计算复杂性理论

傅育熙

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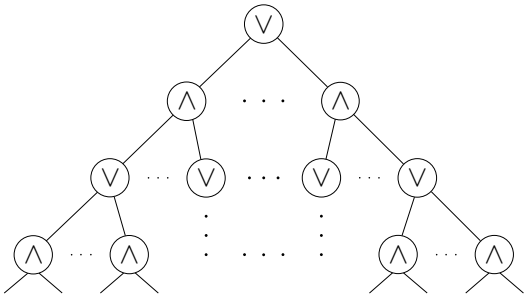
0.1 Hästad Switching Lemma

具有常数高度的电路族可能是唯一一类我们能有信心地说我们完全理解的电路族。在 $\mathbf{NC}^0$ 中的电路族有简单的刻画，也很容易看出包含关系 $\mathbf{NC}^0 \subseteq \mathbf{AC}^0$ 是严格的，见练习**??**。我们希望证明包含关系 $\mathbf{AC}^0 \subseteq \mathbf{NC}^1$ 也是严格的。例**??**中定义的奇偶函数在 $\mathbf{NC}^1$ 中，但直观上没有常数高度的电路族判定，还有很多这样的例子。上世纪八十年代，这个问题被解决了。弗斯特、萨克斯、西普塞 [89] 和阿杰泰 [8] 证明了奇偶函数 $\oplus$ 不在 $\mathbf{AC}^0$ 里。他们的证明使用了随机限制技术 [225]，证明了常数高度的计算奇偶函数的电路的大小有下界 $n^{\Omega(\log n)}$，这个下界大于任何多项式。姚期智沿用了该技术，将下界提高到了约为 $2^{n^{1/(4d)}}$，其中 d 为电路族高度 [254]。哈斯塔德简化了姚的证明并证明了很强的对换引理[108]，利用此引理，哈斯塔德给出了几乎是最优的下界 $2^{n^{1/d}}$。概率方法之外，斯摩伦斯基用代数方法证明了对换引理 [215]，拉兹博罗夫给出了一个用计数方法的简单证明 [187]。我们将介绍的就是拉兹博罗夫的证明。

Furst
Saxe
Sipser

Håstad
Switching Lemma
Smolensky
Razborov

上述所有证明的一个基本思路是利用分配律将一个高度是 $d+1$ 的交替电路（见下图）转换成高度是 d 的交替电路。图1中的上图是交换后的电路，图1中



的下图是将交换后得到的电路的第二层和第三层合并后得到的交替电路。这个简单思路的问题是转换后的交替电路可能变得太大。交替引理给出了一个证明：如果为电路的足够多的输入随机地选取一组输入，会大概率地得到一个交换后门电路个数得到控制的交替电路，这是因为若一个语句（项）中的一个字取 1（0）值，该语句（项）就消失了，一个语句（项）中的一个字取 0（1）值，该字就消失了。用归纳法我们最终得到一个两层电路。两层电路就是析取范式或合取范式，对这个电路的一部分输入确定了一组特定输入后电路的输

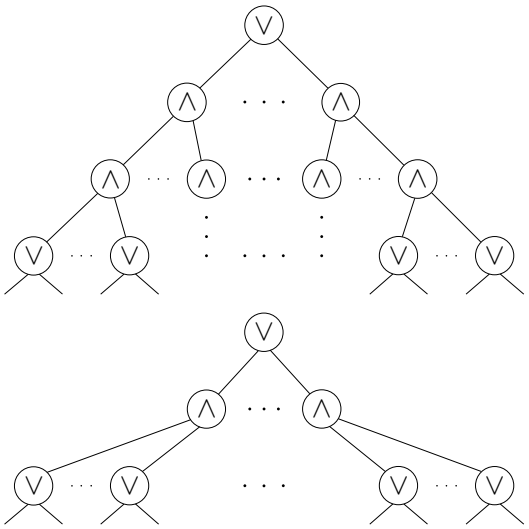


图 1　AC 电路的与或门交换

出值就确定了。这就引起矛盾，因为奇偶函数只有在所有输入值确定后输出才能确定。下面我们就解释如何兑现这一想法。

设 $X = \{x_1, \ldots, x_n\}$。若函数 $\rho : X \to X \cup \{0, 1\}$ 满足：对任意 $x \in X$ 有 $\rho(x) \in \{x, 0, 1\}$，称 ρ 为对 X 中变量的限制。若 $\rho(x) = x$，说明对变量 x 没有限制。称集合 $\{x \mid \rho(x) \neq x\}$ 为 ρ 的支集。一个限制是对 X 中部分变量的赋值，比如 $\rho = [x_1 \leftarrow 0, x_3 \leftarrow 1, x_8 \leftarrow 0]$ 将 x_1 赋值为 0，将 x_3 赋值为 1，将 x_8 赋值为 0，而对其它变量不赋值。一个限制也可看成是一个字的集合，如 $\rho = \{\overline{x_1}, x_3, \overline{x_8}\}$，也可理解成一个合取式，如 $\rho = \overline{x_1} \wedge x_3 \wedge \overline{x_8}$。

restriction

support

对于定义在 X 上的布尔函数 f，f_ρ 是定义在 $\{x \mid x \in X \wedge \rho(x) = x\}$ 上的布尔函数 $f(\rho(x_1), \ldots, \rho(x_n))$。在下面的证明中，我们需要随机地选取限制。设 $0 < n - u \leq u < n$。一个对 $X = \{x_1, \ldots, x_n\}$ 的 u-大小的随机限制 ρ 定义如下：随机地选一个大小为 u 的 X 的子集，对每个被选出的 x，通过投硬币决定其值，即

$$\rho(x) \quad = \quad \begin{cases} 1, & 1/2 \text{ 概率}, \\ 0, & 1/2 \text{ 概率}。 \end{cases}$$

对未被选出的变量 x，定义 $\rho(x) = x$. 用 R^u 表示所有对 u 个变量赋值的限

制。显然

$$|R^u| \quad = \quad \binom{n}{u} 2^u 。$$

在定义**??**中，我们引入了极小项的概念。一个 n-元布尔函数 f 可以有多个长度不等的极小项，我们用符号 $\min(f)$ 表示长度最大的极小项的长度。

引理 0.1. *设 f 为 n-元布尔函数，若 $\min(f) \leq s$，则 f 可表示为 s-析取范式。*

证明. 可以用等式 (**??**) 将 $f(x)$ 写成析取范式 $\bigvee_{f(\alpha)=1} x_\alpha$。设 f 有一个极小项 x_β。一定存在满足 $\beta \subseteq \alpha$ 的 α。使 x_β 为假的真值指派，一定使 x_α 为假，此时 $\bigvee_{f(\alpha')=1}^{\alpha' \neq \alpha} x_{\alpha'}$ 和 $f(x)$ 等价。另一方面，因为 x_β 是极小项，使 x_β 为真的真值指派，一定使 $f(x)$ 为真。结合这两方面的蕴含，得

$$f(x) \Leftrightarrow x_\beta \vee \bigvee_{f(\alpha')=1}^{\alpha' \neq \alpha} x_{\alpha'} 。$$

用归纳法即得引理结论。事实上，f 可表示成所有极小项的析取。 □

若 f 为 s-析取范式，当然有 $\min(f_\rho) \leq s$，结合引理0.1，可知集合

$$\mathrm{Bad}_f(u,s) = \{\rho \in R^u \mid \min(f_\rho) > s\}$$

就是所有使得 f_ρ 不可表示为 s-析取范式的限制 ρ。将引理0.1中的极小项换成极大项，析取范式换成合取范式，结论同样成立。拉兹博罗夫的证明方法基于下述关键引理 [187]。

引理 0.2 (拉兹博罗夫引理)**.** *若 f 是 t-合取范式，则 $|\mathrm{Bad}_f(u,s)| \leq |R^{u+s}| \cdot (4t)^s$。*

证明. 拉兹博罗夫的证明思路出奇地简单：构造从 $\mathrm{Bad}_f(u,s)$ 到 $R^{u+s} \times S$ 的编码函数 $\mathfrak{e}$ 和相应的解码函数，其中 $|S| \leq (4t)^s$。固定字的一个排序，并由此定义 f 中语句的一个序。设 $\rho \in \mathrm{Bad}_f(u,s)$，即 f_ρ 有一个长度 $s' > s$ 的极小项 τ'。去掉 τ' 中任意 $s'-s$ 个字，得到项 τ，即 τ 对这 $s'-s$ 个字不做限制。关于 f_ρ 有几点说明：

- f 中的有些语句消失了（因为取值为 1）；
- f 中的有些语句中的字消失了（因为该字的取值为 0）；

- f_ρ 不可能有语句取值为 0（因为 f_ρ 有极小项）；
- $f_{\rho\tau}$ 不可能是常量（因为 τ' 是 f_ρ 的极小项并且 τ' 的定义域严格包含 τ 的定义域）。

本证明中，我们用 $Lit(\_)$ 表示对象中字的集合，用 $Var(\_)$ 表示对象中变量的集合。设 f 为合取范式，ρ 为对 $Var(f)$ 中变量的限制。设 C_1 为第一个满足条件 $Lit(C_1)\cap Lit(\tau)\neq\emptyset$ 的 f 中的语句。用 τ_1 表示 $Var(C_1)\cap Var(\tau)$，显然 C_1 在限制 τ_1 下取值为 1。用 $\alpha_1\in\{0,1\}^t$ 表示定义在 $Var(C_1)$ 上的集合 τ_1 的特征函数，即若 C_1 的第 i 个变量在 $Var(\tau_1)$ 中，则 $\alpha_1(0^{i-1}10^{t-i})=1$，否则 $\alpha_1(0^{i-1}10^{t-i})=0$。若将 τ_1 中的那些在 $Lit(C_1)\cap Lit(\tau_1)$ 里的字取反，得到唯一的限制 $\overline{\tau_1}$。易见，在限制 $\overline{\tau_1}$ 下 C_1 不再取值 1。将 ρ 换成 $\rho\tau_1$，将 τ 换成 $\tau\setminus\tau_1$，然后重复上述操作，得到 τ_2、$\overline{\tau_2}$、α_2。用归纳法可得 $\tau_3,\tau_3,\ldots,\tau_m$、$\overline{\tau_3},\overline{\tau_4},\ldots,\overline{\tau_m}$、$\alpha_3,\alpha_4,\ldots,\alpha_m$，这里 $m\leq s$ 并且 $\tau=\tau_1\tau_2\ldots\tau_m$。我们需要一些额外的信息指明 $\tau_1,\tau_2,\ldots,\tau_m$ 和 $\overline{\tau_1},\overline{\tau_2},\ldots,\overline{\tau_m}$ 的区别。设向量 $\beta\in\{0,1\}^s$ 指明对 τ 的支集中的变量赋值和 $\overline{\tau}=\overline{\tau_1\tau_2}\ldots\overline{\tau_m}$ 对该变量的赋值是否一致，即若 x 是 $\overline{\tau}$ 中的第 i 个变量，则 $\beta(0^{i-1}10^{t-i})=1$ 当仅当 $\tau(x)=\overline{\tau}(x)$，且 $\beta(0^{i-1}10^{t-i})=0$ 当仅当 $\tau(x)\neq\overline{\tau}(x)$。编码 $\mathfrak{e}$ 的定义如下：

$$\mathfrak{e}(\rho)\ \stackrel{\text{def}}{=}\ \langle\rho\overline{\tau_1\tau_2}\ldots\overline{\tau_m},\alpha_1,\alpha_2,\ldots,\alpha_m,\beta\rangle。$$

限制 $\overline{\tau_1}$ 有如下性质：τ 中的字都不出现在 C_1 之前的那些语句中，否者与 C_1 的定义矛盾；若 τ 中的一个字的否定出现在 C_1 之前的一个语句中，该语句取值 1。由这些性质可知，C_1 是第一个满足如下条件的语句：在限制 $\overline{\tau}$ 下的值不是 1，且对于特征函数 α_1 定义域中的第 i 个字（比如 $\overline{x}$），若 $\alpha_1(x)=1$，则 $\overline{x}$ 在限制 $\overline{\tau}$ 下取假。在 C_1 之前的任何一个取值非 1 的语句中的任何一个字（比如 $\overline{y}$），若 $\overline{y}$ 是该语句中的第 i 个字且 $\alpha_1(0^{i-1}10^{t-i})=1$，则 y 未被 $\overline{\tau}$ 限制。因此，从 f 和 $\rho\overline{\tau}$ 我们可以唯一锁定 C_1。从 C_1 和 α_1 可以恢复 $\overline{\tau_1}$，从 β 和 $\overline{\tau_1}$ 可以恢复 τ_1，并由此得到 $\rho\tau_1\overline{\tau_2}\ldots\overline{\tau_m}$。用归纳法最终得到 $\rho\tau_1\tau_2\ldots\tau_m$ 以及 ρ。因此从 $\mathfrak{e}(\rho)$ 可将 ρ 恢复出来，故 $\mathfrak{e}$ 是单射。

最后，我们估算一下 $\mathfrak{e}(\rho)$ 的值域的大小。首先，$\rho\overline{\tau_1\tau_2}\ldots\overline{\tau_m}\in R^{u+s}$，所以这类可能的限制个数不超过 $|R^{u+s}|$。其次，设 α_i 含有 k_i 个 1，则 $k_i\geq 1$，并且 $k_1+\ldots+k_m=s$。串 $(\alpha_1,\alpha_2,\ldots,\alpha_m)\in\{0,1\}^{mt}$ 的个数不超过 $\prod_{i\in[m]}\binom{t}{k_i}\leq$

$\prod_{i\in[m]} t^{k_i} = t^s$。满足 $k_1 + \ldots + k_m = s$ 的正整数向量 $(k_1, \ldots, k_m)$ 的个数不超过 $\binom{s-1}{m-1} \le 2^s$（见例??）。分量 $\beta \in \{0,1\}^s$ 的个数不超过 2^s，所以 $\mathfrak{e}$ 的值域大小不超过 $|R^{u+s}|\cdot(4t)^s$。 □

拉兹博罗夫引理特别引人注目的一点是其给出的上界不依赖于输入长度，用此特点能比较容易地推出哈斯塔德对换引理。

引理 0.3 (哈斯塔德对换引理)**.** *设 f 为含有 n 个输入变量的 t-合取范式，设 $p < 1/(9t)$。ρ 是大小为 $u = (1-p)n$ 的随机限制。则有*

$$\Pr_{\rho\in_{\mathrm{R}} R^u}[f_\rho \text{不可表示为 } s\text{-析取范式}] \le (9pt)^s。$$

证明. 根据引理0.1，概率值 $\Pr_{\rho\in_{\mathrm{R}} R^u}[f_\rho$不可表示为 s-析取范式$]$ 就是 $\frac{\mathrm{Bad}_f(u,s)}{|R^u|}$。从拉兹博罗夫引理0.2、引理 Razborov-Lemma 和条件 $p < 1/(9t)$ 可得推导：

$$\frac{\mathrm{Bad}_f(u,s)}{|R^u|} \le \frac{\binom{n}{u+s}2^{u+s}(4t)^s}{\binom{n}{u}2^u} \le \left(\frac{n-u}{u}\right)^s (8t)^s \le \left(\frac{p}{1-p}\right)^s (8t)^s \le (9pt)^s。$$

上述第二个不等式之所以成立是因为：

$$\binom{n}{u+s}\Big/\binom{n}{u} = \frac{u!}{(u+s)!}\cdot\frac{(n-u)!}{(n-u-s)!} \le \frac{1}{u^s}\cdot(n-u)^s,$$

最后一个不等式成立是因为

$$1-p > 1-\frac{1}{9t} \ge 1-\frac{1}{9} = \frac{8}{9}。$$

定理得证。 □

哈斯塔德原文 [108] 中的证明用的是条件概率推导，得到了一个更紧的上界 $(5pt)^s$。必须指出的是，上述给出的拉兹博罗夫引理和哈斯塔德对换引理针对的是合取范式，对析取范式相应的结论也成立。接下来我们解释如何从交换引理推出常数高度的计算奇偶函数电路族的指数复杂性下界 [108]。

定理 0.1 (奇偶函数的电路复杂性)**.** *计算 n-元奇偶函数的高度为 $d+1$ 的电路族需 $2^{\Omega(n^{1/d})}$ 个门。*

证明. 给定一个高度是 $d+1$ 的计算 n-元奇偶函数的大小为 $S(n)$ 的电路，我们将利用哈斯塔德对换引理把该电路转换成一个高度是 2 的电路。不失一般性，假定电路的最底层为或门。我们首先将此电路视为一个高度是 $d+2$、最底层为扇入为 1 的与门的交替电路。之所以这样做是因为我们想把电路转换成一个最底层门的扇入数为 $s = 2\log S(n)$ 的电路。取 $p = 1/18$、$t = 1$ 和 $u = (1-p)n$，根据哈斯塔德对换引理，对于大小为 u 的随机限制，每个或门不能表示成一个 s-合取范式的概率不超过 $(9pt)^S(n) = \frac{1}{S(n)^2}$，存在一个或门不能表示成一个 s-合取范式的概率不超过 $\frac{1}{S(n)}$。因此一定存在一个限制，在该限制下，将最下面两层进行交换后，最底层的每个门的扇入为 $s = 2\log S(n)$，并且新电路有 $pn = \frac{n}{18}$ 个输入变量。交换后的电路有 $d+2$ 层，将倒数第二层的与门和倒数第三层的与门合为一层，得到一个 $d+1$ 层的电路。重复上述构造，设 $t = s = 2\log S(n)$、$q = 1/18k$ 和 $v = (1-q)\frac{n}{18}$。在一个 v-大小的随机限制下，倒数第二层的每个与门不能表示成一个 s-析取范式的概率不到 $(9qt)^S(n) = \frac{1}{S(n)^2}$。用同样的推理，我们得知一定存在一个限制，在该限制下，倒数第二层的每个与门可表示成一个 s-析取范式。将得到的新的电路的倒数第二层的或门和倒数第三层的或门合并，得到一个高度是 d 的，最底层与门的扇入数为 $S(n)$，输入变量不超过 $\frac{n}{18}\cdot\frac{1}{18\cdot S(n)} = \frac{n}{18^2 S(n)}$ 的交替电路。

重复上面的操作，最终我们得到一个两层的交替电路，该电路的最底层门的扇入数不超过 $s = 2\log S(n)$，输入变量数不超过 $\frac{n}{18\cdot(18\cdot S(n))^{d-1}} = \frac{n}{O((\log S(n))^{d-1})}$。将此电路中的 $S(n)$ 个变量取特定值后电路的输出为常量（当电路为合取范式时为 0，析取范式时为 1）。因此将原始给定电路的

$$n - \frac{n}{O((\log S(n))^{d-1})} + 2\log S(n)$$

个输入变量取特定值后电路的输出为常量。但 n-元奇偶函数只有在所有 n 个输入确定后才输出常量，因此

$$n \le n - \frac{n}{O((\log S(n))^{d-1})} + 2\log S(n)。$$

由上述不等式即可推得 $S(n) = 2^{\Omega(n^{1/d})}$。 □

定理0.1说，一个解决奇偶函数的常数高度的电路族的大小是指数的，由此得到我们想要的否定结论。

推论 0.1. 奇偶函数不在 $\mathbf{AC}^0$ 中。

threshold function

利用此结论，我们可以通过归约证明其它一些布尔函数不在 $\mathbf{AC}^0$ 中，比如 n-元阈值函数Th_k^n。当 n 个输入中有至少 k 个输入为 1 时，函数 Th_k^n 值为 1，否则值为 0。对任意奇数 $k \in [n]$，容易从计算 Th_k^n 的电路构造计算 $Th_k^n \wedge \neg(Th_{k+1}^n)$ 的电路，此电路可判定输入中是否准确地有 k 个输入为 1。显然，实现下述布尔函数

$$\bigvee_{k\in[n]}^{k\text{为奇数}} \left(Th_k^n \wedge \neg(Th_{k+1}^n)\right)$$

的电路计算奇偶函数。如果阈值函数在 $\mathbf{AC}^0$ 中，奇偶函数就在 $\mathbf{AC}^0$ 中，矛盾。因此阈值函数不在 $\mathbf{AC}^0$ 中。

majority function

另一个著名的在 $\mathbf{NC}^1$ 中但不在 $\mathbf{AC}^0$ 中的函数是多数票函数Maj_n，若 n 个输入中值为 1 的输入个数至少是 $n/2$，函数 Maj_n 输出 1，否则输出 0。我们将证明留给读者。

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