools to estimate the probabilities are needed.

Let $X_1 ... X_n$ be i.i.d. random variables. Define $X = \sum_{i=1}^{n} X_i$. Then as $n \to +\infty$, we have $X \sim N(n\,E(X_i), n\,Var(X_i))$.

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We have $\mathsf{Var}(X_i) = E((X_i - E(X_i))^2) = 1/4$. Therefore, for $n = 1000$ we have

$$\sigma = \sqrt{\mathsf{Var} \sum_i X_i} = \sqrt{\sum_i \mathsf{Var} X_i} = \sqrt{n/4} = \sqrt{250} \approx 16$$

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We can expect to be within 2σ (95% chance), so we probably get 470 to 530 heads. 600 is a big surprise!

Markov's inequality

Let X be a non-negative random variable. It is the case that: $E(X) \geq tPr[y \geq t]$, therefore:

$$Pr[y \geq t] \leq \frac{E(X)}{t}$$

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Chebyshev's inequality:

Let X a random variable of variance σ^2 that can now take negative values. It is the case that:

$$Pr\left[|X - E(X)| > t\sigma\right] \leq \frac{1}{t^2}$$

Applying to Coins

Applying Chebyshev's inequality to n coins such that $E(X) = np$ and $Var(X) = \frac{n}{4}$, we have:

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For $n = 1000$ coins, the probability of $|X - 500| > 2\sigma$ is less than $\frac{1}{4}$. In other words, we expect with probability greater than $3/4$, the number of heads to be between $[468, 516]$.

Randomized Complexity Classes

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Common Complexity Class

P: Deterministic Polynomial Time. We have $L \in \mathbf{P}$ iff there exists a polynomial-time algorithm that decides L.

NP: Nondeterministic Polynomial-Time. We have $L \in \mathbf{NP}$ iff for every input $x \in L$ there exists some solution string y such that a polynomial-time algorithm can accept x if $x \in L$ given the solution y.

ZPP: Zero-error probabilistic Polynomial-Time. $L \in \mathbf{ZPP}$ iff there exists an algorithm that decides L, and the expected value of its running time is polynomial. Note that this class describes the Las Vegas algorithms.

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RP: Randomized polynomial time. This class has tolerance for one-sided error, that is, $L \in \mathbf{RP}$ iff there exists a polynomial-time algorithm A such that:

- if $x \in L$ then A accepts with probability $\geq 1/2$.
- if $x \notin L$ then A rejects.

PP: Probabilistic Polynomial. $L \in \mathbf{PP}$ iff there exists a polynomial-time algorithm A s.t.

- if $x \in L$ then A accepts with probability $\geq 1/2$.
- if $x \notin L$ then A rejects with probability $> 1/2$.

PP: Probabilistic Polynomial. $L \in \mathbf{PP}$ iff there exists a polynomial-time algorithm A s.t.

- if $x \in L$ then A accepts with probability $\geq 1/2$.
- if $x \notin L$ then A rejects with probability $> 1/2$.

BPP: Bounded Probabilistic Polynomial. $L \in \mathbf{BPP}$ iff there exists a poly-time algorithm A s.t.

- if $x \in L$ then A accepts with probability $\geq 2/3$
- if $x \notin L$ then A rejects with probability $\geq 2/3$.

$$\mathbf{P} \subseteq \mathbf{ZPP} \subseteq \mathbf{RP} \subseteq \mathbf{NP} \subseteq \mathbf{PP}$$

$$\mathbf{RP} \subseteq \mathbf{BPP} \subseteq \mathbf{PP}$$

Relations

$$\mathbf{P} \subseteq \mathbf{ZPP} \subseteq \mathbf{RP} \subseteq \mathbf{NP} \subseteq \mathbf{PP}$$

$$\mathbf{RP} \subseteq \mathbf{BPP} \subseteq \mathbf{PP}$$

Conjecture

$$\mathbf{P} = \mathbf{BPP} \subseteq \mathbf{NP}$$