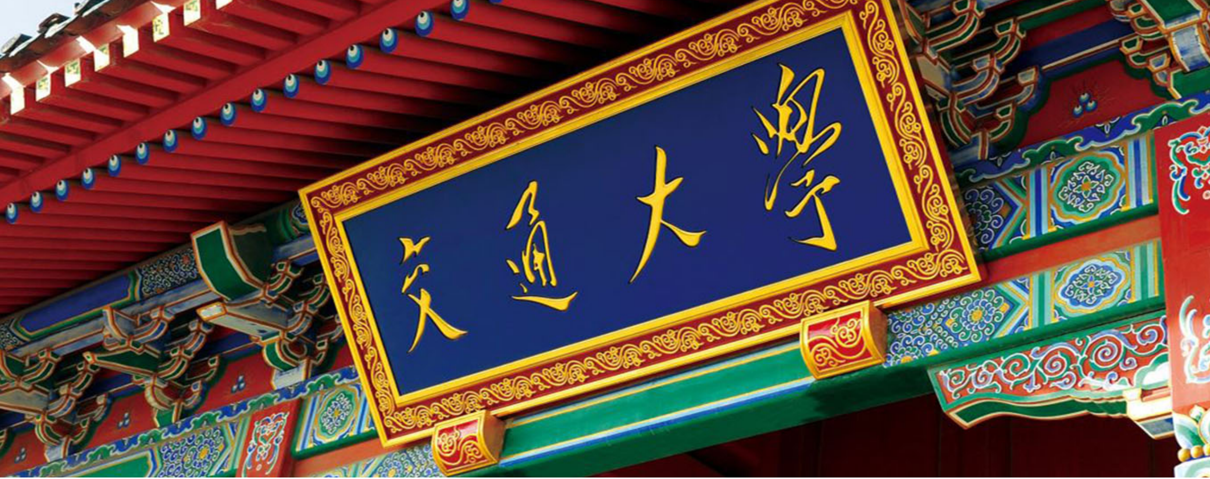

Algorithms Design III

Algorithms with Numbers II

Guoqiang Li
School of Software

SHANGHAI JIAO TONG UNIVERSITY

Primality

Fermat's Little Theorem

Theorem

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$$a^{p-1} \equiv 1 \pmod{p}$$

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multiplying all numbers in each representation, then gives $(p-1)! \equiv a^{(p-1)}.(p-1)! \pmod{p}$, and thus

$$1 \equiv a^{(p-1)} \pmod{p}$$

A (Problematic) Algorithm for Testing Primality

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```
PRIMALITY(N)
Positive integer N;

Pick a positive integer a < N at random;
if a^(N-1) ≡ 1 (mod N) then
    return yes;
    else return no;
end
```

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Our best hope: for composite N, most values of a will fail the test.

Rather than fixing an arbitrary value of a, we should choose it randomly from $\{1, \ldots, N-1\}$.

Carmichael Number

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There are composite numbers N *such that for every* $a < N$ *relatively prime to* N,

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Example:

$$561 = 3 \cdot 11 \cdot 17$$

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If $a^{N-1} \not\equiv 1 \pmod{N}$ for some a relatively prime to N, then it must hold for at least half the choices of $a < N$.

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Assume some $b < N$ satisfies $b^{N-1} \equiv 1(\mod N)$, then

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The one-to-one function $b \mapsto a \cdot b(\mod N)$ shows that at least as many elements fail the test as pass it.

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Therefore, (for non-Carmichael numbers)

- $Pr(\texttt{PRIMALITY}$ returns yes when N is prime$) = 1$
- $Pr(\texttt{PRIMALITY}$ returns yes when N is not prime$) \leq 1/2$

Primality Testing with Low Error Probability


```
PRIMALITY2(N)
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Positive integer N;

Pick positive integers $a_1, \ldots, a_k < N$ at random;

if $a_i^{N-1} \equiv 1 \mod N$ *for all* $1 \leq i \leq k$ **then**

  `return` yes;

  else `return` no;

end

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Generating Random Primes

Lagrange's Prime Number Theorem

Let $\pi(x)$ be the number of primes $\leq x$. Then $\pi(x) \approx x/ln(x)$, or more precisely,

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Such abundance makes it simple to generate a random n-bit prime:

- Pick a random n-bit number N.
- Run a primality test on N.
- If it passes the test, output N; else repeat the process.

Generating Random Primes

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- Exercise 1.34!

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- Examples: Quicksort, Hashing

Cryptography

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Even Ida, an intruder, will break the rules of communications positively.

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IOW, knowing $e(x)$ tells her little or nothing about what x might be.

Private VS. Public Schemes

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Bob is able to implement a digital lock, to which only he has the key. Now by making this digital lock public, he gives Alice a way to send him a secure message.

Private-Key Schemes: One-Time Pad

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$$e : \langle messages \rangle \to \langle encoded\ \ messages \rangle$$

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- The function e_r is a bijection, and it is its own inverse:

$$e_r(e_r(x)) = (x \oplus r) \oplus r = x \oplus 0 = x$$

Why Secure?

Alice and Bob pick r at random.

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This will ensure that if Eve intercepts the encoded message $y = e_r(x)$, she gets no information about x.

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- 128-bit fixed size.
- repeatedly use
- no techniques to break are better than brute-force.

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Eve is welcome to see as many encrypted messages, but she will not be able to decode them, under certain assumptions.

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Pick up two primes p and q and let $N = pq$.

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Bob retain the value d as his secret key, with which he can decode all messages that come to him by simply raising them to the d-th power modulo N.

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$x^{1+k(p-1)(q-1)} - x$ is divisible by p (since $x^{p-1} \equiv 1 \mod p$) and likewise by q. Since p and q are primes, this expression must be divisible by $N = pq$.

RSA protocols

Bob chooses his public and secret keys:

- He starts by picking two large (n-bit) random primes p and q.
- His public key is (N, e) where $N = pq$ and e is a $2n$-bit number relatively prime to $(p-1)(q-1)$.
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Alice wishes to send message x to Bob

- She looks up his public key (N, e) and sends him $y = (x^e \mod N)$.
- He decodes the message by computing $y^d \mod N$.

Security Assumption of RSA

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The security of RSA hinges upon a simple assumption

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How might Eve try to guess x

she could try to factor N to retrieve p and q, and then figure out d by inverting e modulo $(p-1)(q-1)$, but we believe factoring to be hard.

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A signing algorithm that, given a message and a private key, produces a signature.

Digital Signature

A digital signature scheme is a mathematical scheme for demonstrating the authenticity of a digital message or document.

In a digital signature scheme, there are two algorithms, signing and verifying.

A signing algorithm that, given a message and a private key, produces a signature.

A signature verifying algorithm that, given a message, public key and a signature, either accepts or rejects the message's claim to authenticity.

Is Communication Safe?

Is a communication safe in the internet when cryptography is unbreakable?

Is Communication Safe?

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Is a communication safe in the internet when cryptography is unbreakable?

- No!

The NSPK Protocol

$$
\begin{aligned}
A \longrightarrow B: &\quad \{A, N_A\}_{+K_B} \\
B \longrightarrow A: &\quad \{N_A, N_B\}_{+K_A} \\
A \longrightarrow B: &\quad \{N_B\}_{+K_B}
\end{aligned}
$$

An Attack

$$
\begin{array}{rcll}
A & \longrightarrow & I: & \{A, N_A\}_{+K_I} \\
I(A) & \longrightarrow & B: & \{A, N_A\}_{+K_B} \\
B & \longrightarrow & I(A): & \{N_A, N_B\}_{+K_A} \\
I & \longrightarrow & A: & \{N_A, N_B\}_{+K_A} \\
A & \longrightarrow & I: & \{N_B\}_{+K_I} \\
I(A) & \longrightarrow & B: & \{N_B\}_{+K_B}
\end{array}
$$

The Fixed NSPK Protocol

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$$\begin{aligned} A \longrightarrow B: &\quad \{A, N_A\}_{+K_B} \\ B \longrightarrow A: &\quad \{B, N_A, N_B\}_{+K_A} \\ A \longrightarrow B: &\quad \{N_B\}_{+K_B} \end{aligned}$$

$$\begin{array}{rcll} A & \longrightarrow & I: & \{A, N_A\}_{+K_I} \\ I(A) & \longrightarrow & B: & \{A, N_A\}_{+K_B} \\ B & \longrightarrow & I(A): & \{B, N_A, N_B\}_{+K_A} \\ I & \not\longrightarrow & A: & \{I, N_A, N_B\}_{+K_A} \end{array}$$

Homework

Homework

- Assignment 1 (1 week). Exercises 0.1, 0.2, 1.14, 1.20, 1.31 and 1.35.