A Introduction to Game Theory

Xiuting Tao

Outline

- 1st a brief introduction of Game theory
- 2nd Strategic games
- 3rd Extensive games

Game theory and CS

- There has been a remarkable increase in work at the interface of computer science and game theory in the past decade.

----Joseph Y. Halpern

《Computer Science and Game Theory: A Brief Survey》

- 1^{st} a brief introduction of Game theory

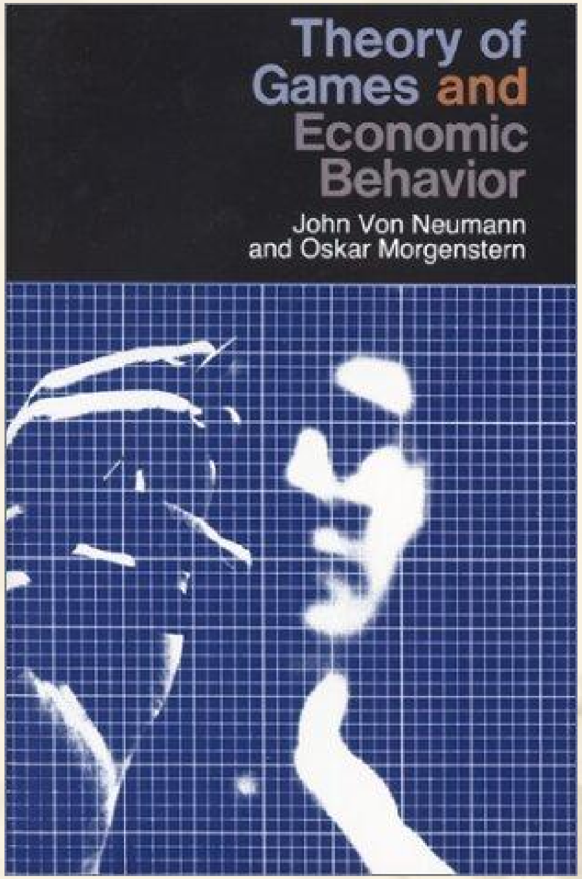
In the beginning

- Some game-theoretic ideas traced back to the 18-th century.
- Emile Borel (1871~1956) and John von Neumann (1903~1957) began the major development of game theory.

The book

- Game theory became a field since the book of John von Neumann and Oskar Morgenstern published in 1944.



John Nash(1928-)

- Received his Ph.D. from Princeton University with a 28-page thesis on his 22-nd birthday.
  - Invented the notion of Nash equilibrium.
- Wrote a seminal paper on bargain theory.

What is Game Theory?

Game theory is a study of how to mathematically determine the best strategy for given conditions in order to optimize the outcome

Rationality

Assumptions:

- **humans are rational beings**
- **humans always seek the best alternative in a set of possible choices**

Why assume rationality?

- narrow down the range of possibilities
- predictability

Types of Games

- **Perfect** vs. **Imperfect information**
- **Sequential** vs. **Simultaneous moves**
- **Zero** vs. **non-zero sum**
- **Cooperative** vs. **conflict**

Types of game and Equilibrium

Information \ order	Strategic game	Extensive game
perfect	Nash Equilibrium (John Nash)	subgame perfect equilibrium (Reinhard Selten)
imperfect	Bayesian equilibrium (John Harsanyi)	perfect Bayesian equilibrium (Reinhard Selten)

Two examples

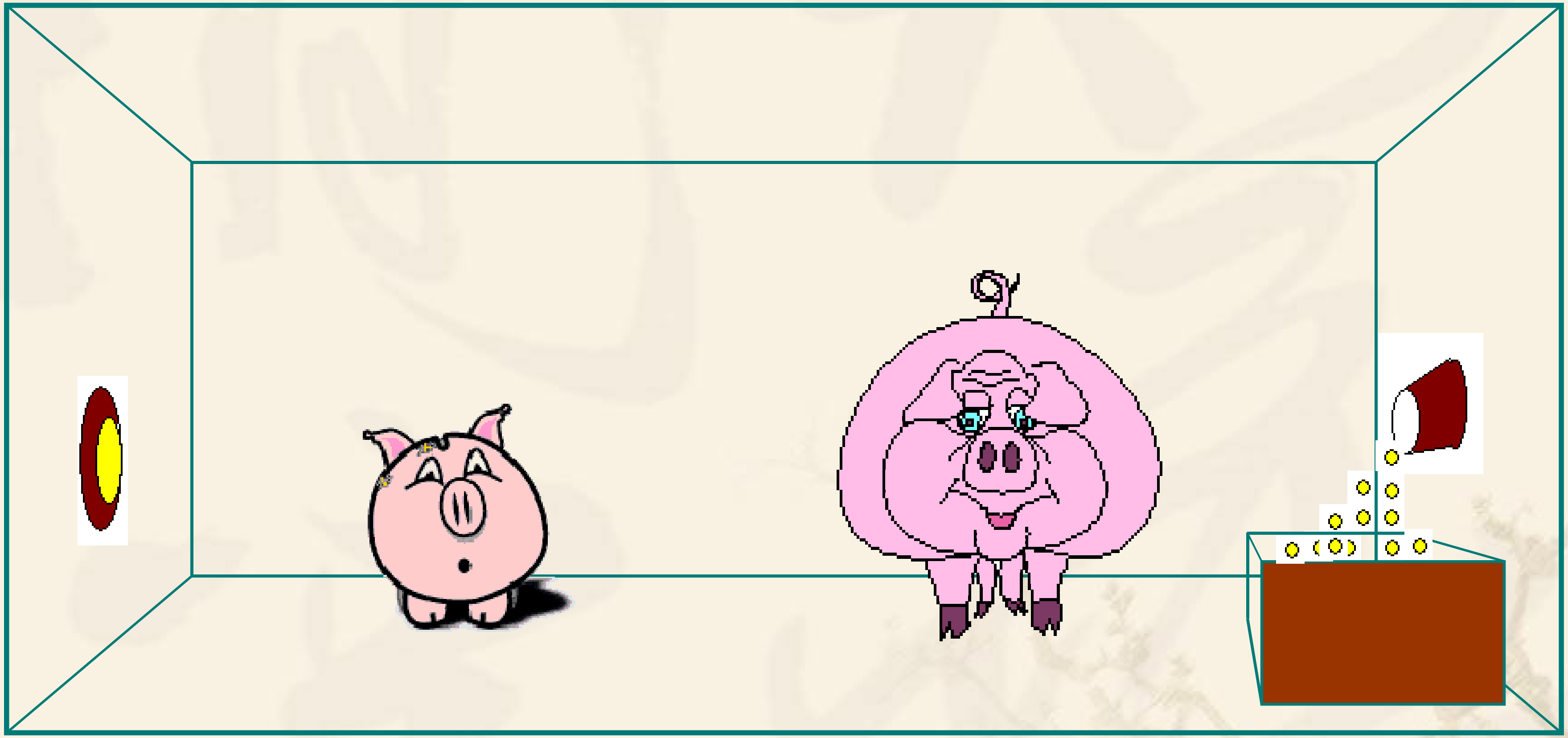
- 1) Prisoner’s Dilemma
- 2) Boxed Pigs

Prisoner’s Dilemma

Prisoner’s Dilemma

		Prisoner 2	
		Blame	Don't
Prisoner 1	Blame	10 , 10	0 , 20
	Don't	20 , 0	1 , 1

Boxed Pigs



Cost to press button = 2 units

When button is pressed, food given = 10 units

Decisions, decisions...

Little Pig

Big Pig

	Press	Wait
Press	5 , 1	4 , 4
Wait	9 , -1	0 , 0

2^{nd} Strategic game

- < N, (Ai), (ui) >

A **strategic game** consists of

- a finite set N (the set of **players**)
- for each player $i \in N$ a nonempty set A_i (the set of **actions** available to player i)
- for each player $i \in N$ a preference relation $\succsim_i$ on $A = \times_{j \in N} A_j$ (the **preference relation** of player i).

If the set A_i of actions of every player i is finite then the game is *finite*.

Nash Equilibrium

“If there is a set of strategies with the property that no player can benefit by changing her strategy while the other players keep their strategies unchanged, then that set of strategies and the corresponding payoffs constitute the Nash Equilibrium. "

Nash Equilibrium

A Nash equilibrium of a strategic game $\langle N, (A_i), (\succsim_i) \rangle$, is a profile $a^* \in A$ of actions with the property that for every player $i \in N$ we have

$$(a^*_{-i}, a^*_i) \succsim_i (a^*_{-i}, a_i) \text{ for all } a_i \in A_i.$$

Imperfect Information

- Partial or no information concerning the opponent is given in advance to the player's decision.

Bayesian game

A **Bayesian game** consists of $\langle N, \Omega, (A_i), (T_i), (\tau_i), (p_i), (\succsim_i) \rangle$

- a finite set N (the set of **players**)
- a finite set Ω (the set of **states**)

and for each player $i \in N$

- a set A_i (the set of **actions** available to player i)
- a finite set T_i (the set of **signals** that may be observed by player i) and a function $\tau_i : \Omega \to T_i$ (the **signal function** of player i)
- a probability measure p_i on Ω (the **prior belief** of player i) for which $p_i(\tau_i^{-1}(t_i)) > 0$ for all $t_i \in T_i$
- a preference relation $\succsim_i$ on the set of probability measures over $A \times \Omega$ (the **preference relation** of player i), where $A = \times_{j \in N} A_j$.

BoS

Two people wish to go out together to a concert of music by either Bach or Stravinsky. Their main concern is to go out together, but one person prefers Bach and the other person prefers Stravinsky. Representing the individuals' preferences by payoff functions

	Bach	*Stravinsky*
Bach	2, 1	0, 0
Stravinsky	0, 0	1, 2

Variant of *BoS* with imperfect information

Consider another variant of the situation modeled by *BoS*, in which neither player knows whether the other wants to go out with her.

Specifically, suppose that player 1 thinks that with probability ½ player 2 wants to go out with her, and with probability ½ player 2 wants to avoid her, and player 2 thinks that with probability 2/3 player 1 wants to go out with her and with probability 1/3 player 1 wants to avoid her.

Assume that each player knows her own preferences.

Variant of *BoS* with imperfect information

Players The pair of people.

States The set of states is $\{yy, yn, ny, nn\}$.

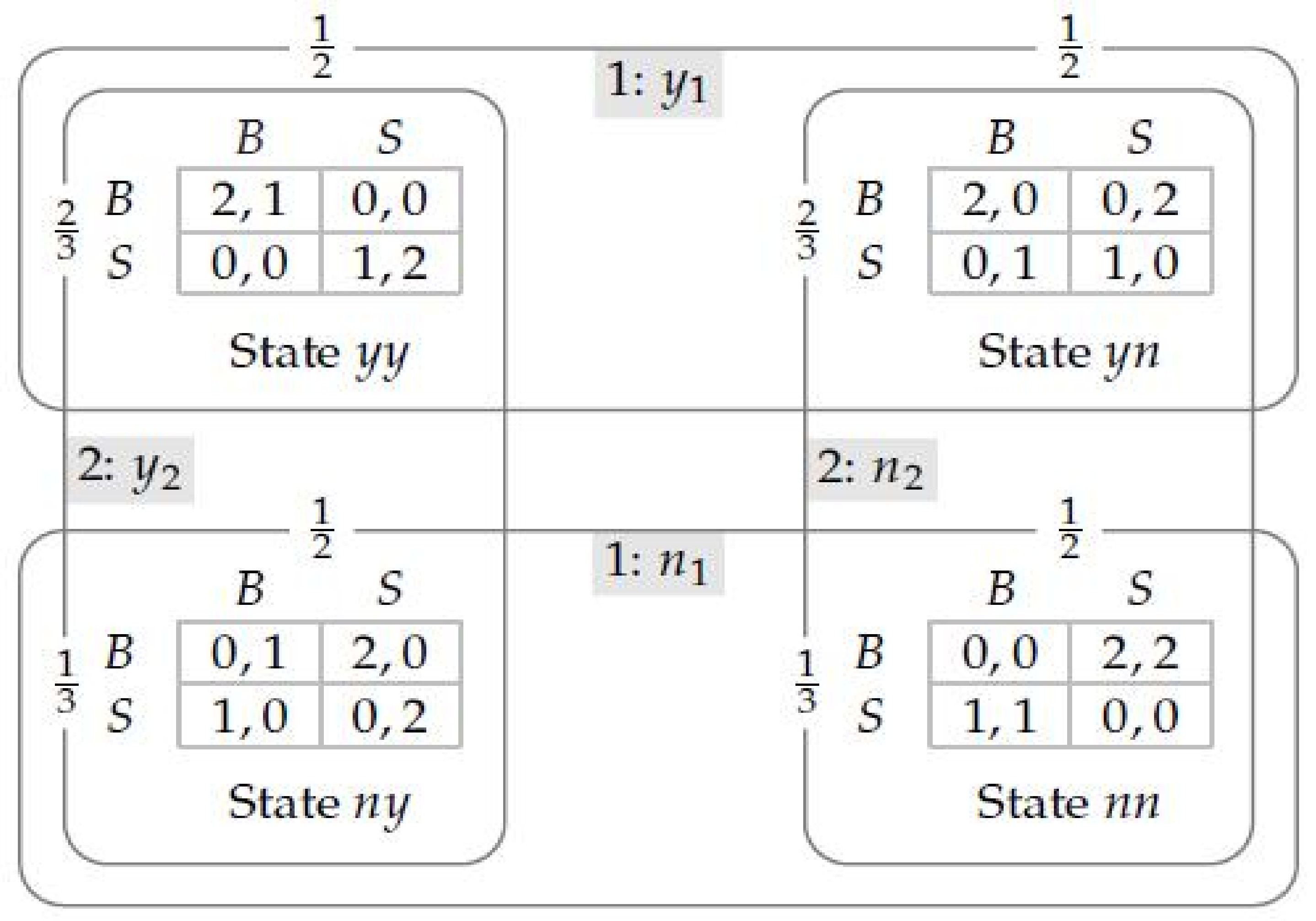
Actions The set of actions of each player is $\{B, S\}$.

Signals Player 1 receives one of two signals, y_1 and n_1; her signal function τ_1 satisfies $\tau_1(yy) = \tau_1(yn) = y_1$ and $\tau_1(ny) = \tau_1(nn) = n_1$. Player 2 receives one of two signals, y_2 and n_2; her signal function τ_2 satisfies $\tau_2(yy) = \tau_2(ny) = y_2$ and $\tau_2(yn) = \tau_2(nn) = n_2$.

Variant of *BoS* with imperfect information

Beliefs Player 1 assigns probability $\frac{1}{2}$ to each of the states yy and yn after receiving the signal y_1 and probability $\frac{1}{2}$ to each of the states ny and nn after receiving the signal n_1. Player 2 assigns probability $\frac{2}{3}$ to the state yy and probability $\frac{1}{3}$ to the state ny after receiving the signal y_2, and probability $\frac{2}{3}$ to the state yn and probability $\frac{1}{3}$ to the state nn after receiving the signal n_2.

Payoffs



1/2
1: y_1
1/2
B S
2/3 B 2, 1 0, 0
S 0, 0 1, 2
State yy
B S
2/3 B 2, 0 0, 2
S 0, 1 1, 0
State yn
2: y_2
2: n_2
1/2
1: n_1
1/2
B S
1/3 B 0, 1 2, 0
S 1, 0 0, 2
State ny
B S
1/3 B 0, 0 2, 2
S 1, 1 0, 0
State nn

Bayesian Equilibrium

A **Nash equilibrium of a Bayesian game** $\langle N, \Omega, (A_i), (T_i), (\tau_i), (p_i), (\succsim_i) \rangle$ is a Nash equilibrium of the strategic game defined as follows.

- The set of players is the set of all pairs (i, t_i) for $i \in N$ and $t_i \in T_i$.
- The set of actions of each player (i, t_i) is A_i.
- The preference ordering $\succsim^*_{(i,t_i)}$ of each player (i, t_i) is defined by

$$a^* \succsim^*_{(i,t_i)} b^* \text{ if and only if } L_i(a^*, t_i) \succsim_i L_i(b^*, t_i),$$

where $L_i(a^*, t_i)$ is the lottery over $A \times \Omega$ that assigns probability $p_i(\omega)/p_i(\tau_i^{-1}(t_i))$ to $((a^*(j, \tau_j(\omega)))_{j \in N}, \omega)$ if $\omega \in \tau_i^{-1}(t_i)$, zero otherwise.

Bayesian Equilibrium

- (B, y) (B, y)
- (B, y) (S, n)
- (B, n) (B, y)
- (B, n) (S, n)

- 3rd Extensive games

Extensive game with perfect information

An **extensive game with perfect information** has the following components.

- A Set N (the set of **players**).
- A set H of sequences (finite or infinite)that satisfies the following three properties
- The empty sequence Ø is a member of H.
- $(a^k)_{k=1,\dots,K} \in H$ (where K may be infinite) and $L < K$ then $(a^k)_{k=1,\dots,L} \in H$
- If an infinite sequence $(a^k)_{k=1}^{\infty}$ satisfies $(a^k)_{k=1,\dots,L} \in H$ satisfies $(a^k)_{k=1,\dots,L} \in H$ for every positive integer L then $(a^k)_{k=1}^{\infty} \in H$.

An example

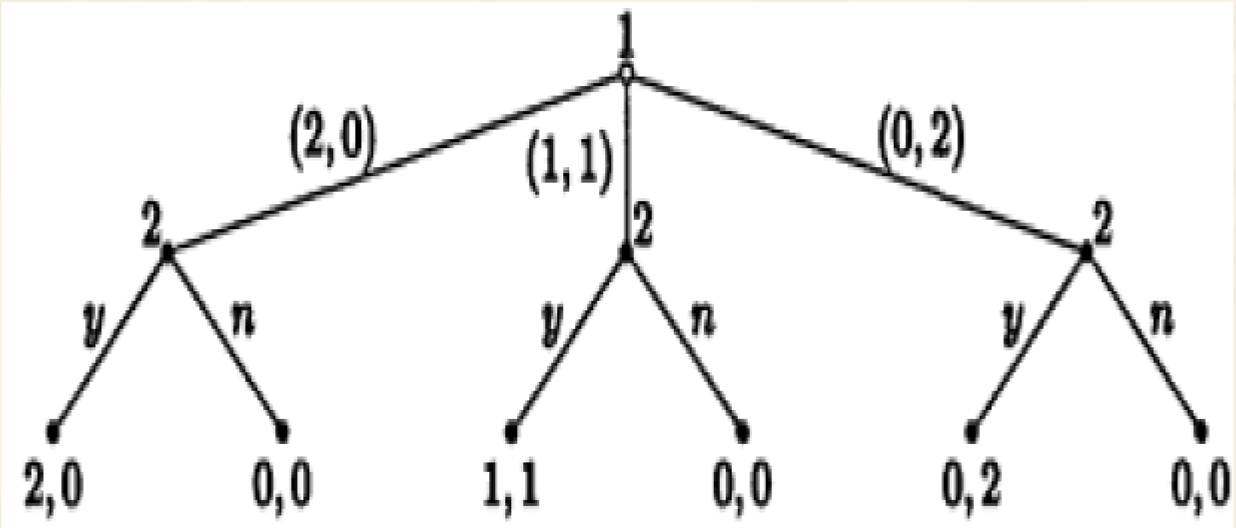
Two people use the following procedure to share two desirable identical indivisible objects. One of them proposes an allocation, which the other then either accepts or rejects. In the event of rejection, neither person receives either of the objects. Each person cares only about the number of objects he obtains.

allocating two identical indivisible objects

An extensive game that models the individuals' predicament is $\langle N, H, P, (\succsim_i)\rangle$ where

- $N = \{1,2\}$;
- H consists of the ten histories Ø, (2, 0), (1, 1), (0, 2), ((2, 0), y), ((2, 0), n), ((1, 1), y), ((1, 1), n), ((0, 2), y), ((0, 2), n);
- P(Ø) = and $P(h)$ = 2 for every nonterminal history h $h \neq \varnothing$.
- $((2,0),y) \succ_1 ((1,1),y) \succ_1 ((0,2),y) \sim_1 ((2,0),n) \sim_1 ((1,1),n) \sim_1 ((0,2),n)$ and $((0,2),y) \succ_2 ((1,1),y) \succ_2 ((2,0),y) \sim_2 ((0,2),n) \sim_2 ((1,1),n) \sim_2 ((2,0),n)$.

Payoff tree



Strategy

A strategy of player $i \in N$ in an extensive game with perfect information $\langle N, H, P, (\succsim_i) \rangle$ is a function that assigns an action in $A(h)$ to each nonterminal history $h \in H \setminus Z$ for which $P(h) = i$.

In the just example:

The strategies of player 1: {(2,0),(1,1),(0,2)}
The strategies of player 2:{yyy,yyn,yny,ynn,nyy,nyn,nny,nnn}

Nash Equilibrium

For each strategy profile $s = (s_i)_{i \in N}$ in the extensive game $\langle N, H, P, (\succsim_i) \rangle$ we define the **outcome** $O(s)$ of s to be the terminal history that results when each player $i \in N$ follows the precepts of s. That is, $O(s)$ is the (possibly infinite) history $(a^1, \ldots, a^K) \in Z$ such that for $0 \leq k < K$ we have $s_{P(a^1,\ldots,a^k)}(a^1, \ldots, a^k) = a^{k+1}$.

A **Nash equilibrium of an extensive game with perfect information** $\langle N, H, P, (\succsim_i) \rangle$ is a strategy profile s^* such that for every player $i \in N$ we have

$$O(s^*_{-i}, s^*_i) \succsim_i O(s^*_{-i}, s_i) \text{ for every strategy } s_i \text{ of player } i.$$

a problem

- The Nash equilibria of the example game are ((2, 0), *yyy*), ((2, 0), *yyn*), ((2, 0), *yny*), ((2, 0), *ynn*), ((1, 1),*nyy*), ((1, 1), *nyn*), ((0, 2), *nny*), ((2, 0), *nny*), and ((2, 0), *nnn*).

Subgame extensive game

The subgame of the extensive game with perfect information $\Gamma = \langle N, H, P, (\succsim_i)\rangle$ that follows the history h is the extensive game $\Gamma(h) = \langle N, H|_h, P|_h, (\succsim_i|_h)\rangle$, where $H|h$ is the set of sequences h' of actions for which $(h, h') \in H$, is defined by $h' \in H|_h$, for each $\succsim_i|_h$ and $h' \succsim_i|_h h''$ is defined by $(h, h') \succsim_i (h, h'')$. if and only if $h' \in H|_h$);

Subgame Perfect Equilibrium

A subgame perfect equilibrium of an extensive game with perfect information $i \in N$ is a strategy profile s^* such that for every player $h \in H \setminus Z$ and every nontermlnal history $O_h(s^*_{-i}|_h, s^*_i|_h) \succsim_i|_h O_h(s^*_{-i}|_h, s_i)$ for which $P(h)$ $=i$ we have

$$O_h(s^*_{-i}|_h, s^*_i|_h) \succsim_i|_h O_h(s^*_{-i}|_h, s_i)$$

for every strategy s_i of player i in the subgame $F(h)$.

Extensive game with imperfect information

$\langle N, H, P, f_c, (\mathcal{I}_i), (\succsim_i)\rangle$

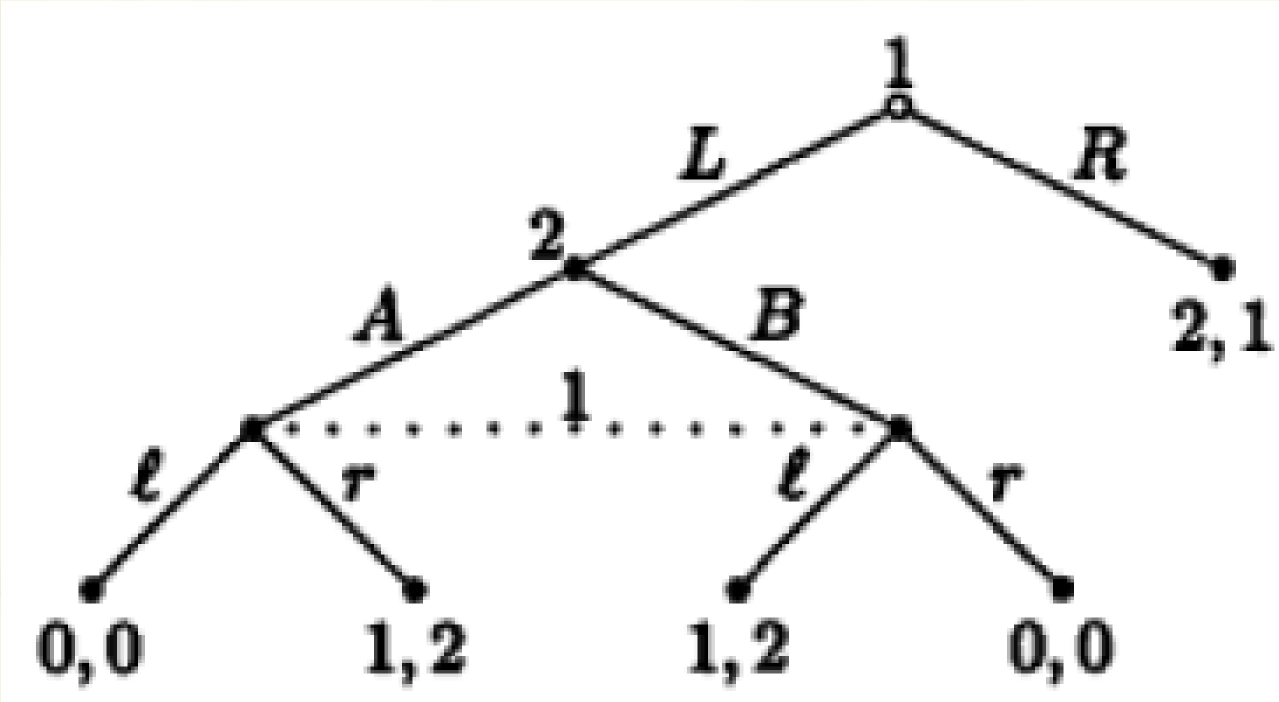
- A function P that assigns to each nonterminal history (each member of $H \setminus Z$) a member of $N \cup \{c\}$. (P is the **player function**. $P(h)$ being the player who takes an action after the history h. If $P(h) = c$ then chance determines the action taken after the history h.)

- A function f_c that associates with every history h for which $P(h) = c$ a probability measure $f_c(\cdot|h)$ on $A(h)$, where each such probability measure is independent of every other such measure. ($f_c(a|h)$ is the probability that a occurs after the history h.)

Extensive game with imperfect information

- For each player $i \in N$ a partition $\mathcal{I}_i$ of $\{h \in H: P(h) = i\}$ with the property that $A(h) = A(h')$ whenever h and h' are in the same member of the partition. For $I_i \in \mathcal{I}_i$ we denote by $A(I_i)$ the set $A(h)$ and by $P(I_i)$ the player $P(h)$ for any $h \in I_i$. ($\mathcal{I}_i$ is the **information partition** of player i; a set $I_i \in \mathcal{I}_i$ is an **information set** of player i.)

An example

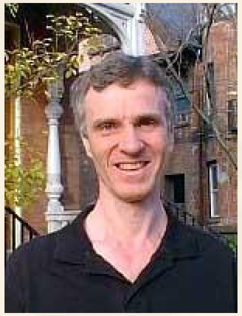


Nash equilibrium

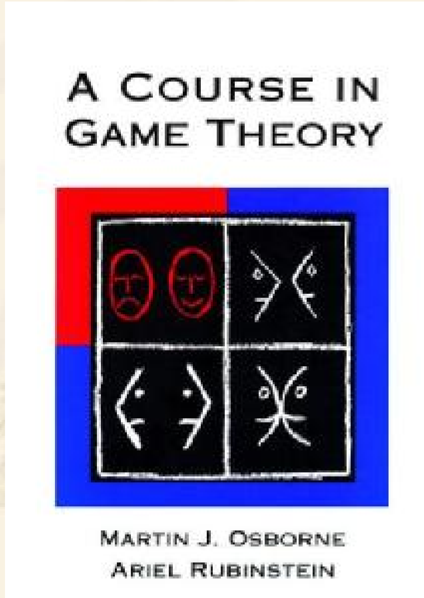
A **Nash equilibrium in mixed strategies** of an extensive game is (as before) a profile σ^* of mixed strategies with the property that for every player $i \in N$ we have

$O(\sigma^*_{-i}, \sigma^*_i) \succsim_i O(\sigma^*_{-i}, \sigma_i)$ for every mixed strategy σ_i of player i.

Four parts of the textbook




- Strategic games (Part I)
- Extensive games with perfect information (Part II)
- Extensive games without perfect information (Part III)
- Coalitional games (Part IV)



Thank you !